

Iterative regularization of non-smooth equations

Christian Clason Vu Huu Nhu

Fakultät für Mathematik, Universität Duisburg-Essen

Workshop on Numerical Methods for
Optimal Control and Inverse Problems
Garching, March 12, 2019

Inverse problem: find

- unknown parameter u^\dagger
e.g., heat source, diffusion constant, thermal conductivity, heat capacity, latent heat density, ...

given

- measurement y
- model $S : u \mapsto y$, e.g., solution of PDE

\rightsquigarrow solve

$$S(u) = y$$

Problem: measurement $y = y^\delta$ noisy, range of S not closed

\rightsquigarrow ill-posed, needs regularization

Solve approximate, **stable** problem:

- 1 Tikhonov regularization \rightsquigarrow optimal control
- 2 iterative regularization, e.g., **Landweber iteration**

$$u_{n+1} = u_n + w_n S'(u_n)^* (y^\delta - S(u_n)) \quad n = 1, \dots, N$$

- $S'(u)$ **Fréchet derivative**, $S'(u)^*$ adjoint
- **stopping index** $N = N(\delta) < \infty$ regularization parameter
- **regularization**: $N(\delta) \rightarrow \infty$, $u_{N(\delta)} \rightarrow u^\dagger$ for $\delta \rightarrow 0$

Here: S solution operator for **non-smooth PDE**

\rightsquigarrow **not** Fréchet differentiable

- 1 Motivation
- 2 Non-smooth equation
- 3 Bouligand–Landweber iteration
- 4 Bouligand–Levenberg–Marquardt iteration
- 5 Numerical examples

$$-\Delta y + \max(0, y) = u$$

solution operator $S : u \mapsto y$ ($:= S(u)$)

- well-posed (in suitable – standard – spaces)
- Lipschitz continuous
- completely continuous (\rightsquigarrow ill-posed)
- **not** Fréchet differentiable unless $\{x : y(x) = 0\} = \emptyset$
- model for membrane under water
- can be extended to arbitrary $f(y)$ piecewise differentiable
- simplified model for sharp phase transition (Stefan problem)

$$-\Delta y + \max(0, y) = u$$

solution operator $S : u \mapsto y$ ($:= S(u)$)

- **not** Fréchet differentiable unless $|\{x : y(x) = 0\}| = 0$
- but: **directionally differentiable**

Directional derivative $S'(u; h) =: \eta$ solves

$$-\Delta \eta + \mathbb{1}_{\{y=0\}} \max(0, \eta) + \mathbb{1}_{\{y>0\}} \eta = h$$

not linear in $h \rightsquigarrow$ not useful for algorithm

Bouligand subdifferential

$$\partial_B S(u) := \left\{ G \text{ linear} \mid \begin{array}{l} \text{there is } \{u_n\} \text{ Gâteaux differentiable with} \\ u_n \rightarrow u \text{ and } S'(u_n; h) \rightarrow Gh \text{ for all } h \end{array} \right\}$$

$$-\Delta\eta + \mathbb{1}_{\{y>0\}}\eta = h$$

$$G_u : h \mapsto \eta$$

- $G_u \in \partial_B S(u)$ **Bouligand derivative** [Christof/Meyer/Walter/C.]
- $u \mapsto G_u$ uniformly bounded (in right spaces)
- linear \rightsquigarrow **use for Landweber** in place of $S'(u)$

- 1 Motivation
- 2 Non-smooth equation
- 3 Bouligand–Landweber iteration**
- 4 Bouligand–Levenberg–Marquardt iteration
- 5 Numerical examples

$$u_{n+1}^{\delta} = u_n^{\delta} + w_n G_{u_n^{\delta}}^* \left(y^{\delta} - S(u_n^{\delta}) \right), \quad n = 0, 1, 2, \dots, N(\delta)$$

- $S : u \mapsto y$ non-smooth
- y^{δ} with $\|y^{\delta} - y^{\dagger}\| \leq \delta$, $y^{\dagger} = S(u^{\dagger})$ (assume unique)
- $u_0^{\delta} = u_0$ starting point
- w_n step sizes
- stopping index $N(\delta)$ by discrepancy principle:

$$\|y^{\delta} - S(u_{N(\delta)}^{\delta})\|_Y \leq \tau\delta < \|y^{\delta} - S(u_n^{\delta})\|_Y, \quad 0 \leq n < N(\delta)$$

(modified Landweber iteration [Scherzer '95])

Assume:

- 1 $\{G_u\}$ uniformly bounded
- 2 generalized tangential cone condition (GTCC)

$$\|S(u') - S(u) - G_u(u' - u)\| \leq \mu \|S(u') - S(u)\| \quad \text{for all } u, u' \in B_\rho(u^\dagger)$$

non-smooth PDE: satisfied for $1 > \mu > C(\{x : y^\dagger(x) = 0\})$

- 3 $u_0 \in B_\rho(u^\dagger)$

Then (under conditions on μ, τ, w_n):

- $u_n^\delta \in B_\rho(u^\dagger)$ for all $n \leq N(\delta)$
- $\delta > 0$: $N(\delta) < \infty$ and $\|u_n^\delta - u^\dagger\| < \|u_{n-1}^\delta - u^\dagger\|$ for $n \leq N(\delta)$
- $\delta = 0$: $N(\delta) = \infty$ and $u_n^0 \rightarrow u^\dagger$ for $n \rightarrow \infty$

Goal: show that $u_{N(\delta)}^\delta \rightarrow u^\dagger$ for $\delta \rightarrow 0$

Standard proof: combine

- 1 monotonicity: $\|u_n^\delta - u^\dagger\| < \|u_{n-1}^\delta - u^\dagger\|$ for $n \leq N(\delta)$
- 2 stability: $u_n^\delta \rightarrow u_n^0$ for all $n = 1, \dots$

Problem:

- stability requires continuity of $u \mapsto G_u$
- $u \mapsto G_u$ **not continuous** for S non-smooth
- \rightsquigarrow use **asymptotic** stability

Definition

Iterative method generating $\{u_n^\delta\}_{n \leq N(\delta)}$ **asymptotically stable** for $\delta \rightarrow 0$ if exists subsequence $\{\delta_k\}$ with:

- For all $0 \leq n \leq \bar{N} := \lim_{k \rightarrow \infty} N(\delta_k) \in \mathbb{N} \cup \{\infty\}$

$$u_n^{\delta_k} \rightarrow \tilde{u}_n \quad \text{as } k \rightarrow \infty$$

for some $\tilde{u}_n \in \bar{B}_U(u^\dagger, \rho)$

- If $\bar{N} = \infty$,

$$\tilde{u}_n \rightarrow u^\dagger \quad \text{as } n \rightarrow \infty$$

- \tilde{u}_n generated by **perturbed** noise-free iteration
- perturbation needs to vanish for $n \rightarrow \infty$

Bouligand–Landweber iteration

$$u_{n+1}^{\delta} = u_n^{\delta} + w_n G_{u_n^{\delta}}^* \left(y^{\delta} - S(u_n^{\delta}) \right), \quad n = 0, 1, 2, \dots, N(\delta)$$

- asymptotically stable for non-smooth PDE
(proof uses GTCC and compact embedding for $\mathcal{R}(G_u^*)$)
- \rightsquigarrow regularization (under conditions on μ, τ, w_n):

$$u_{N(\delta)}^{\delta} \rightarrow u^{\dagger} \quad \text{for} \quad \delta \rightarrow 0$$

- 1 Motivation
- 2 Non-smooth equation
- 3 Bouligand–Landweber iteration
- 4 Bouligand–Levenberg–Marquardt iteration**
- 5 Numerical examples

$$\begin{aligned}u_{n+1}^\delta &= \operatorname{argmin}_{u \in D(S)} \|S'(u_n^\delta)(u - u_n^\delta) - y^\delta - S(u_n^\delta)\|^2 + \alpha_n \|u - u_n^\delta\|^2 \\ &= u_n^\delta + \left(\alpha_n I + S'(u_n^\delta)^* S'(u_n^\delta)\right)^{-1} S'(u_n^\delta)^* \left(y^\delta - S(u_n^\delta)\right)\end{aligned}$$

- $\alpha_n = \alpha_0 r^n, r < 1$
- stopping by discrepancy principle
- TCC + transfer operator property

$$S'(u_2) = Q(u_1, u_2) S'(u_1) \quad Q \text{ linear, near identity}$$

- \rightsquigarrow stable, convergent regularization
- $N(\delta) = \mathcal{O}(1 + |\log \delta|)$

$$\begin{aligned} u_{n+1}^\delta &= \operatorname{argmin}_{u \in D(S)} \|G_{u_n^\delta}(u - u_n^\delta) - y^\delta - S(u_n^\delta)\|^2 + \alpha_n \|u - u_n^\delta\|^2 \\ &= u_n^\delta + \left(\alpha_n I + G_{u_n^\delta}^* G_{u_n^\delta}\right)^{-1} G_{u_n^\delta}^* \left(y^\delta - S(u_n^\delta)\right) \end{aligned}$$

- $\alpha_n = \alpha_0 r^n, r < 1$
- stopping by discrepancy principle
- **GTCC** + transfer operator property (holds for non-smooth PDE)

$$G_{u_2} = Q(u_1, u_2)G_{u_1} \quad Q \text{ linear, near identity}$$

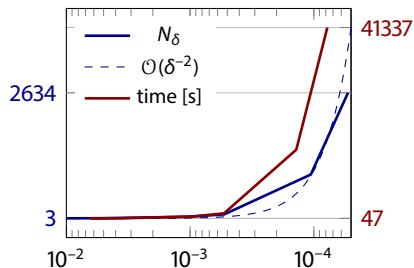
- \rightsquigarrow **asymptotically** stable, convergent regularization
- $N(\delta) = \mathcal{O}(1 + |\log \delta|)$

- 1 Motivation
- 2 Non-smooth equation
- 3 Bouligand–Landweber iteration
- 4 Bouligand–Levenberg–Marquardt iteration
- 5 Numerical examples**

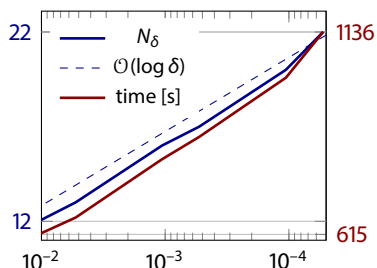
$$-\Delta y + \max(0, y) = u$$

- finite element discretization
- semismooth Newton method for solution (evaluation of S)
- constructed exact solution u^\dagger with $|\{x : y^\dagger(x) = 0\}| > 0$
- random Gaussian noise: $\|y^\delta - y^\dagger\| = \delta$
- $\mu = 0.1, \quad \tau = 1.4, \quad \rho = 5, \quad w_n = \frac{2-2\mu}{L^2}, \quad \bar{L} = 5 \times 10^{-2}$
- compare two starting points:
 - 1 $u_0 \equiv 0$
 - 2 \bar{u}_0 satisfying $u^\dagger - u_0 \in \mathcal{R}(G_{u^\dagger}^*)$ (generalized source condition)

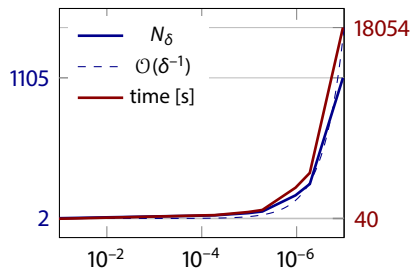
Numerical example: results with u_0



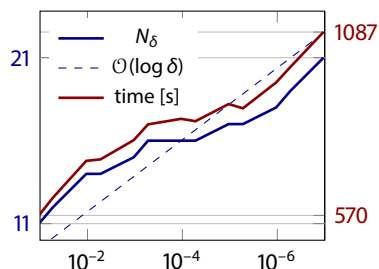
(a) Bouligand-Landweber



(b) Bouligand-Levenberg-Marquardt



(a) Bouligand-Landweber



(b) Bouligand-Levenberg-Marquardt

Summary

- iterative regularization using **Bouligand derivatives**
- inverse source problems for **non-smooth PDEs**
- convergence under **asymptotic stability**

Outlook

- **convergence rates** under source condition
- **accelerated Landweber iteration**
- other non-smooth equations, **variational inequalities**
- **coefficient inverse problems**

Preprint, Python/Julia codes:

http://www.uni-due.de/mathematik/agclason/clason_pubs.php