Optimal control of elliptic PDEs with positive measures

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Motivation

Positive control problem

$$\min_{y,u} \frac{1}{2}\|y-z\|_{L^2}^2 \quad \text{s.t.} \quad u \geq 0, \quad Ay = u$$

- A second order elliptic operator
- unilateral pointwise (a.e.) bounds on control
- no control cost, only tracking

Sufficient for compactness, but (intuitively)

- y bounded in L^2 only implies $u = Ay$ bounded in H^{-2}
- positive cone must have interior points

$\rightsquigarrow$ seek control in space of Radon measures $\mathcal{M} \hookrightarrow H^{-2}$

Problem formulation

$$\text{(P)} \qquad \min_{y\in L^2, u\in\mathcal{M}(Q)} \frac{1}{2}\|y-z\|_{L^2}^2 \quad \text{s.t.} \quad u \geq 0, \quad Ay = Bu$$

Assumptions:

- A isomorphism from $W^{1,p}$ to $W^{-1,p}$ for $p < n$
  $\rightsquigarrow$ state equation well-posed in $\mathcal{M}(\Omega) \hookrightarrow W^{-1,p}$
  $\rightsquigarrow$ adjoint equation has solution in $W^{1,p} \hookrightarrow C(\bar{\Omega})$
- $B : \mathcal{M}(Q) \to \mathcal{M}(\Omega)$ control operator, $Q \subset \bar{\Omega}$

Approach: apply Fenchel duality to obtain both

- existence of solution
- optimality system

Fenchel duality

Fenchel conjugate of $\mathcal{F} : V \to \bar{\mathbb{R}}$

$$\mathcal{F}^* : V^* \to \bar{\mathbb{R}}, \qquad \mathcal{F}^*(q) = \sup_{v \in V} \langle q, v \rangle_{V^*,V} - \mathcal{F}(v)$$

In particular:

$$\mathcal{F}(v) = \frac{1}{2}\|v\|_{L^2}^2 \quad \Rightarrow \quad \mathcal{F}^*(q) = \frac{1}{2}\|q\|_{L^2}^2$$

$$\mathcal{F}(v) = \delta_{\{w \geq 0\}}(v) \quad \Rightarrow \quad \mathcal{F}^*(q) = \delta_{\{w \leq 0\}}(q) := \begin{cases} 0 & q \leq 0 \\ \infty & \text{else} \end{cases}$$

Fenchel duality

1. $\mathcal{F} : V \to \overline{\mathbb{R}}, \mathcal{G} : Y \to \overline{\mathbb{R}}$ convex and lower semicontinuous,
2. $\Lambda : V \to Y$ linear operator, adjoint $\Lambda^* : Y^* \to V^*$
3. $\exists v_0 \in V$: $\mathcal{F}(v_0) < \infty, \mathcal{G}(\Lambda v_0) < \infty, \mathcal{G}$ continuous at Λv_0:

Then:

- $\inf_{v\in V} \mathcal{F}(v) + \mathcal{G}(\Lambda v) = \sup_{q\in Y^*} -\mathcal{F}^*(\Lambda^* q) - \mathcal{G}^*(-q)$
- primal problem has solution $\bar{v} \in V$ (by convexity and 3)
- dual problem has solution $\bar{q} \in Y^*$ (by duality theorem)
- $(\bar{v}, \bar{q})$ satisfy extremality relations

$$\begin{cases} \Lambda^*\bar{q} \in \partial\mathcal{F}(\bar{v}), \\ -\bar{q} \in \partial\mathcal{G}(\Lambda\bar{v}), \end{cases}$$

Fenchel duality

Apply Fenchel duality to predual problem

$$(\mathrm{P}^*) \qquad \min_{q\in \mathrm{L}^2} \frac{1}{2}\|q+z\|_{\mathrm{L}^2}^2 - \frac{1}{2}\|z\|_{\mathrm{L}^2}^2 + \delta_{\{w\geq 0\}}(B^*(A^*)^{-1}q)$$

Set

$$\begin{aligned}
\mathcal{F} &: \mathrm{L}^2(\Omega) \to \mathbb{R}, && q \mapsto \tfrac{1}{2}\|q+z\|_{\mathrm{L}^2}^2 - \tfrac{1}{2}\|z\|_{\mathrm{L}^2}^2 \\
\mathcal{G} &: \mathrm{C}(\bar{Q}) \to \bar{\mathbb{R}}, && q \mapsto \delta_{\{w\geq 0\}}(q) \\
\Lambda &: \mathrm{L}^2(\Omega) \to \mathrm{C}(\bar{Q}), && q \mapsto B^*(A^*)^{-1}q
\end{aligned}$$

$\rightsquigarrow$ dual problem of (P^*) is original problem (P)

Fenchel duality

Assumption 3 : $\delta_{\{w\geq 0\}}$ finite, continuous at $B^*(A^*)^{-1}v_0$, i.e.,

(A) There is $v_0 \in L^2(\Omega)$ with $B^*(A^*)^{-1}v_0 \geq \varepsilon > 0$

Examples:

- control on subdomain: $Q \subsetneq \Omega$, B^* restriction operator
- Neumann boundary control: $Q \subset \partial\Omega$, B^* trace operator

Counterexample:

- distributed control: $Q = \Omega$, B^* identity,
  homogeneous Dirichlet conditions
  (limit of minimizing sequence violates boundary conditions)

Fenchel duality

If assumption (A) holds,

- predual problem (P^*) has solution $\bar{q} \in L^2(\Omega)$
- primal problem (P) has (unique) solution $\bar{u} \in \mathcal{M}(Q)$
- $(\bar{u}, \bar{q})$ satisfy extremality relations

$$\begin{cases} A^{-1}B\bar{u} = \bar{q} + z, \\ \quad -\bar{u} \in \partial\delta_{\{w\geq 0\}}(B^*(A^*)^{-1}\bar{q}) \end{cases}$$

Optimality conditions

Introduce:

- state $\bar{y} = A^{-1}B\bar{u}$
- adjoint state $\bar{p} = (A^*)^{-1}\bar{q} = (A^*)^{-1}(y - z)$
- complementarity formulation of $\bar{u} \in \partial\delta_{\{w \geq 0\}}(B^*\bar{p})$

Optimality system

$$\begin{cases} A\bar{y} = B\bar{u} \\ A^*\bar{p} = \bar{y} - z \\ -\bar{u} = \min(0, -\bar{u} + B^*\bar{p}) \end{cases}$$

Regularization

Regularized control problem

$$\min_{y\in L^2, u\in L^2(Q)} \frac{1}{2}\|y-z\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \quad \text{s.t.} \quad u \geq 0, \quad Ay = Bu$$

Dual problem

$$\min_{q\in L^2} \frac{1}{2}\|q+z\|_{L^2}^2 - \frac{1}{2}\|z\|_{L^2}^2 + \frac{1}{2\alpha}\left\|\min(0, B^*(A^*)^{-1}q)\right\|_{L^2}^2$$

Regularization

Regularized control problem

$$\min_{y\in L^2, u\in L^2(Q)} \frac{1}{2}\|y-z\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \quad \text{s.t.} \quad u \geq 0, \quad Ay = Bu$$

Optimality system

$$\begin{cases} A\bar{y} = B\bar{u} \\ A^*\bar{p} = \bar{y} - z \\ -\bar{u} = \min(0, \frac{1}{\alpha}B^*\bar{p}) \end{cases}$$

Discretization

Goal:

- conforming discretization $U_h \subset \mathcal{M}(Q)$
- discrete optimality system in terms of basis coefficients
  $\rightsquigarrow$ numerical solution of measure space problem
- (discretize-then-optimize = optimize-then-discretize)

Approach: (see talk by E. Casas)

1. choose (adjoint-consistent) Galerkin discretization $Y_h \subset C(\bar{Q})$
   $\rightsquigarrow$ continuous piecewise linear finite elements
2. construct dual $U_h = Y_h^*$ with respect to discrete topology
   $\rightsquigarrow$ linear combinations of Dirac measures at nodes

Numerical solution

Discrete optimality system

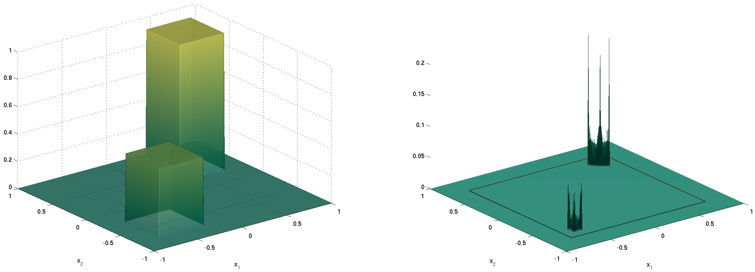
$$\begin{cases} A_h\vec{y}_h = B\vec{u}_h \\ A_h^T\vec{p}_h = M_h(\vec{y}_h - \vec{z}) \\ -\vec{u}_h = \min(0, -\vec{u}_h + B^*\vec{p}_h) \end{cases}$$

- $\vec{y}_h, \vec{p}_h, \vec{u}_h$ vectors of basis coefficients
- A_h stiffness matrix, M_h mass matrix
- semismooth in $\mathbb{R}^{N_h}$ $\rightsquigarrow$ semismooth Newton method
- start with regularization and homotopy $\alpha \to 0$ (globalization)

Numerical examples

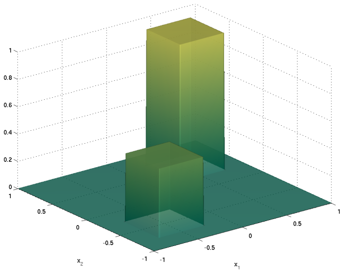
- $A = -\Delta$, homogeneous Dirichlet conditions
- $\Omega = (-1, 1)^2$, structured triangular grid, $N_h = 256^2$
- compare:
  1. subdomain control: $Q = [-\frac{3}{4}, \frac{3}{4}]^2$
  2. Neumann boundary control: $Q = \partial\Omega$

Numerical example: subdomain control

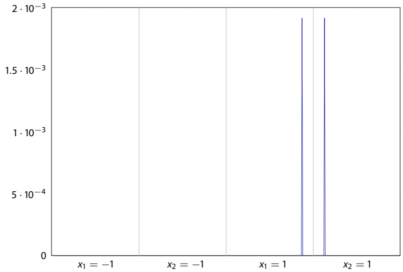


(a) target (b) control

Numerical example: Neumann control



(c) target



(d) control

Conclusion

- Positive control problems well-posed in measure space
- Existence and optimality conditions via Fenchel duality
- Numerical solution by conforming discretization and semismooth Newton methods
- Measure controls are sparse (if target non-attainable)