# Inverse problems for PDEs with uniform noise

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# Motivation

## Inverse problem

$$Kx = y^\delta$$

Here: data $y^\delta$ subject to uniformly distributed noise

- frequently used in synthetic test data for algorithms
- statistical model for quantization errors in digital data acquisition and processing (e.g., in wireless sensor networks)
- appropriate Tikhonov functional involves $L^\infty$ norm discrepancy
- more robust than $L^2$ fitting, but not differentiable
  $\rightsquigarrow$ semi-smooth Newton method for numerical solution

# Problem formulation

$$\min_{x \in X} \frac{1}{p} \|Kx - y^\delta\|_{L^\infty}^p + \frac{\alpha}{2} \|x\|_X^2$$

- $K : X \to L^\infty(\Omega)$ bounded linear operator (here: solution operator for PDE $Ay = x$)
- $y^\delta \in L^\infty(\Omega)$
- $p \in [1, \infty)$
- $X$ Hilbert space
- Assumption: $x_n \rightharpoonup x^\dagger$ in $X$ implies $Kx_n \to Kx^\dagger$ in $L^\infty(\Omega)$

# Well-posedness

## Theorem

(i) *Existence of unique minimizer* $x_\alpha^\delta$

(ii) *Stability:* $y_n \to y^\delta$ *in* $L^\infty$ *implies* $x_\alpha^n \to x_\alpha^\delta$ *(up to subsequence)*

(iii) *Convergence: If* $\alpha = \alpha(\delta)$ *satisfies*

$$\lim_{\delta \to 0} \alpha(\delta) = \lim_{\delta \to 0} \frac{\delta}{\alpha(\delta)} = 0,$$

*then* $x_{\alpha(\delta)}^\delta$ *converges to true solution* $x^\dagger$ *(up to subsequence)*

(iv) *Convergence rates (under source condition)*

Proof: standard [Engl/Hanke/Neubauer '96, Engl/Kunisch/Neubauer '89, Hofmann/Kaltenbacher/Pöschl/Scherzer '07]

# Automatic parameter choice

Noise level $\delta$ unknown: choose $\alpha^*$ such that

## Balancing principle

$$\sigma \|Kx^\delta_{\alpha^*} - y^\delta\|_{L^\infty} = \frac{\alpha^*}{2} \|x^\delta_{\alpha^*}\|^2_X$$

is satisfied ( $\sigma$ fixed, depends on $K$, $X$, not noise)

## Fixed point iteration

$$\alpha_{k+1} = \sigma \frac{\|Kx^\delta_{\alpha_k} - y^\delta\|_{L^\infty}}{\frac{1}{2}\|x^\delta_{\alpha_k}\|^2_X}$$

# Automatic parameter choice

## Theorem

*If initial guess $\alpha_0$ satisfies*

$$\sigma \|Kx_{\alpha_0}^{\delta} - y^{\delta}\|_{L^\infty} - \frac{\alpha_0}{2}\|x_{\alpha_0}^{\delta}\|_X^2 < 0,$$

*sequence $\{\alpha_k\}$*

- *is monotonically decreasing,*
- *converges to solution of balancing equation*

Constructive: Fix $\alpha_0$, choose $\sigma$ sufficiently small

# Relaxation

$$\min_{(x,c)\in X\times\mathbb{R}} \frac{c^p}{p} + \frac{\alpha}{2}\|x\|_X^2 \quad \text{subject to} \quad \|Kx - y^\delta\|_{L^\infty(\Omega)} \le c.$$

- equivalent reformulation
  [Grund/Rösch '01, Prüfert/Schiela '09, C/Ito/Kunisch '10]
- unique minimizer $(x^*, c^*)$, strictly convex for $p = 2$
- optimality conditions (Maurer–Zowe regular point condition)
  but: Lagrange multipliers are in $L^\infty(\Omega)^*$

# Moreau–Yosida approximation

$$\min_{(x,c)\in X\times\mathbb{R}} \frac{c^2}{2} + \frac{\alpha}{2}\|x\|_X^2 + \frac{\gamma}{2}\|\max(0, Kx - y^\delta - c)\|_{L^2(\Omega)}^2 + \frac{\gamma}{2}\|\min(0, Kx - y^\delta + c)\|_{L^2(\Omega)}^2,$$

- unique solution $(x_\gamma, c_\gamma) \in X \times \mathbb{R}$
- strong convergence to $(x^*, c^*)$ as $\gamma \to \infty$

# Moreau–Yosida approximation

Set

$$\lambda_\gamma^1 = \gamma \max(0, Kx_\gamma - y^\delta - c_\gamma), \quad \lambda_\gamma^2 = \gamma \min(0, Kx_\gamma - y^\delta + c_\gamma),$$

## Optimality system

$$\begin{cases} \alpha(\frac{1}{2}\|\cdot\|_X^2)'(x_\gamma) + K^*\left(\lambda_\gamma^1 + \lambda_\gamma^2\right) = 0, \\ c_\gamma + \int_\Omega -\lambda_\gamma^1 + \lambda_\gamma^2 \, dt = 0. \end{cases}$$

- optimality system is semi-smooth
  $\rightsquigarrow$ path-following semi-smooth Newton method
- reformulate in terms of $y_\gamma = Kx_\gamma$ (avoid inversion of $K = A^{-1}$)

# Test problem: Inverse source problem

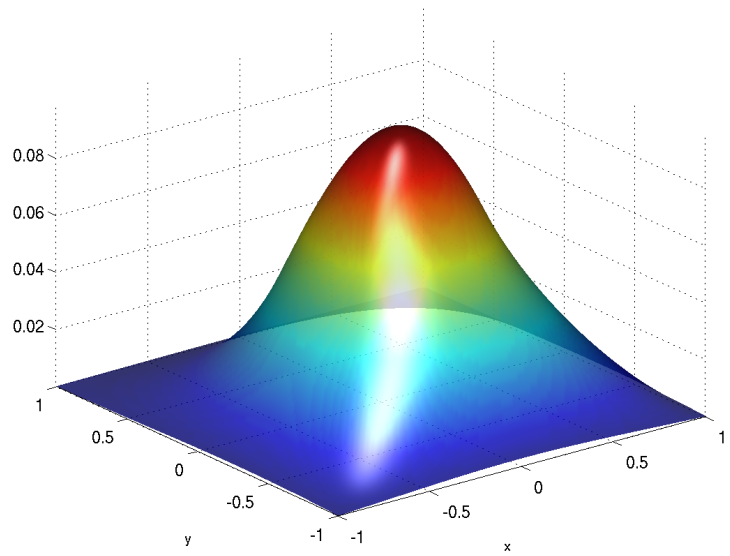
- $X = \mathrm{L}^2(\Omega), \Omega = (0,1)^2$,
- $K = A^{-1}$ is solution operator for PDE

$$Ay = -\Delta y - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \nabla y$$

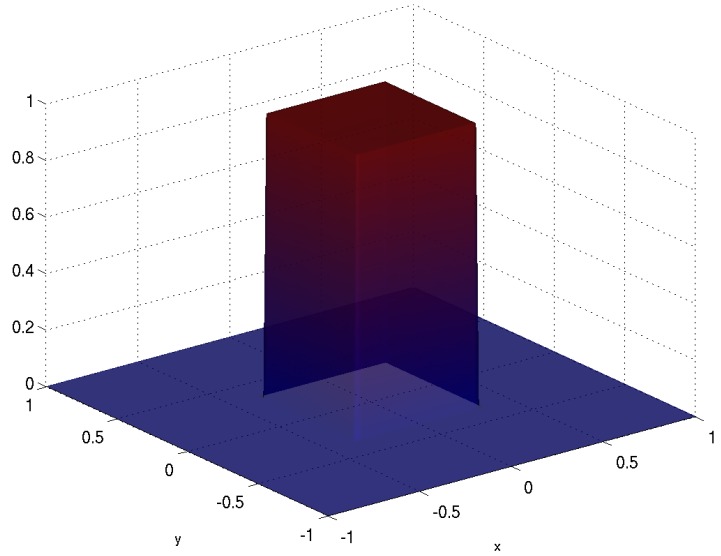
with homogeneous Dirichlet conditions

- illustrate structure with deterministic uniform "noise":
  1. Quantization (rounding to $n_b$ nearest values)
  2. Checkerboard ("best case noise")

# Data

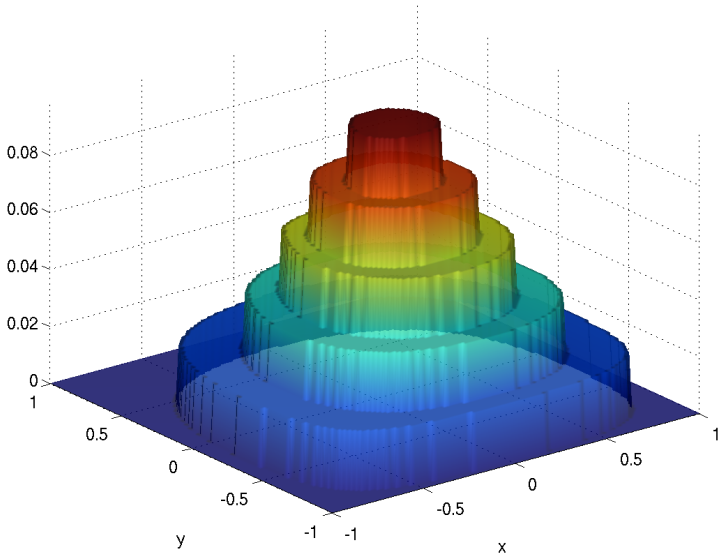



(a) true data $Kx^\dagger$

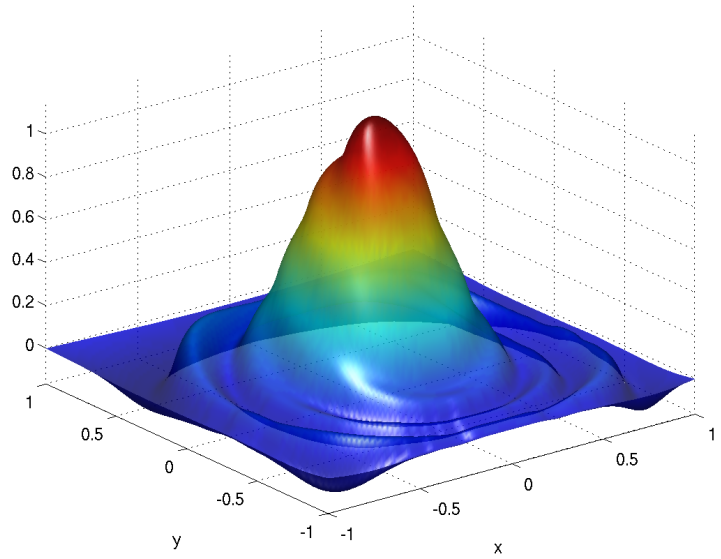



(b) true solution $x^\dagger$

# Quantization ($n_b = 5$)

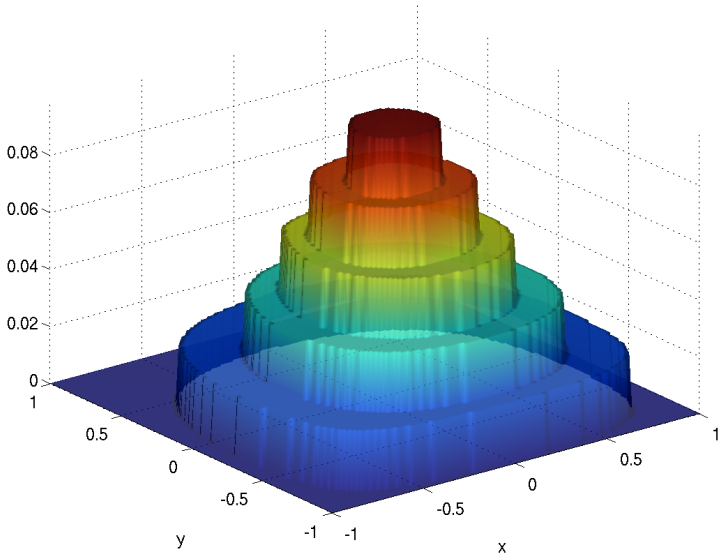



(a) data $y^\delta$

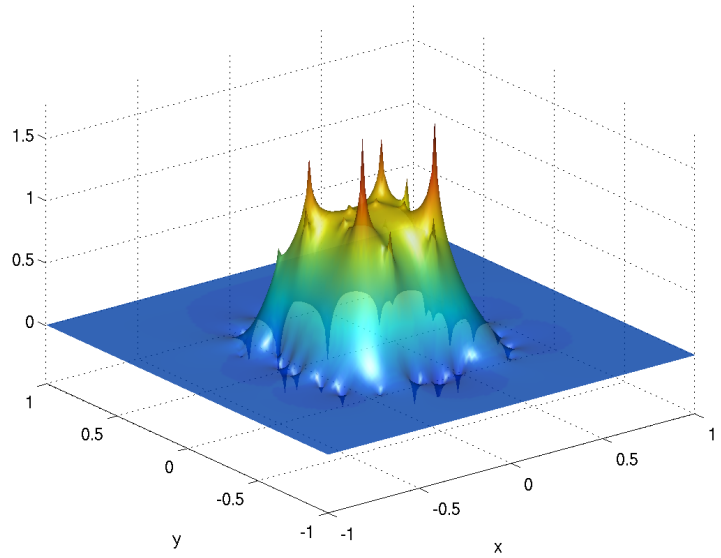



(b) $L^2$ reconstruction $x_\alpha$

# Quantization ($n_b = 5$)




(a) data $y^\delta$




(b) $L^\infty$ reconstruction $x_\alpha$

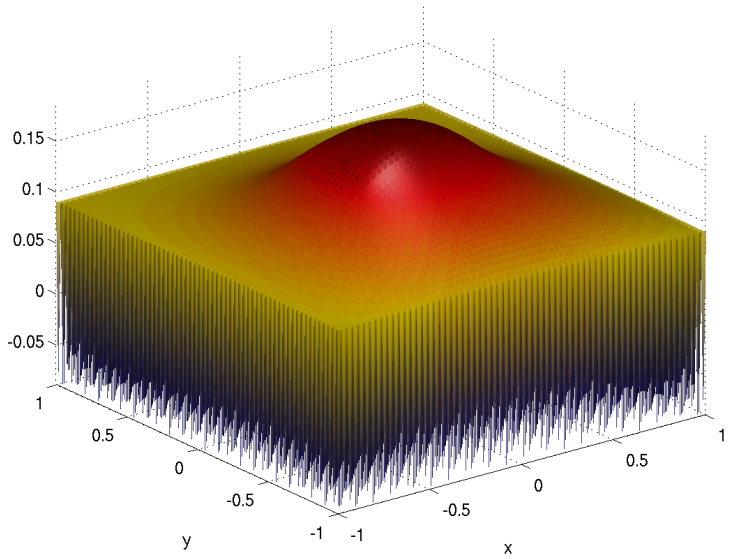
# Checkerboard noise

- Observation: Reconstruction error is large in regions where noise does not change sign
- $\rightsquigarrow$ "Best case noise": constant magnitude, alternating sign

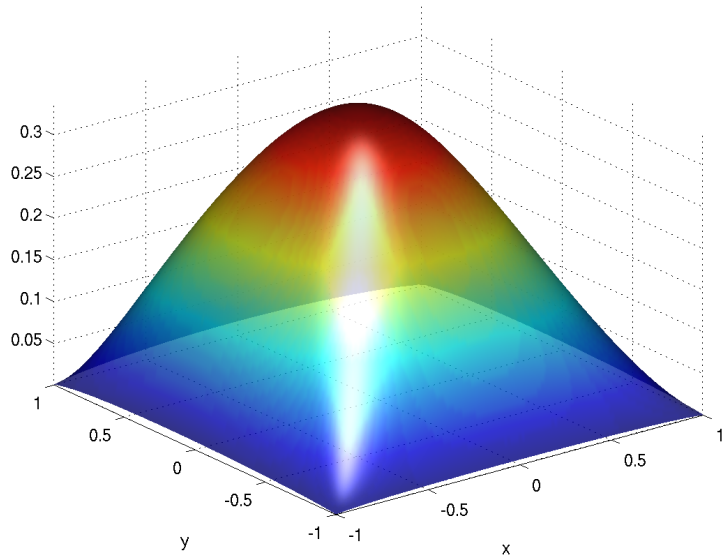
  Grid points $t_{ij} = (t_{1,i}, t_{2,j}), 1 \leq i, j \leq N$,

$$y^\delta(t_{ij}) = y^\dagger(t_{ij}) + (-1)^{i+j} d \, \|y^\dagger\|_\infty$$

# Checkerboard ($d = 0.9$)

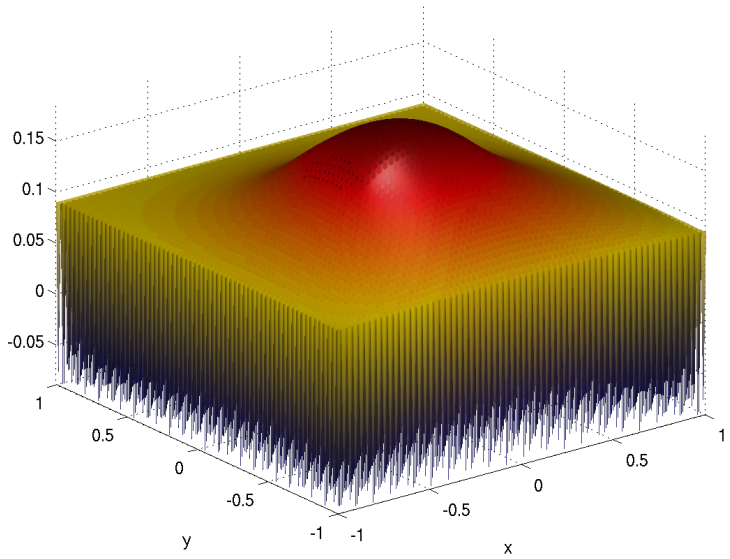



(a) data $y^\delta$

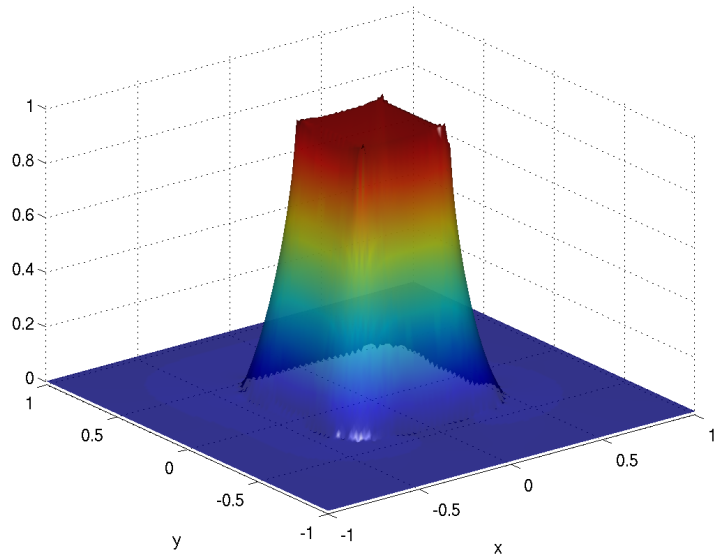



(b) $L^2$ reconstruction $x_\alpha$

# Checkerboard ($d = 0.9$)




(a) data $y^\delta$




(b) $L^\infty$ reconstruction $x_\alpha$

# Test problem: Nonlinear parameter identification

- $X = \mathrm{L}^2(\Omega), \Omega = (0,1)^2$,
- $S : \mathrm{L}^2(\Omega) \to \mathrm{L}^\infty(\Omega)$ is nonlinear solution operator $S(x) = y$,

$$-\Delta y + xy = f$$

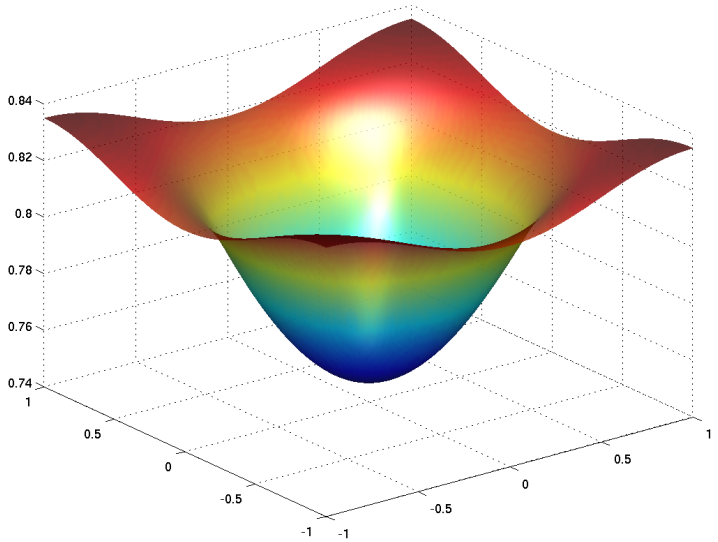
with homogeneous Neumann conditions

- $S$ strictly differentiable $\rightsquigarrow$ use chain rule

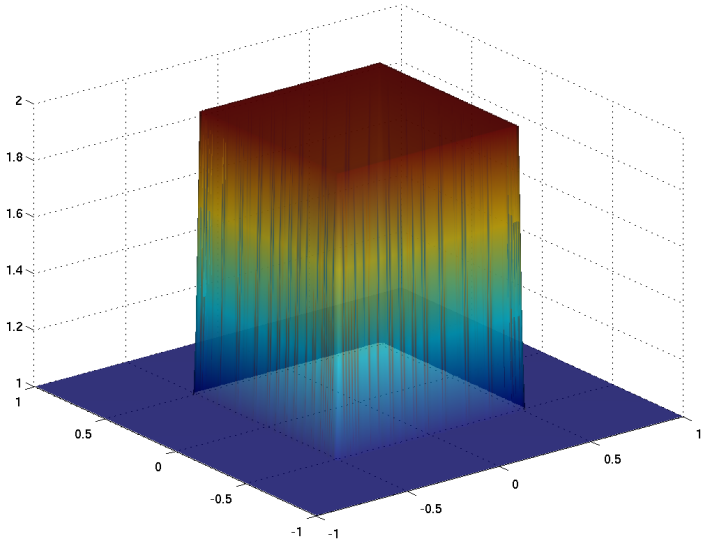
$$D_N \max(0, S(x))h = \begin{cases} [S'(x)h](t) & [S(x)](t) > 0, \\ 0 & \text{else.} \end{cases}$$

$\rightsquigarrow$ matrix-free Krylov method for Newton step

# Data

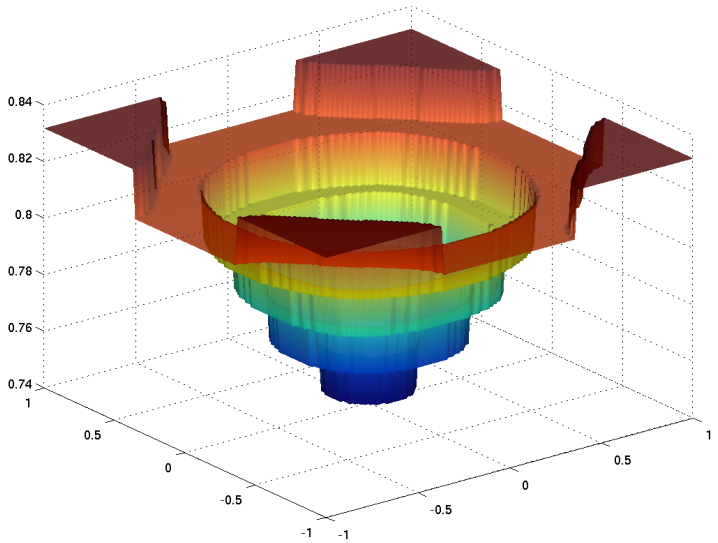



(a) true data $Kx^\dagger$

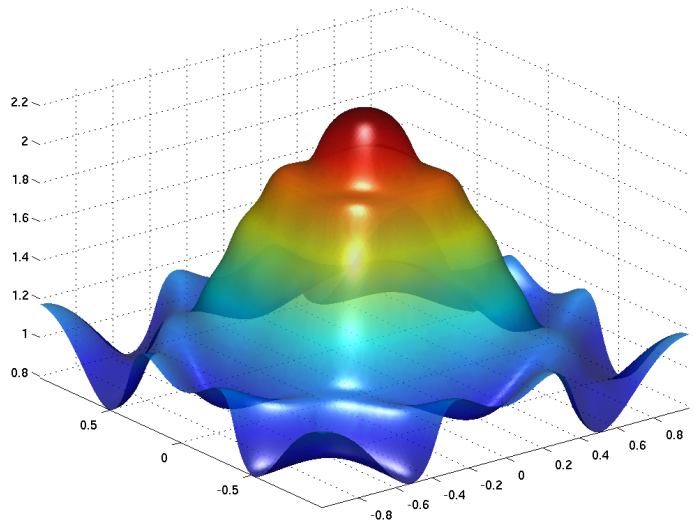



(b) true solution $x^\dagger$

# Quantization ($n_b = 5$)

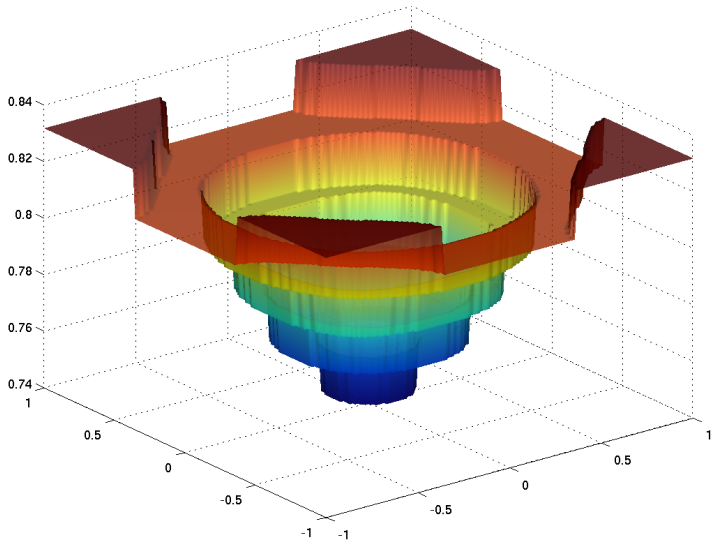



(a) data $y^\delta$

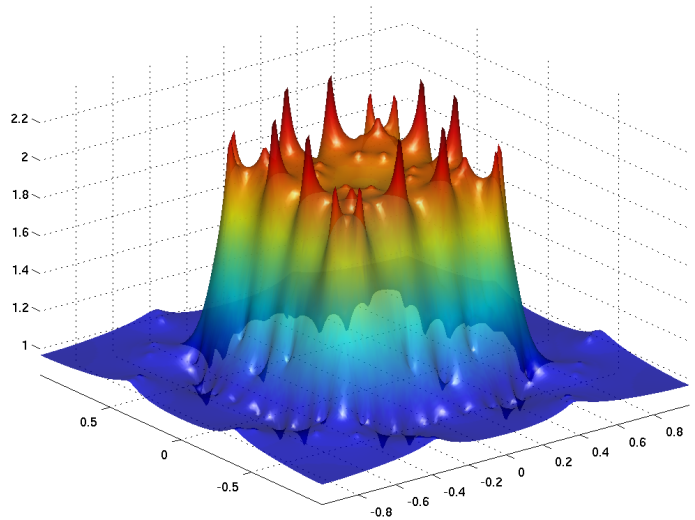



(b) $L^2$ reconstruction $x_\alpha$

# Quantization ($n_b = 5$)




(a) data $y^\delta$




(b) $L^\infty$ reconstruction $x_\alpha$

# Conclusion

- $L^\infty$ fitting more robust for uniform noise
- Solution by semi-smooth Newton method
- For non-Gaussian noise: structure more important than level

## Outlook

- Nonlinear parameter identification
- $L^\infty$-$L^1$ functional ("Dantzig selector")

Preprint, MATLAB/Python code:

`http://www.uni-graz.at/~clason/publications.html`