

Inverse problems with uniform noise

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Problems

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Motivation

Inverse problem

$$Kx = y^\delta$$

Here: data y^δ subject to **uniformly distributed** noise

- frequently used in synthetic test data for algorithms
- statistical model for **quantization errors** in digital data acquisition and processing
- appropriate Tikhonov functional involves L^∞ discrepancy
- more robust than L^2 , but not differentiable
 \rightsquigarrow **semi-smooth Newton method** for numerical solution

Problem formulation

$$\min_{x \in X} \frac{1}{p} \|Kx - y^\delta\|_{L^\infty}^p + \frac{\alpha}{2} \|x\|_X^2$$

- $K : X \rightarrow L^\infty(\Omega)$ bounded linear operator (e.g., solution operator for PDE)
- $y^\delta \in L^\infty(\Omega), 1 \leq p < \infty$
- X Hilbert space
- Assumption: $x_n \rightharpoonup x^\dagger$ in X implies $Kx_n \rightarrow Kx^\dagger$ in $L^\infty(\Omega)$

Well-posedness

Theorem

- (i) Existence of unique minimizer x_{α}^{δ} ;
- (ii) Stability: $y_n \rightarrow y^{\delta}$ in L^{∞} implies $x_{\alpha}^n \rightarrow x_{\alpha}^{\delta}$ (subsequence);
- (iii) Convergence: If $\alpha = \alpha(\delta)$ satisfies

$$\lim_{\delta \rightarrow 0} \alpha(\delta) = \lim_{\delta \rightarrow 0} \frac{\delta}{\alpha(\delta)} = 0,$$

then $x_{\alpha(\delta)}^{\delta}$ converges to true solution x^{\dagger} (subsequence);

- (iv) Convergence rates (under source condition)

Proof: standard [Engl/Hanke/Neubauer '96, Engl/Kunisch/Neubauer '89, Hofmann/Kaltenbacher/Pöschl/Scherzer '07]

Automatic parameter choice

Noise level δ **unknown**: choose α^* such that

Balancing principle

$$\sigma \|Kx_{\alpha^*}^{\delta} - y^{\delta}\|_{L^{\infty}} = \frac{\alpha^*}{2} \|x_{\alpha^*}^{\delta}\|_X^2$$

(σ fixed, depends on K, X , **not noise**)

Fixed point iteration

$$\alpha_{k+1} = \sigma \frac{\|Kx_{\alpha_k}^{\delta} - y^{\delta}\|_{L^{\infty}}}{\frac{1}{2} \|x_{\alpha_k}^{\delta}\|_X^2}$$

Automatic parameter choice

Theorem

If initial guess α_0 satisfies

$$\sigma \|Kx_{\alpha_0}^\delta - y^\delta\|_{L^\infty} - \frac{\alpha_0}{2} \|x_{\alpha_0}^\delta\|_X^2 < 0,$$

sequence $\{\alpha_k\}$

- is monotonically decreasing,
- converges to solution of balancing equation

Constructive: Fix α_0 , choose σ sufficiently small

Relaxation

$$\min_{(x,c) \in X \times \mathbb{R}} \frac{c^2}{2} + \frac{\alpha}{2} \|x\|_X^2 \quad \text{subject to} \quad \|Kx - y^\delta\|_{L^\infty(\Omega)} \leq c.$$

- equivalent reformulation for $p = 2$
[Grund/Rösch '01, Prüfert/Schiela '09, C/Ito/Kunisch '10]
- unique minimizer (x^*, c^*)
- optimality conditions (Maurer–Zowe regular point condition)
but: Lagrange multipliers are in $L^\infty(\Omega)^*$

Moreau–Yosida approximation

$$\min_{(x,c) \in X \times \mathbb{R}} \frac{c^2}{2} + \frac{\alpha}{2} \|x\|_X^2 + \frac{\gamma}{2} \|\max(0, Kx - y^\delta - c)\|_{L^2(\Omega)}^2 \\ + \frac{\gamma}{2} \|\min(0, Kx - y^\delta + c)\|_{L^2(\Omega)}^2,$$

- unique solution $(x_\gamma, c_\gamma) \in X \times \mathbb{R}$
- strong convergence to (x^*, c^*) as $\gamma \rightarrow \infty$
- optimality system is semi-smooth
- \rightsquigarrow path-following semi-smooth Newton method

Morozov regularization

Noise level δ known: fix $c = \delta \rightsquigarrow$ approach can be applied for

Morozov functional

$$\min_{x \in X} \frac{1}{2} \|x\|_X^2 \quad \text{subject to} \quad \|Kx - y^\delta\|_{L^\infty(\Omega)} \leq \delta$$

Optimality system

$$F(x) = \alpha j(x) + \gamma K^* \left(\max(0, Kx - y^\delta - \delta) + \min(0, Kx - y^\delta + \delta) \right)$$

Newton derivative

$$D_N F(x) \delta x = (\alpha j'(x) + \gamma K^* D_{\mathcal{A}} K) \delta x$$

Test problem: Inverse heat conduction

- $X = L^2(0, 1)$, K is Volterra integral operator of the first kind:

$$(Kx)(t) = \int_0^t k(s, t)x(s) ds, \quad k(s, t) = \frac{(s-t)^{-\frac{3}{2}}}{2\sqrt{\pi}} e^{-\frac{1}{4(s-t)}}$$

- noise:

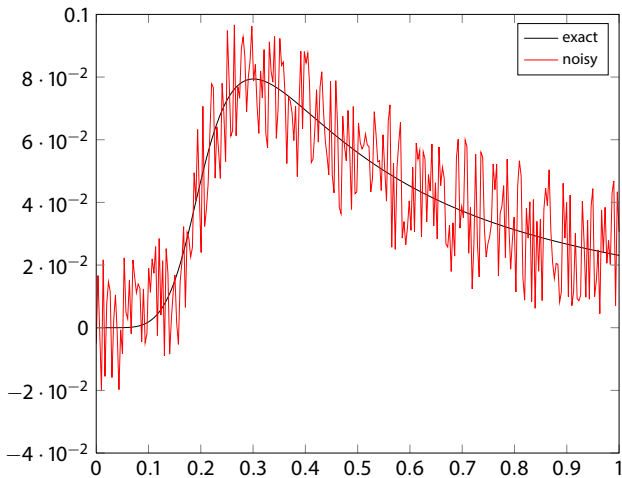
$$y^\delta(t) = (Kx^\dagger)(t) + \xi(t) \|Kx^\dagger\|_{L^\infty}$$

$\xi(t)$ uniformly distributed between $\left[-\frac{d}{2}, \frac{d}{2}\right]$

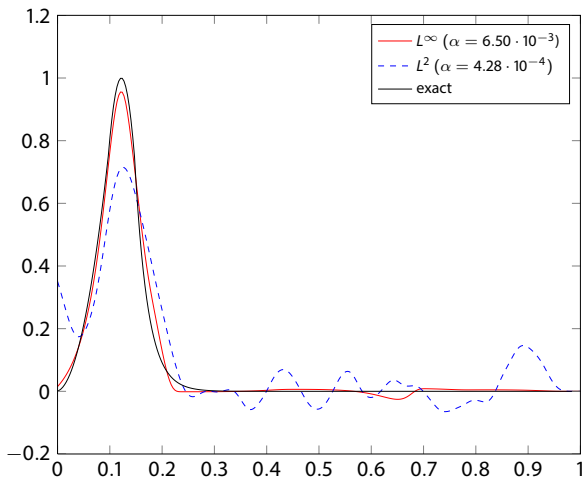
- Comparison with L^2 fitting:

$$x_{L^2} = (\alpha I + K^*K)^{-1}(K^*y^\delta)$$

Data ($d = 0.3$)



Reconstruction ($d = 0.3$)



Test problem: Inverse source problem

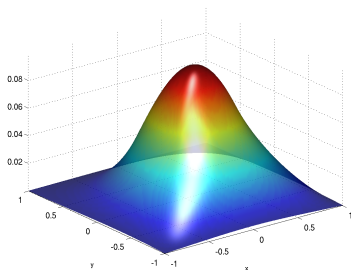
- $X = L^2(\Omega), \Omega = (0, 1)^2,$
- $K = A^{-1}$ is solution operator for PDE

$$Ay = -\Delta y - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \nabla y$$

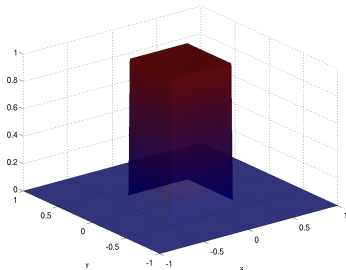
with homogeneous Dirichlet conditions

- reformulation in terms of y, c (avoid inversion of A^{-1})
- illustrate structure with deterministic uniform “noise”:
 - 1 Quantization (rounding to n_b nearest values)
 - 2 Checkerboard (“best case noise”)

Inverse source problem: Data

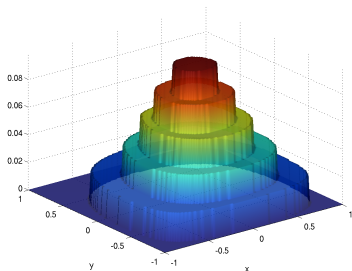


(a) true data Kx^\dagger

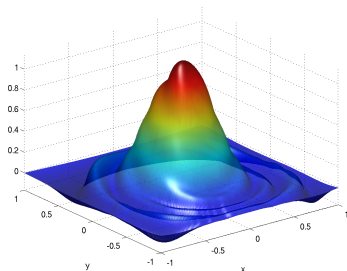


(b) true solution x^\dagger

Quantization ($n_b = 5$)

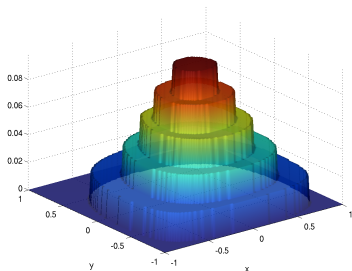


(a) data y^δ

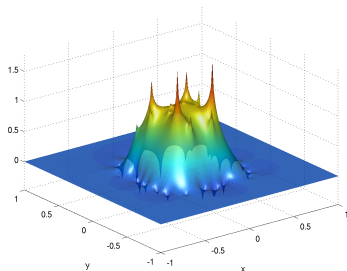


(b) L^2 reconstruction x_α

Quantization ($n_b = 5$)



(a) data y^δ



(b) L^∞ reconstruction x_α

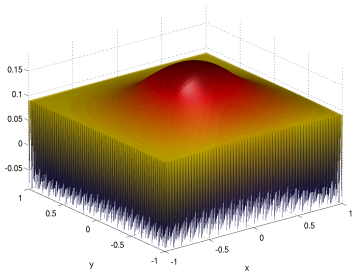
Checkerboard noise

- Observation: Reconstruction error is large in regions where noise does **not change sign**
- \rightsquigarrow “Best case noise”: constant magnitude, alternating sign

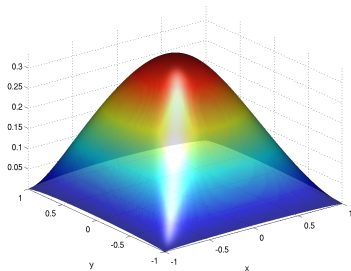
Grid points $t_{ij} = (t_{1,i}, t_{2,j}), 1 \leq i, j \leq N,$

$$y^\delta(t_{ij}) = y^\dagger(t_{ij}) + (-1)^{i+j} d \|y^\dagger\|_\infty$$

Checkerboard ($d = 0.9$)

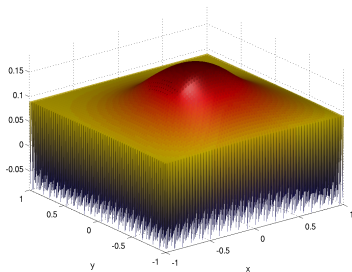


(a) data y^δ

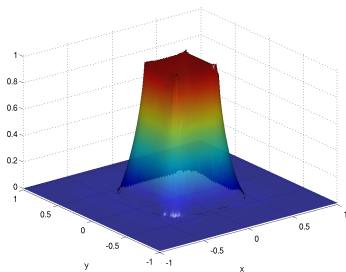


(b) L^2 reconstruction x_α

Checkerboard ($d = 0.9$)



(a) data y^δ



(b) L^∞ reconstruction x_α

Conclusion

- L^∞ fitting more robust for uniform noise
- Solution by semi-smooth Newton method
- For non-Gaussian noise: structure more important than level

Outlook

- Nonlinear parameter identification
- L^∞ - L^1 functional (“Dantzig selector”)

Preprint, MATLAB code:

<http://www.uni-graz.at/~clason/publications.html>