A predual formulation for PDE constrained optimization with total variation regularization

Christian Clason, Karl Kunisch

Institute for Mathematics and Scientific Computing
Karl-Franzens-Universität Graz

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Problem statement

Optimal control problem

(P)
$$\begin{cases} \min\limits_{u \in \mathrm{BV}(\Omega)} \frac{1}{2}\int_\Omega |y-z|^2\,dx + \beta\int_\Omega |Du|\,dx \\ \text{s.t.} \quad Ay = u \end{cases}$$

- A is partial differential operator (with b.c.) on $\Omega \subset \mathbb{R}^d$
- Assumption: A is isometry between state and control space (otherwise add $L^2(\Omega)$ regularization for u)
- Motivation: Control cost proportional to control changes

Bounded Variation

Functions of bounded variation

$$\mathrm{BV}(\Omega) = \left\{ u \in \mathrm{L}^2(\Omega);\ \int_\Omega |Du|\, dx < \infty \right\}$$

$$\int_\Omega |Du|\, dx = \sup \left\{ \int_\Omega u \operatorname{div} \vec{v}\, dx\ ;\ \vec{v} \in (\mathrm{C}_0^\infty(\Omega))^d,\ |\vec{v}|_{\ell^\infty} \leq 1 \right\}$$

- Nondifferentiable, but convex
- Optimality conditions via Fenchel duality
- Dual of BV(Ω): space of measures
- BV(Ω) not reflexive $\Rightarrow$ use predual of BV(Ω) (Hilbert space)

Formal Fenchel duality

Set

- $\Lambda(u) = Du$
- $\mathcal{F}(u) = \frac{1}{2}\int_\Omega |A^{-1}u - z|^2\,dx$
- $\mathcal{G}(\vec{q}) = \beta\int_\Omega |\vec{q}|\,dx$

Fenchel duality

Problem (P) equivalent to

(P^*) $$\sup -\mathcal{F}^*(\Lambda^* u^*) - \mathcal{G}^*(-u^*)$$

Solutions $\overline{u}, \overline{u}^*$ related by

$$\Lambda^*\overline{u}^* = \mathcal{F}'(\overline{u})$$

Formal Fenchel dual

Fenchel conjugates:

- $\Lambda^*(u^*) = -\operatorname{div} u^*$
- $\mathcal{F}^*(p^*) = \frac{1}{2}\|A^* p^*\|^2 + \langle A^* p^*; z\rangle$
- $\mathcal{G}^*(\vec{q}^*) = \mathrm{I}_\beta := \begin{cases} 0 & \text{if } |\vec{q}(x)|_{\ell^\infty} \leq \beta \text{ for a.e. } x \in \Omega, \\ \infty & \text{otherwise.} \end{cases}$

Formal dual problem

$$(P^*) \qquad \begin{cases} \inf_{u^*} \frac{1}{2}\|A^* \operatorname{div} u^*\|^2 - \langle A^* \operatorname{div} u^*, z\rangle \\ \quad \text{s.t. } |u^*|_{\ell^\infty} \leq \beta \text{ for a.e. } x \in \Omega. \end{cases}$$

Primal solution from extremality relations:

$$u = Az - AA^* \operatorname{div} u^*$$

Predual Problem

Constant functions in kernel of div: Add $\mathrm{L}^2(\Omega)$ regularization for u^*:

Regularized predual problem (P^*_γ)

$$\begin{cases} \min\limits_{u^* \in \mathrm{H}^A_{\mathrm{div}}} \frac{1}{2}\|A^* \operatorname{div} u^*\|^2_{\mathrm{L}^2(\Omega)} + \frac{\gamma}{2}\|u^*\|^2_{(\mathrm{L}^2(\Omega))^d} - \langle A^* \operatorname{div} u^*, z\rangle_{\mathrm{L}^2(\Omega)} \\ \quad \text{s.t. } |u^*|_{\ell^\infty} \leq \beta \ \text{ for a.e. } x \in \Omega. \end{cases}$$

$$\mathrm{H}^A_{\mathrm{div}} := \left\{\vec{v} \in (\mathrm{L}^2(\Omega))^d ; \operatorname{div} \vec{v} \in \mathcal{D}(A^*),\ \vec{v} \cdot \nu = 0 \text{ on } \partial\Omega\right\}$$

$\Rightarrow$ (P^*_γ) has unique solution for $\gamma > 0$, dual of (P^*_0) is (P)

Predual Regularization

Predual problem: Minimization problem with box constraints
$\Rightarrow$ solve with semismooth Newton method

- Insufficient regularity of Lagrange multipliers for box constraints
  $\Rightarrow$ Moreau-Yosida-regularization of box constraint
- $\max(0,\cdot), \min(0,\cdot) : L^q \to L^p$ Newton diff'able only if $q > p$
  $\Rightarrow$ add H^1-regularization for u^*

Optimality System

Regularized predual problem

$$\min_{u^*\in(H_0^1(\Omega))^d\cap H_{\mathrm{div}}^A} \frac{1}{2}\|A^*\operatorname{div} u^*\|^2_{L^2(\Omega)} - \langle A^*\operatorname{div} u^*, z\rangle_{L^2(\Omega)}$$
$$+\frac{\gamma}{2}\|u^*\|^2_{L^2(\Omega)} + \frac{\delta}{2}\|\nabla u^*\|^2_{L^2(\Omega)}$$
$$+\frac{1}{2c}|\max(0, c(u^*-\vec{\beta}))|^2 + \frac{1}{2c}|\min(0, c(u^*+\vec{\beta}))|^2$$

Optimality conditions

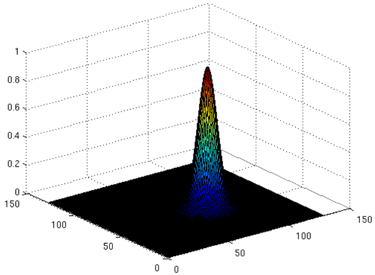
$$(-\operatorname{div})^* AA^*\operatorname{div}\overline{u}^* + \gamma\overline{u}^* - \delta\Delta\overline{u}^*$$
$$+\max(0, c(\overline{u}^*-\vec{\beta})) + \min(0, c(\overline{u}^*+\vec{\beta})) = (-\operatorname{div})^* Az$$

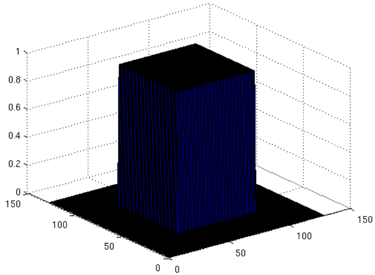
Numerical Examples

- Model Problem $Ay = y - \Delta y$ in $\Omega = [-1, 1]^2 \subset \mathbb{R}^2$ with Neumann boundary conditions
- Discretization by finite differences $((-\operatorname{div})_h = (\nabla_h)^T)$
- Comparison with $\mathrm{H}^1(\Omega)$-regularized problem

$$\begin{cases} \min\limits_{u \in \mathrm{H}^1(\Omega)} \dfrac{1}{2} \displaystyle\int_\Omega |y - z|^2 \, dx + \dfrac{\beta}{2} \displaystyle\int_\Omega |\nabla u|^2 \, dx \\ \text{s.t.} \quad Ay = u \end{cases}$$

Target state

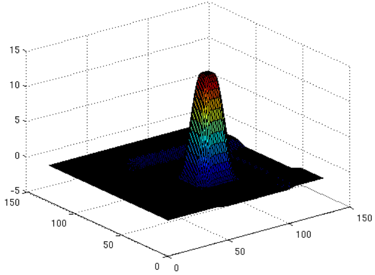


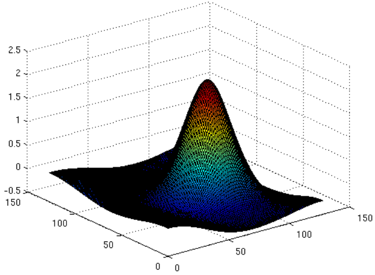


$$z_a(x, y) = e^{-50[(x-0.2)^2+(y+0.1)^2)]}$$

$$z_b(x, y) = \chi_{\{|x|<\frac{1}{2},|y|<\frac{1}{2}\}}$$

Optimal control for target z_a: BV vs. H^1 penalty

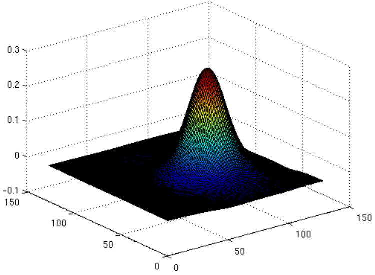




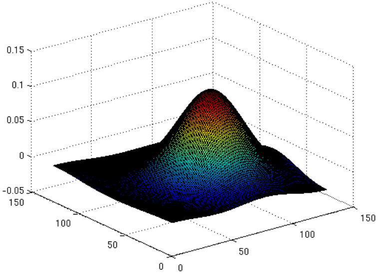
BV(Ω) H^1(Ω)

$$(\beta = 10^{-4}, \gamma = \delta = 10^{-1}, c = 10^6)$$

Optimal state for target z_a: BV vs. H^1 penalty



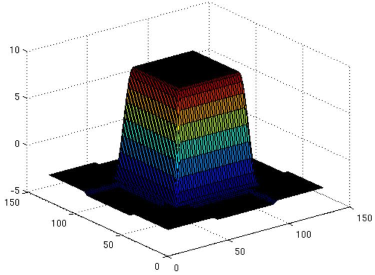
$BV(\Omega)$



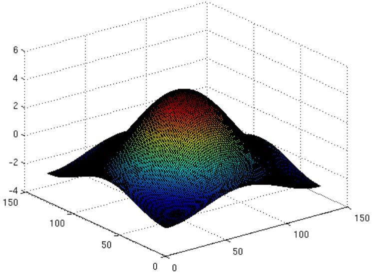
$H^1(\Omega)$

$$(\beta = 10^{-4}, \gamma = \delta = 10^{-1}, c = 10^6)$$

Optimal control for target z_b: BV vs. H^1 penalty



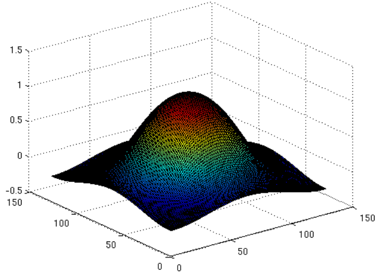
BV(Ω)

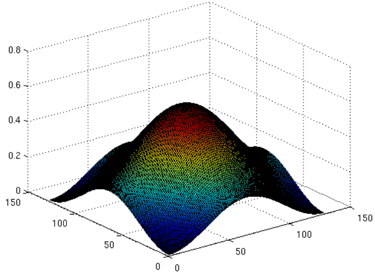


$H^1(\Omega)$

$$(\beta = 10^{-3}, \gamma = \delta = 10^{-1}, c = 10^6)$$

Optimal state for target z_b: BV vs. H^1 penalty





BV(Ω) $H^1(\Omega)$

$$(\beta = 10^{-3}, \gamma = \delta = 10^{-1}, c = 10^6)$$

Comparison with L^1-Regularization

Cost of control proportional to magnitude of control: L^1-Penalty

Optimal control problem

$$\begin{cases} \min\limits_{u \in \mathrm{L}^1(\Omega)} \dfrac{1}{2} \displaystyle\int_\Omega |y - z|^2 \, dx + \beta \int_\Omega |u| \, dx \\ \text{s.t.} \quad Ay = u \end{cases}$$

Comparison with L^1-Regularization

Set $\Lambda(y,u) = (0,u) \Rightarrow \Lambda^*(y^*,u^*) = (0,u^*)$, use Fenchel duality

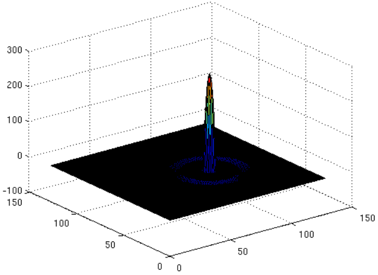
Predual problem

$$\begin{cases} \inf\limits_{u^*} \dfrac{1}{2} \|A^* u^*\|^2_{L^2(\Omega)} + \langle A^* u^*, z\rangle_{L^2(\Omega)} \\ \quad \text{s.t. } |u^*| \leq \beta \ \text{ for a.e. } x \in \Omega. \end{cases}$$

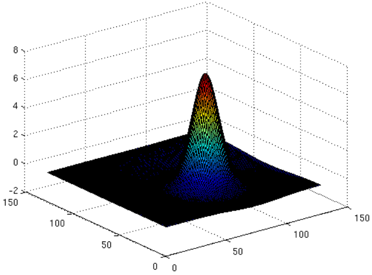
Optimality conditions of Moreau-Yosida regularization

$$AA^*\overline{u}^* + \max(0, c(\overline{u}^* - \vec{\beta})) + \min(0, c(\overline{u}^* + \vec{\beta})) = Az$$

Optimal control for target z_a: L^1 vs. L^2 penalty



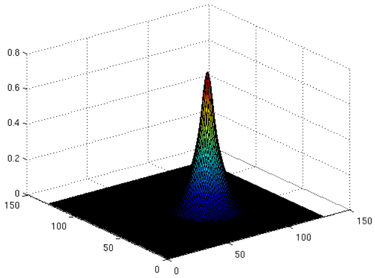
$L^1(\Omega)$

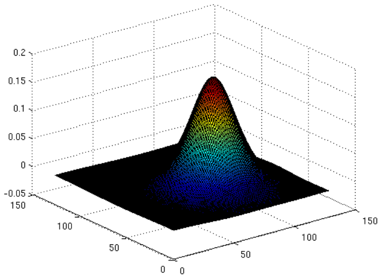


$L^2(\Omega)$

$$(\beta = 10^{-3}, c = 10^6)$$

Optimal state for target z_a: L^1 vs. L^2 penalty

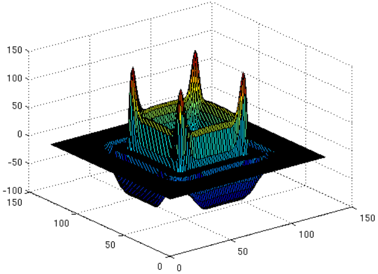




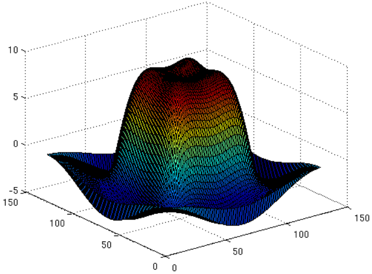
$L^1(\Omega)$ $L^2(\Omega)$

$$(\beta = 10^{-3}, c = 10^6)$$

Optimal control for target z_b: L^1 vs. L^2 penalty



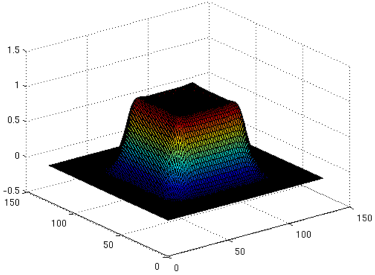
$L^1(\Omega)$



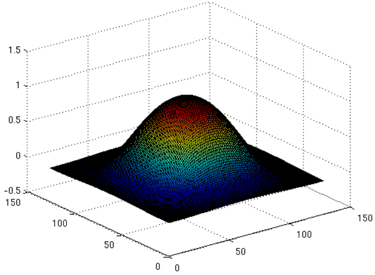
$L^2(\Omega)$

$$(\beta = 10^{-3}, c = 10^6)$$

Optimal state for target z_b: L^1 vs. L^2 penalty



$L^1(\Omega)$



$L^2(\Omega)$

$$(\beta = 10^{-3}, c = 10^6)$$