Second-order analysis of pointwise subdifferentials and applications to nonsmooth optimization

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Motivation

Primal-dual extragradient method:

- first-order algorithm for nonsmooth convex problems with linear operators [Chambolle/Pock 2011]
- *very* popular in imaging (TV denoising, deblurring, …)
- version for nonlinear operators [Valkonen 2014]

Goal:

- application to parameter identification for PDEs
- $\rightsquigarrow$ analysis in function space

Difficulty:

- convergence proof requires set-valued analysis in infinite-dimensional spaces (metric regularity)
- $\rightsquigarrow$ pointwise finite-dimensional analysis

Model problems

Inverse problem with L^1-fitting

$$\min_{u\in U} \|S(u) - y^\delta\|_{L^1} + \frac{\alpha}{2}\|u\|_{L^2}^2$$

Optimal control with state constraints

$$\min_{u\in U} \frac{1}{2}\|S(u) - y^d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \quad \text{s. t.} \quad S(u) \leqslant c \quad \text{in } \Omega$$

$S : U \subset L^2(\Omega) \to L^2(\Omega), \quad u \mapsto y$ satisfying

$$\begin{cases} -\Delta y + uy = f \\ \partial_\nu y = 0 \end{cases}$$

Problem statement

$$\min_{u \in X} F(K(u)) + G(u)$$

- $F : Y \to \overline{\mathbb{R}}, G : X \to \overline{\mathbb{R}}$ convex, lower semicontinuous
- X, Y Hilbert spaces
- $K \in C^2(X, Y)$ (e.g., $K(u) = S(u) - y^\delta$)
- $\rightsquigarrow$ problem non-smooth and non-convex
- $\rightsquigarrow$ control of nonlinearity: metric regularity

Metric regularity

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Saddle-point problem

$$\min_{u \in X} \sup_{v \in Y^*} G(u) + \langle K(u), v \rangle - F^*(v)$$

Primal-dual optimality conditions

$$\begin{cases} K(\bar{u}) \in \partial F^*(\bar{v}) \\ -K'(\bar{u})^* \bar{v} \in \partial G(\bar{u}) \end{cases}$$

F^* Fenchel conjugate, ∂F^* convex subdifferential
$K'(u)$ Fréchet derivative, $K'(u)^*$ adjoint

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Primal-dual optimality conditions

$$\begin{cases} K(\bar{u}) \in \partial F^*(\bar{v}) \\ -K'(\bar{u})^*\bar{v} \in \partial G(\bar{u}) \end{cases}$$

Set inclusion for $H : L^2(\Omega)^2 \rightrightarrows L^2(\Omega)^2$

$$0 \in H_{\bar{u}}(\bar{u}, \bar{v}) := \begin{pmatrix} \partial G(\bar{u}) + K'(\bar{u})^*\bar{v} \\ \partial F^*(\bar{v}) - K(\bar{u}) \end{pmatrix}$$

Metric regularity

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Set inclusion for $H : L^2(\Omega)^2 \rightrightarrows L^2(\Omega)^2$

$$0 \in H_{\bar u}(\bar u, \bar v) := \begin{pmatrix} \partial G(\bar u) + K'(\bar u)^* \bar v \\ \partial F^*(\bar v) - K(\bar u) \end{pmatrix}$$

Metric regularity at $\bar q := (\bar u, \bar v)$

$$\inf_{q:\, w \in H_{\bar u}(q)} \|q - (\bar u, \bar v)\| \leqslant L\|w\| \quad \text{for all } \|w\| \leqslant \rho$$

- interpretation: small perturbation w of 0
  $\Rightarrow$ small perturbation q of saddle point $(\bar u, \bar v)$
- Lipschitz property for set-valued $H_{\bar u}^{-1}$ at $((\bar u, \bar v), 0)$

Metric regularity

Metric regularity at $\bar{q} := (\bar{u}, \bar{v})$

$$\inf_{q:\, w \in H_{\bar{u}}(q)} \|q - (\bar{u}, \bar{v})\| \leqslant L\|w\| \quad \text{for all } \|w\| \leqslant \rho$$

Consequences:

- stability of $(\bar{u}, \bar{v})$ w.r.t. data perturbation
- stability of $(\bar{u}, \bar{v})$ w.r.t. regularization parameter
- convergence of primal-dual extragradient algorithm

Metric regularity

Metric regularity at $\bar{q} := (\bar{u}, \bar{v})$

$$\inf_{q:\, w \in H_{\bar{u}}(q)} \|q - (\bar{u}, \bar{v})\| \leqslant L\|w\| \quad \text{for all } \|w\| \leqslant \rho$$

Mordukhovich criterion

$$L_H = \inf_{t>0} \sup \left\{ \|\widehat{D}^* H(q'|w')\| \;\middle|\; q' \in B((\bar{u}, \bar{v}), t),\ w' \in H(q') \cap B(w, t) \right\}$$

- Aubin constant L_H is minimal choice of L
- $\widehat{D}^* H$ regular coderivative of H (cf. $L = \|\nabla f\|$ for $f \in C^1$)
- $\rightsquigarrow$ set-valued analysis in function spaces

Set-valued analysis in function space

Difficulties:

- multiple non-equivalent concepts (regular, limiting)
- calculus not tight

Here:

- set-valued mappings from subdifferentials of pointwise functionals
- $\rightsquigarrow$ infinite-dimensional (regular) derivatives pointwise via nice finite-dimensional (regular, graphical) derivatives
- cf. pointwise Fenchel conjugates, subdifferentials [Ekeland]

Derivatives of set-valued mappings

Q, W Hilbert space, $R : Q \rightrightarrows W$

Regular coderivative

$$\widehat{D}^* R(q|w)(\Delta w) = \left\{ \Delta q \in Q \mid (\Delta q, -\Delta w) \in \widehat{N}((q,w); \operatorname{Graph} R) \right\}$$

Graphical derivative

$$DR(q|w)(\Delta q) = \left\{ \Delta w \in W \mid (\Delta q, \Delta w) \in T((q,w); \operatorname{Graph} R) \right\}$$

- $\widehat{N}(u; U)$ regular normal cone to $U \subset X$
- $T(u; U)$ tangent cone to $U \subset X$

Derivatives of set-valued mappings

For Q, W finite-dimensional:

$$\widehat{D}^* R(q|w)(\Delta w) = [DR(q|w)]^{*+} = [\widetilde{DR}(q|w)]^{*+}$$

- upper adjoint

$$J^{*+}(\Delta w) = \left\{\Delta q \in Q \mid \langle \Delta q, \Delta q' \rangle \leqslant \langle \Delta w, \Delta w' \rangle \text{ for } \Delta w' \in J(\Delta q')\right\}$$

- convexification

$$\text{Graph}\, \widetilde{DR}(q|w) = \text{conv}\, \text{Graph}[DR(q|w)]$$

- (A linear operator: $\quad DA = \widetilde{DA} = A, \quad \widehat{D}^* A = A^*$)

Pointwise characterization

Assume $g : \Omega \times \mathbb{R}^m \to \overline{\mathbb{R}}$ normal, proper, convex,
proto-differentiable a.e.,

$$G : L^2(\Omega) \to \overline{\mathbb{R}} \qquad G(u) = \int_\Omega g(x, u(x))\, dx$$

Then:

$$\widehat{D}^*[\partial G](u|\xi)(\Delta\xi) = \left\{ \Delta u \in L^2(\Omega; \mathbb{R}^m) \,\middle|\, \Delta u(x) \in \widehat{D}^*[\partial g(x, \cdot\,)](u(x)|\xi(x))(\Delta\xi(x)) \right\}$$

$$D[\partial G](u|\xi)(\Delta u) = \left\{ \Delta\xi \in L^2(\Omega; \mathbb{R}^m) \,\middle|\, \Delta\xi(x) \in D[\partial g(x, \cdot\,)](u(x)|\xi(x))(\Delta u(x)) \right\}$$

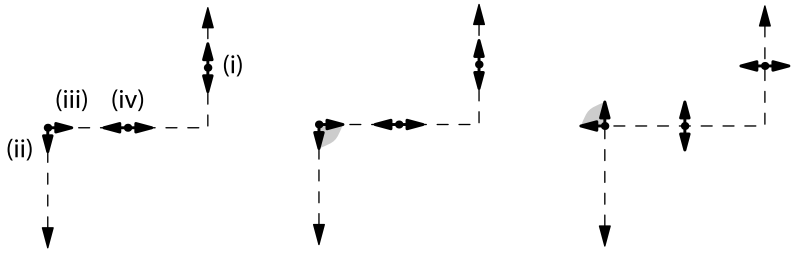
$$\widehat{D}^*[\partial G](u|\xi) = [D[\partial G](u|\xi)]^{*+} = [\widetilde{D[\partial G]}(u|\xi)]^{*+}$$

Examples: L^1 fitting

$$\min_u \|S(u) - y^\delta\|_{L^1} + \frac{\alpha}{2}\|u\|_{L^2}^2$$

- $K(u) = S(u) - y^\delta$
- $F(y) = \int_\Omega |y(x)|\,dx \quad \rightsquigarrow \quad f(z) = |z|$
- $f^*(z) = \delta_{[-1,1]}(z) = \begin{cases} 0 & |z| \leqslant 1 \\ \infty & |z| > 1 \end{cases}$
- $\partial f^*(z) = \begin{cases} [0,\infty)z & |z| = 1 \\ \{0\} & |z| < 1 \\ \emptyset & \text{otherwise} \end{cases}$

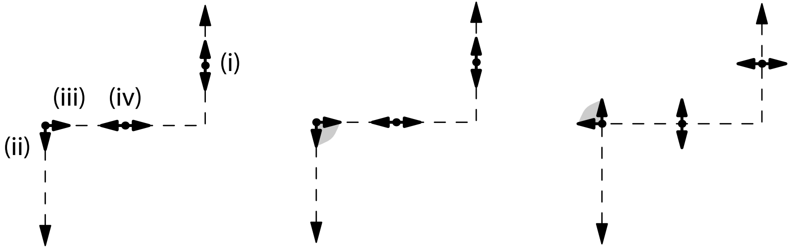
Examples: L^1 fitting (conjugate)



(a) $D(\partial f)(z|\zeta)$ (b) $\widetilde{D(\partial f)}(z|\zeta)$ (c) $\widehat{D}^*(\partial f)(z|\zeta)$

$$D(\partial f)(z|\zeta)(\Delta z) = \begin{cases} \mathbb{R} & |z| = 1,\ \zeta \in (0,\infty)z,\ \Delta z = 0 \\ [0,\infty)z & |z| = 1,\ \zeta = 0,\ \Delta z = 0 \\ \{0\} & |z| = 1,\ \zeta = 0,\ z\Delta z < 0 \\ \{0\} & |z| < 1,\ \zeta = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

Examples: L^1 fitting (conjugate)



(a) $D(\partial f)(z|\zeta)$ (b) $\widetilde{D(\partial f)}(z|\zeta)$ (c) $\widehat{D}^*(\partial f)(z|\zeta)$

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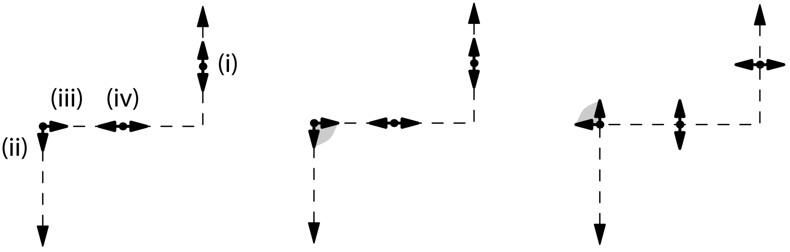
Examples: L¹ fitting (conjugate)



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Examples: L^1 fitting (conjugate)

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$$F : L^2(\Omega) \to \overline{\mathbb{R}} \qquad F(u) = \int_\Omega f(u(x))\,dx$$

$$\widetilde{D[\partial F]}(v|\eta)(\Delta v) = \begin{cases} V_{\partial F}(v|\eta)^\circ & \Delta v \in V_{\partial F}(v|\eta) \text{ and } \eta \in \partial F(v) \\ \emptyset & \text{otherwise} \end{cases}$$

$$\widehat{D}^*[\partial F](v|\eta)(\Delta\eta) = \begin{cases} V_{\partial F}(v|\eta)^\circ & -\Delta\eta \in V_{\partial F}(v|\eta) \text{ and } \eta \in \partial F(v) \\ \emptyset & \text{otherwise} \end{cases}$$

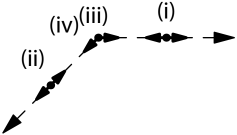
$$V_{\partial F}(v|\eta) = \left\{ z \in L^2(\Omega) \mid z(x) = 0 \text{ if } |v(x)| = 1 \right\}$$
$$V_{\partial F}(v|\eta)^\circ = \left\{ z \in L^2(\Omega) \mid z(x)v(x) \geqslant 0 \text{ if } \eta(x) = 0 \text{ and } z(x) = 0 \text{ if } |v(x)| < 1 \right\}$$

State constraints

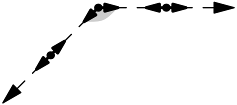
$$\min_{u\in L^2(\Omega)} \frac{1}{2}\|S(u)-y^d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \quad \text{s.t.} \quad S(u) \leqslant c \quad \text{in } \Omega$$

- $K(u) = S(u)$
- $F(y) = \int_\Omega |y(x) - y^d(x)|\, dx \quad \rightsquigarrow \quad f(x,z) = \frac{1}{2}|z - y^d(x)|^2 + \delta_{(-\infty,c)}(z)$
- $f^*(x,v) = \begin{cases} cv - \frac{1}{2}|c - y^d(x)|^2 & v > c - y^d(x) \\ \frac{1}{2}|v|^2 + vy^d(x) & v \leqslant c - y^d(x) \end{cases}$
- $\partial f^*(x,z) = \begin{cases} \{c\} & v > c - y^d(x) \\ \{v + y^d(x)\} & v \leqslant c - y^d(x) \end{cases}$

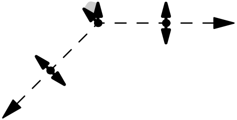
Examples: state constraints (conjugate)



(a) $D(\partial f)(z|\zeta)$



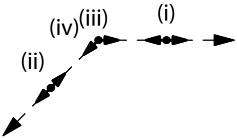
(b) $\widetilde{D(\partial f)}(z|\zeta)$



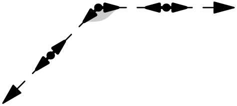
(c) $\widehat{D}^*(\partial f)(z|\zeta)$

$$D(\partial f)(v|\zeta)(\Delta v) = \begin{cases} 0 & v > c - y^d,\ \zeta = c \\ \Delta v & v < c - y^d,\ \zeta = v + y^d \\ 0 & v = c - y^d,\ \zeta = c,\ \Delta v \geqslant 0 \\ \Delta v & v = c - y^d,\ \zeta = c,\ \Delta v < 0 \end{cases}$$

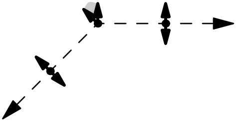
Examples: state constraints (conjugate)



(a) $D(\partial f)(z|\zeta)$



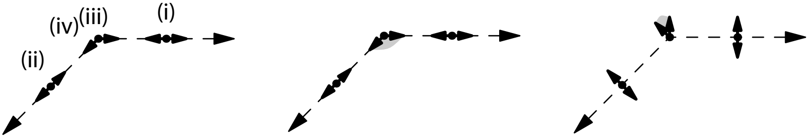
(b) $\widetilde{D(\partial f)}(z|\zeta)$



(c) $\widehat{D}^*(\partial f)(z|\zeta)$

$$\widetilde{D(\partial f)}(z|\zeta)(\Delta z) = \begin{cases} 0 & v > c - y^d,\ \zeta = c \\ \Delta v & v < c - y^d,\ \zeta = v + y^d \\ (-\infty, 0] & v = c - y^d,\ \zeta = c,\ \Delta v \geqslant 0 \\ \Delta v + (-\infty, 0] & v = c - y^d,\ \zeta = c,\ \Delta v < 0 \end{cases}$$

Examples: state constraints (conjugate)



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Examples: state constraints (conjugate)

$$F : L^2(\Omega) \to \overline{\mathbb{R}} \qquad F(u) = \int_\Omega f(u(x))\,dx$$

$$\widetilde{D[\partial F]}(v|\eta)(\Delta v) = \begin{cases} T_{F,v}\Delta v + V_{\partial F}(v|\eta)^\circ & \Delta v \in V_{\partial F}(v|\eta),\ \eta \in \partial F(v) \\ \emptyset & \text{otherwise} \end{cases}$$

$$\widehat{D}^*[\partial F](v|\eta)(\Delta \eta) = \begin{cases} T^*_{F,v}\Delta \eta + V_{\partial F}(v|\eta)^\circ & -\Delta \eta \in V_{\partial F}(v|\eta),\ \eta \in \partial F(v) \\ \emptyset & \text{otherwise} \end{cases}$$

$$V_{\partial F}(v|\eta) = L^2(\Omega) \qquad V_{\partial F}(v|\eta)^\circ = \{0\} \subset L^2(\Omega)$$

$$[T_{F,v}\Delta v](x) = t_v(x)\Delta v(x) \qquad t_v(x) = \begin{cases} 0 & v(x) > c - y^d(x) \\ 1 & v(x) < c - y^d(x) \end{cases}$$

Saddle point inclusion

$$0 \in H_{\bar u}(\bar u, \bar v) = \begin{pmatrix} \partial G(\bar u) + K'(\bar u)^* \bar v \\ \partial F^*(\bar v) - K(\bar u) \end{pmatrix}$$

$$DH_{\bar u}(q|w)(\Delta q) = \begin{pmatrix} D[\partial G](u|\xi - K'(\bar u)^* v)(\Delta u) + K'(\bar u)^* \Delta v \\ D[\partial F^*](v|\eta + K'(\bar u)u + c_{\bar u})(\Delta v) - K'(\bar u)\Delta u \end{pmatrix}$$

$$\widetilde{DH_{\bar u}}(q|w)(\Delta q) = \begin{cases} T_q \Delta q + V(q|w)^\circ & \Delta q \in V(q|w) \\ \emptyset & \text{otherwise} \end{cases}$$

- $q = (u, v), \quad w = (\xi, \eta), \quad c_{\bar u} = K(\bar u) - K'(\bar u)\bar u$
- T_q linear Operator (independent of w), $\quad V(q|w)$ cone
- $\widehat{D}^* H_{\bar u}(q|w) = [\widetilde{DH_{\bar u}(q|w)}]^{*+}$

Mordukhovich criterion

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$$\widetilde{DH_{\bar{u}}}(q|w)(\Delta q) = \begin{cases} T_q\Delta q + V(q|w)^\circ & \Delta q \in V(q|w) \\ \emptyset & \text{otherwise} \end{cases}$$

Then: Aubin constant $L_H \leqslant c < \infty$ **iff**

$$\sup_{t>0} \inf_{\substack{(\Delta w, z) \in W^t(q|w), \\ \|\Delta w\| > 0}} \frac{\|T_q^* \Delta w - z\|}{\|\Delta w\|} \geqslant c^{-1} > 0$$

$$W^t(q|w) = \bigcup \left\{ V(q'|w') \times V(q'|w')^\circ \;\middle|\; \begin{array}{l} w' \in H_{\bar{u}}(q'), \\ \|q' - q\| < t,\ \|w' - w\| < t \end{array} \right\}$$

Application to PDEs

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Primal-dual optimality conditions

$$\begin{cases} K(\bar{u}) \in \partial F^*(\bar{v}) \\ -S'(\bar{u})^*\bar{v} = \alpha\bar{u} \end{cases}$$

Metric regularity around $(\bar{u}, \bar{v})$ if either

1. $\sup_{t>0} \inf \left\{ \frac{\|S'(\bar{u})S'(\bar{u})^* z - v\|}{\|z\|} \,\middle|\, (z, v) \in V^t_{\partial F^*}(\bar{v}| - K(\bar{u})),\ \|z\| > 0 \right\} > 0$

   $\rightsquigarrow \|S'(\bar{u})^* z\| \geqslant c\|z\|$ for $z \in V^t_{\partial F^*}(\bar{v}| - K(\bar{u}))$ necessary!

2. Moreau–Yosida regularization: $F^* \mapsto F^*_\gamma = F^* + \frac{\gamma}{2}\|\cdot\|^2$

3. finite-dimensional data: $Y \mapsto Y_h$

Application to PDEs

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L^1 fitting: $z \in V^t_{\partial F^*}(v|\eta)$ if

$$z(x) \in \begin{cases} \{0\} & |v'(x)| = 1,\ \eta'(x) \neq 0 \\ -\operatorname{sign} v'(x)[0,\infty) & |v'(x)| = 1,\ \eta'(x) = 0 \\ \mathbb{R} & |v'(x)| < 1,\ \eta'(x) = 0 \end{cases}$$

for some $\|v' - \bar{v}\| \leqslant t$, $\|\eta' - \bar{\eta}\| \leqslant t$

- S compact operator: $\|S'(\bar{u})^* z\| \geqslant c\|z\|$ only holds for $z = 0$
- $\bar{\eta} = S(\bar{u}) - y^\delta, \quad \bar{v} \in \operatorname{sign} \bar{\eta}$
- $\rightsquigarrow$ in general not satisfied!
- similar for state constraints ($z \neq 0 \in L^2(\Omega)$, $\bar{\eta} = S(\bar{u})$)

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Metric regularity in function space:

- explicit characterization via pointwise set-valued analysis
- requires Moreau–Yosida regularization
- $\rightsquigarrow$ mesh independent stability

Outlook:

- convergence of primal-dual algorithms
- pointwise limiting coderivatives
- partial stability (w.r.t. primal variable only)
- pointwise set-valued analysis for bilevel problems

Preprints:

http://www.uni-due.de/mathematik/agclason/clason_pub.php