

# Multi-bang control of elliptic equations

**Christian Clason**<sup>1</sup>    **Karl Kunisch**<sup>2</sup>

<sup>1</sup> Fakultät für Mathematik, Universität Duisburg-Essen

<sup>2</sup> Institut für Mathematik und wissenschaftliches Rechnen, Karl-Franzens-Universität  
Graz

Numerical Methods for Optimal Control and Inverse Problems  
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# Motivation

Optimal control of elliptic PDE  $Ay = u$

## Bang-bang control:

- control constraints  $u_1 \leq u(x) \leq u_2$ , no control costs
- $\rightsquigarrow$  control  $u(x) \in \{u_1, u_2\}$  almost everywhere

## Multi-bang control

- control  $u(x) \in \{u_1, \dots, u_d\}$  almost everywhere
- motivation: control by discrete voltages, velocities, ...
- hybrid **discrete–continuous problem**, combinatorial
- solve using **continuous relaxation**  $\rightsquigarrow$  linear complexity in  $d$

# Formulation

$$\begin{cases} \min_{u,y} \frac{1}{2} \|y - z\|_{L^2}^2 + \frac{\alpha}{2} \|u\|_{L^2}^2 + \beta \int_{\Omega} \prod_{i=1}^d |u(x) - u_i|_0 dx \\ \text{s. t. } Ay = u, \quad u_1 \leq u(x) \leq u_d \end{cases}$$

- $u_1 < \dots < u_d, d \geq 2$ , desired control states
- $|t|_0 := \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t \neq 0 \end{cases}$       binary penalty
- $\rightsquigarrow$  non-smooth, non-convex, not lower-semicontinuous
- $A : V \rightarrow V^*$  isomorphism for Hilbert space  $V \hookrightarrow L^2(\Omega) \hookrightarrow V^*$

# Approach

Consider  $\mathcal{F}$  convex,  $\mathcal{G}$  convex

$$J(\bar{u}) := \mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u}) = \min_u \mathcal{F}(u) + \mathcal{G}(u)$$

## Necessary optimality conditions

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{p} \in \partial\mathcal{G}(\bar{u}) \end{cases}$$

- Fermat, sum rule for subdifferentials (under regularity condition)

# Approach

Consider  $\mathcal{F}$  convex,  $\mathcal{G}$  convex

$$J(\bar{u}) := \mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u}) = \min_u \mathcal{F}(u) + \mathcal{G}(u)$$

## Necessary optimality conditions

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{u} \in \partial\mathcal{G}^*(\bar{p}). \end{cases}$$

- $\mathcal{G}^*(p) = \sup_u \langle u, p \rangle - \mathcal{G}(u)$  Fenchel conjugate

# Approach

Consider  $\mathcal{F}$  convex,  $\mathcal{G}$  non-convex

$$\min_u \mathcal{F}(u) + \mathcal{G}(u)$$

## Sufficient(?) optimality conditions

$$\begin{cases} -\bar{p} \in \partial \mathcal{F}(\bar{u}) \\ \bar{u} \in \partial \mathcal{G}^*(\bar{p}). \end{cases}$$

- $\mathcal{G}^*$  Fenchel conjugate: **always convex**
- $\rightsquigarrow$  well-defined, **unique solution  $\bar{u}$**  ( $= \arg \min \mathcal{F}(u) + \mathcal{G}^{**}(u)$ )
- but:  $\bar{u}$  in general not minimizer of  $J \rightsquigarrow$  **sub-optimal**

# Optimality system

Here:

$$\mathcal{F} : L^2(\Omega) \rightarrow \mathbb{R}, \quad u \mapsto \frac{1}{2} \|A^{-1}u - z\|_{L^2}^2$$

$$\mathcal{G} : L^2(\Omega) \rightarrow \overline{\mathbb{R}}, \quad u \mapsto \int_{\Omega} \left( \frac{\alpha}{2} u(x)^2 + \beta \prod_{i=1}^d |u(x) - u_i|_0 \right) dx + \delta_U(u)$$

- $\delta_U$  indicator function of

$$U := \{u \in L^2(\Omega) : u_1 \leq u(x) \leq u_d \quad \text{a. e.}\}$$

- $\mathcal{G}$  defined pointwise  $\rightsquigarrow$  compute Fenchel conjugate, subdifferential pointwise

# Fenchel conjugate

$$\begin{aligned}
 g : \mathbb{R} &\rightarrow \overline{\mathbb{R}}, & v &\mapsto \frac{\alpha}{2}v^2 + \beta \prod_{i=1}^d |v - u_i|_0 + \delta_{[u_1, u_d]}(v) \\
 g^* : \mathbb{R} &\rightarrow \overline{\mathbb{R}}, & q &\mapsto \sup_v qv - g(v)
 \end{aligned}$$

Case differentiation: sup attained at  $\bar{v}$ ,

$$g^*(q) = \begin{cases} qu_i - \frac{\alpha}{2}u_i^2 & \bar{v} = u_i, & 1 \leq i \leq d \\ \frac{1}{2\alpha}q^2 - \beta & \bar{v} \neq u_i, & 1 \leq i \leq d \end{cases}$$



# Fenchel conjugate

$$g^*(q) = \begin{cases} qu_i - \frac{\alpha}{2}u_i^2 & q \in \bar{P}_i, \quad 1 \leq i \leq d \\ \frac{1}{2\alpha}q^2 - \beta & q \in \bar{P}_0 \end{cases}$$

$$P_1 := \left\{ q : q - \alpha u_1 < \sqrt{2\alpha\beta} \wedge q < \frac{\alpha}{2}(u_1 + u_2) \right\}$$

$$P_i := \left\{ q : |q - \alpha u_i| < \sqrt{2\alpha\beta} \wedge \frac{\alpha}{2}(u_{i-1} + u_i) < q < \frac{\alpha}{2}(u_i + u_{i+1}) \right\}$$

$$P_d := \left\{ q : q - \alpha u_d > \sqrt{2\alpha\beta} \wedge \frac{\alpha}{2}(u_d + u_{d-1}) < q \right\}$$

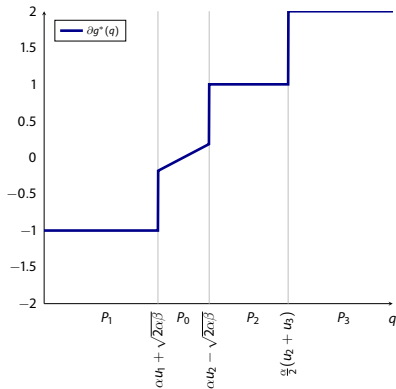
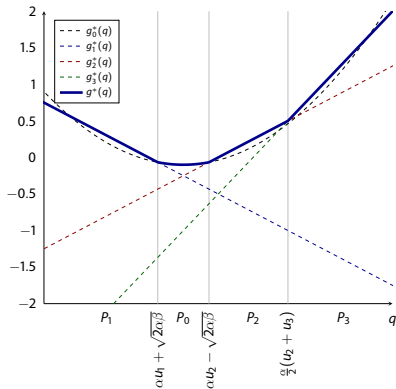
$$P_0 := \left\{ q : |q - \alpha u_j| > \sqrt{2\alpha\beta} \text{ for all } j \wedge \alpha u_1 < q < \alpha u_d \right\}$$

# Fenchel conjugate

$$g^*(q) = \begin{cases} qu_i - \frac{\alpha}{2}u_i^2 & q \in \bar{P}_i, \quad 1 \leq i \leq d \\ \frac{1}{2\alpha}q^2 - \beta & q \in \bar{P}_0 \end{cases}$$

$$\partial g^*(q) = \begin{cases} \{u_i\} & q \in P_i, \quad 1 \leq i < d \\ \{\frac{1}{\alpha}q\} & q \in P_0 \\ [u_i, u_{i+1}] & q \in \bar{P}_i \cap \bar{P}_{i+1}, \quad 1 \leq i < d \\ [\min\{u_i, \frac{1}{\alpha}q\}, \max\{u_i, \frac{1}{\alpha}q\}] & q \in \bar{P}_i \cap \bar{P}_0, \quad 1 \leq i \leq d \end{cases}$$

# Fenchel conjugate: sketch



# Optimality system

$$-\bar{p} = A^{-*}(A^{-1}\bar{u} - z)$$

$$\bar{u} \in \partial\mathcal{G}^*(\bar{p})$$

$$= \begin{cases} \{u_i\} & \bar{p}(x) \in P_i \\ \{\frac{1}{\alpha}\bar{p}(x)\} & \bar{p}(x) \in P_0 \\ [u_i, u_{i+1}] & \bar{p}(x) \in \bar{P}_i \cap \bar{P}_{i+1} \\ [\min(u_i, \frac{1}{\alpha}\bar{p}(x)), \max\{u_i, \frac{1}{\alpha}\bar{p}(x)\}] & \bar{p}(x) \in \bar{P}_i \cap \bar{P}_0 \end{cases}$$

- necessary conditions for  $\min_u \mathcal{F}(u) + \mathcal{G}^{**}(u)$  (convex, l.s.c.)
- $\rightsquigarrow$  **unique solution**  $(\bar{u}, \bar{p})$

# Structure of solution

$$\Omega = \mathcal{A} \cup \mathcal{F} \cup \mathcal{S}$$

- multi-bang arc  $\mathcal{A} = \bigcup_{i=1}^d \{x : \bar{u}(x) = u_i\}$
- free arc  $\mathcal{F} = \{x : \bar{u}(x) = \frac{1}{\alpha} \bar{p}(x) \neq u_i\}$
- singular arc  $\mathcal{S} = \{x : \bar{u}(x) \notin \{u_i, \frac{1}{\alpha} \bar{p}(x)\}\}$

# Generalized multi-bang principle

- if  $\beta$  sufficiently large:  $P_0 = \emptyset$ , free arc

$$\mathcal{F} \subset \{\bar{p}(x) \in P_0\} = \emptyset$$

- singular arc corresponds to set-valued subdifferential:

$$\begin{aligned} \mathcal{S} &= \{\bar{p}(x) \in \bigcup_{i=1}^{d-1} (\bar{P}_i \cap \bar{P}_{i+1}) \cup \bigcup_{i=1}^d (\bar{P}_i \cap \bar{P}_0)\} \\ &\subset \{\bar{p}(x) \in \left\{ \frac{\alpha}{2}(u_i + u_{i+1}), \alpha u_i - \sqrt{2\alpha\beta}, \alpha u_i + \sqrt{2\alpha\beta} \right\}\} \end{aligned}$$

- for suitable  $A$ ,  $\bar{p}(x)$  constant implies  $(A^*\bar{p})(x) = (\bar{y} - z)(x) = 0$ ,

$$\rightsquigarrow |\{\bar{y} = z\}| = 0 \Rightarrow \bar{u} \in \{u_1, \dots, u_d\} \text{ a. e., true multi-bang control}$$

# (Sub)optimality

- duality gap for non-convex  $\mathcal{G}$ :

$$\mathcal{G}(\bar{u}) + \mathcal{G}^*(\bar{p}) - \langle \bar{p}, \bar{u} \rangle \leq \beta |\mathcal{S}|$$

(pointwise gap of  $\beta$  where  $\partial g^*(\bar{p}(x))$  set-valued)

- $\rightsquigarrow$  in general:  $\bar{u}$  sub-optimal:

$$J(\bar{u}) \leq J(u) + \beta |\mathcal{S}| \quad \text{for all } u$$

- but:  $\bar{u}$  true multi-bang  $\rightsquigarrow |\mathcal{S}| = 0 \rightsquigarrow \bar{u}$  optimal

# Numerical solution of optimality system

$$A\bar{y} = \bar{u}$$

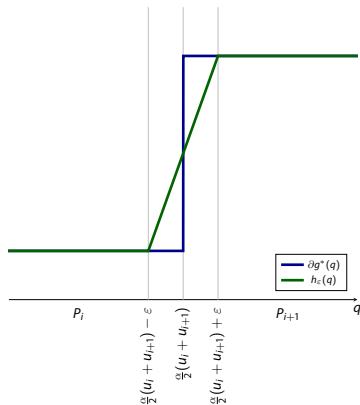
$$A^*\bar{p} = z - \bar{y}$$

$$\bar{u} \in \begin{cases} \{u_i\} & \bar{p}(x) \in P_i \\ \{\frac{1}{\alpha}\bar{p}(x)\} & \bar{p}(x) \in P_0 \\ [u_i, u_{i+1}] & \bar{p}(x) \in \bar{P}_i \cap \bar{P}_{i+1} \\ [\min(u_i, \frac{1}{\alpha}\bar{p}(x)), \max\{u_i, \frac{1}{\alpha}\bar{p}(x)\}] & \bar{p}(x) \in \bar{P}_i \cap \bar{P}_0 \end{cases}$$

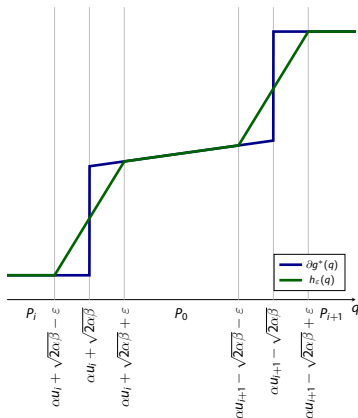
- set-valued, not differentiable
- $\rightsquigarrow$  replace set-valued  $\partial g^*$  by piecewise linear  $h_{\varepsilon}$ , let  $\varepsilon \rightarrow 0$



# Regularization: sketch



(a)  $q \in \bar{P}_i \cap \bar{P}_{i+1}$  (no free arc)



(b)  $q \in \bar{P}_i \cap \bar{P}_0$  (free arc)

# Regularization

## Regularized system

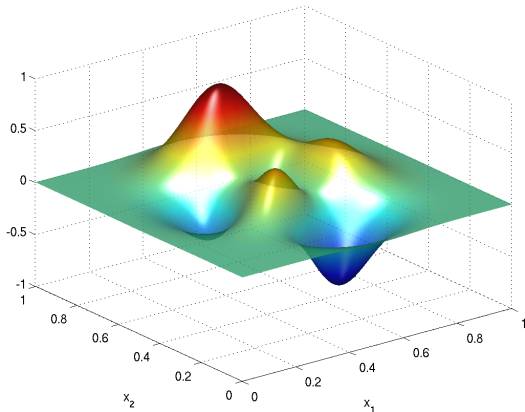
$$\begin{cases} Ay_\varepsilon = u_\varepsilon \\ A^*p_\varepsilon = z - y_\varepsilon \\ u_\varepsilon = H_\varepsilon(p_\varepsilon) \end{cases}$$

- $H_\varepsilon$  maximal monotone  $\rightsquigarrow$  unique solution  $(u_\varepsilon, p_\varepsilon, y_\varepsilon)$
- convergence  $(u_\varepsilon, p_\varepsilon, y_\varepsilon) \rightarrow (\bar{u}, \bar{p}, \bar{y})$  as  $\varepsilon \rightarrow 0$
- $h_\varepsilon$  Lipschitz continuous, norm gap  $\rightsquigarrow H_\varepsilon$  **semismooth**
- only number of sets  $P_i$  depends on  $d \rightsquigarrow$  **linear complexity**

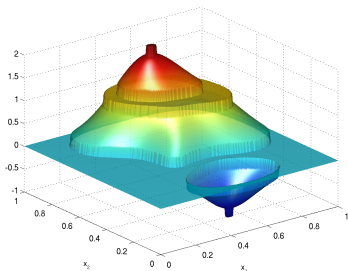
# Numerical examples

- $\Omega = [0, 1]^2$ ,  $A = -\Delta$
- finite element discretization: uniform grid,  $256 \times 256$  nodes
- state, adjoint: piecewise linear
- control: eliminated (variational discretization)
- $d = 5$ ,  $(u_1, \dots, u_5) = (-2, 1, 0, 1, 2)$
- $\varepsilon = 0$ : regularized active sets empty, true multi-bang  
 $\varepsilon > 0$ : terminated with 2–21 nodes in regularized active sets

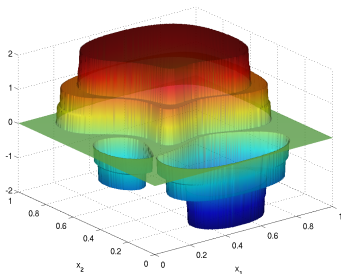
# Numerical examples: target



# Controls: $\beta = 10^{-4}$ (free arcs)

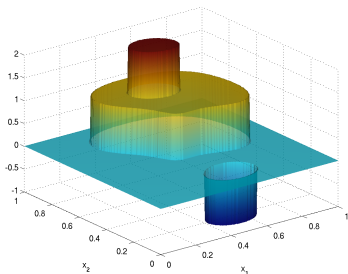


(a)  $\alpha = 5 \cdot 10^{-3}$  ( $\varepsilon = 0$ )

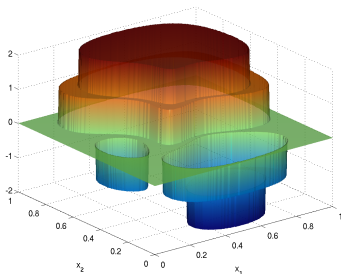


(b)  $\alpha = 10^{-3}$  ( $\varepsilon \approx 10^{-8}$ )

Controls:  $\beta = 10^{-3}$  (no free arcs)

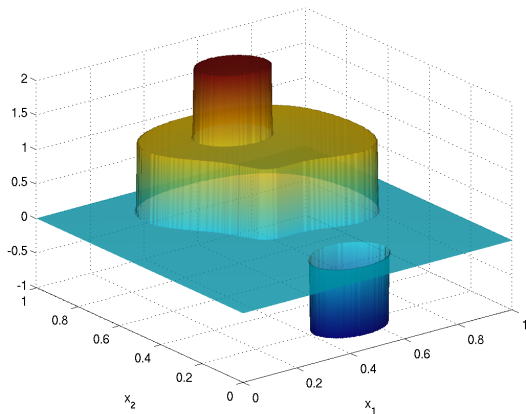


(c)  $\alpha = 5 \cdot 10^{-3}$  ( $\varepsilon = 0$ )



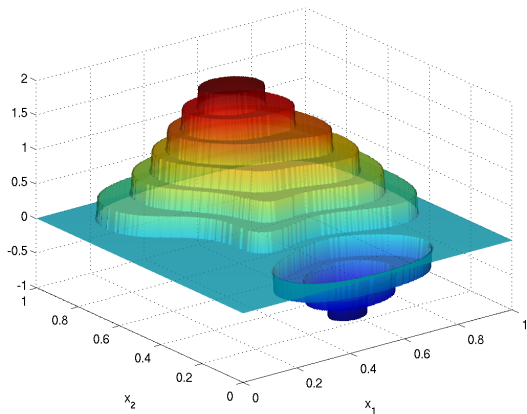
(d)  $\alpha = 10^{-3}$  ( $\varepsilon \approx 10^{-7}$ )

## Controls: effect of $d$



(a)  $d = 5$  ( $\varepsilon = 0$ )

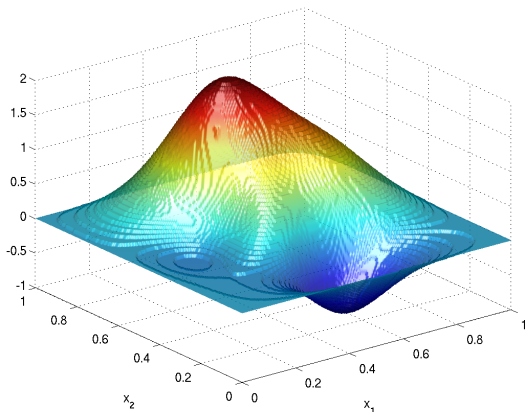
## Controls: effect of $d$



(b)  $d = 15$  ( $\varepsilon = 0$ )

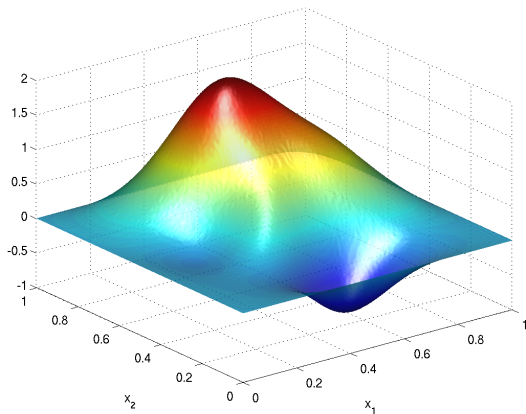


# Controls: effect of $d$



(c)  $d = 101$  ( $\varepsilon \approx 10^{-9}$ )

## Controls: effect of $d$



(d)  $d = 1001$  ( $\varepsilon \approx 10^{-11}$ )

## Conclusion

(Non-)convex relaxation of discrete control problem:

- well-posed primal-dual optimality system
- controls optimal under reasonable assumptions
- linear complexity in number of desired states
- $\rightsquigarrow$  efficient numerical solution (superlinear convergence)

Outlook:

- inverse problems (identification of known tissue types)
- nonlinear control-to-state mapping
- other hybrid discrete–continuous problems (e.g., switching)

Preprint, MATLAB/Python codes:

<http://www.uni-due.de/mathematik/agclason/publications.php>