Inverse problems for PDEs with L^∞ data fitting

Christian Clason

Institut für Mathematik und wissenschaftliches Rechnen
Karl-Franzens-Universität Graz

Inverse Problems: Modeling and Simulation
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Motivation

Inverse problem

$$Kx = y^\delta$$

Here: data y^δ subject to uniformly distributed noise

- frequently used in synthetic test data for algorithms
- statistical model for quantization errors in digital data acquisition and processing
- appropriate Tikhonov functional involves L^∞ discrepancy
- more robust than L^2, but not differentiable
  $\rightsquigarrow$ semi-smooth Newton method for numerical solution

Problem formulation

$$\min_{x \in X} \frac{1}{p} \|Kx - y^\delta\|_{L^\infty}^p + \frac{\alpha}{2} \|x\|_X^2$$

- $K : X \to L^\infty(\Omega)$ bounded linear operator (here: solution operator for PDE $Ay = x$)
- $y^\delta \in L^\infty(\Omega), 1 \leq p < \infty$
- X Hilbert space
- Assumption: $x_n \rightharpoonup x^\dagger$ in X implies $Kx_n \to Kx^\dagger$ in $L^\infty(\Omega)$

Well-posedness

Theorem

(i) *Existence of unique minimizer* x_α^δ;

(ii) *Stability:* $y_n \to y^\delta$ *in* L^∞ *implies* $x_\alpha^n \to x_\alpha^\delta$ *(subsequence);*

(iii) *Convergence: If* $\alpha = \alpha(\delta)$ *satisfies*

$$\lim_{\delta\to 0} \alpha(\delta) = \lim_{\delta\to 0} \frac{\delta}{\alpha(\delta)} = 0,$$

then $x_{\alpha(\delta)}^\delta$ *converges to true solution* $x^\dagger$ *(subsequence);*

(iv) *Convergence rates (under source condition)*

Proof: standard [Engl/Hanke/Neubauer '96, Engl/Kunisch/Neubauer '89, Hofmann/Kaltenbacher/Pöschl/Scherzer '07]

Automatic parameter choice

Noise level δ unknown: choose α^* such that

Balancing principle

$$\sigma \|Kx_{\alpha^*}^{\delta} - y^{\delta}\|_{L^\infty} = \frac{\alpha^*}{2} \|x_{\alpha^*}^{\delta}\|_X^2$$

is satisfied (σ fixed, depends on K, X, not noise)

Fixed point iteration

$$\alpha_{k+1} = \sigma \frac{\|Kx_{\alpha_k}^{\delta} - y^{\delta}\|_{L^\infty}}{\frac{1}{2}\|x_{\alpha_k}^{\delta}\|_X^2}$$

Automatic parameter choice

Theorem

If initial guess α_0 satisfies

$$\sigma \|Kx^\delta_{\alpha_0} - y^\delta\|_{L^\infty} - \frac{\alpha_0}{2}\|x^\delta_{\alpha_0}\|^2_X < 0,$$

sequence $\{\alpha_k\}$

- *is monotonically decreasing,*
- *converges to solution of balancing equation*

Constructive: Fix α_0, choose σ sufficiently small

Relaxation

$$\min_{(x,c)\in X\times\mathbb{R}} \frac{c^2}{2} + \frac{\alpha}{2}\|x\|_X^2 \quad \text{subject to} \quad \|Kx - y^\delta\|_{L^\infty(\Omega)} \le c.$$

- equivalent reformulation for $p = 2$
  [Grund/Rösch '01, Prüfert/Schiela '09, C/Ito/Kunisch '10]
- unique minimizer (x^*, c^*)
- optimality conditions (Maurer–Zowe regular point condition)
  but: Lagrange multipliers are in $L^\infty(\Omega)^*$

Moreau–Yosida approximation

$$\min_{(x,c)\in X\times\mathbb{R}} \frac{c^2}{2} + \frac{\alpha}{2}\|x\|_X^2 + \frac{\gamma}{2}\|\max(0, Kx - y^\delta - c)\|_{L^2(\Omega)}^2 + \frac{\gamma}{2}\|\min(0, Kx - y^\delta + c)\|_{L^2(\Omega)}^2,$$

- unique solution $(x_\gamma, c_\gamma) \in X \times \mathbb{R}$
- strong convergence to (x^*, c^*) as $\gamma \to \infty$

Moreau–Yosida approximation

Set

$$\lambda_\gamma^1 = \gamma \max(0, Kx_\gamma - y^\delta - c_\gamma), \quad \lambda_\gamma^2 = \gamma \min(0, Kx_\gamma - y^\delta + c_\gamma),$$

Optimality system

$$\begin{cases} \alpha(\frac{1}{2}\|\cdot\|_X^2)'(x_\gamma) + K^*\left(\lambda_\gamma^1 + \lambda_\gamma^2\right) = 0, \\ c_\gamma + \int_\Omega -\lambda_\gamma^1 + \lambda_\gamma^2 \, ds = 0. \end{cases}$$

- optimality system is semi-smooth
  $\rightsquigarrow$ path-following semi-smooth Newton method
- reformulate in terms of $y_\gamma = Kx_\gamma$ (avoid inversion of $K = A^{-1}$)

Test problem: Inverse heat conduction

- $X = \mathrm{L}^2(0,1)$, K is Volterra integral operator of the first kind:

$$(Kx)(t) = \int_0^t k(s,t)x(s)\,ds, \quad k(s,t) = \frac{(s-t)^{-\frac{3}{2}}}{2\sqrt{\pi}} e^{-\frac{1}{4(s-t)}}$$

- noise:

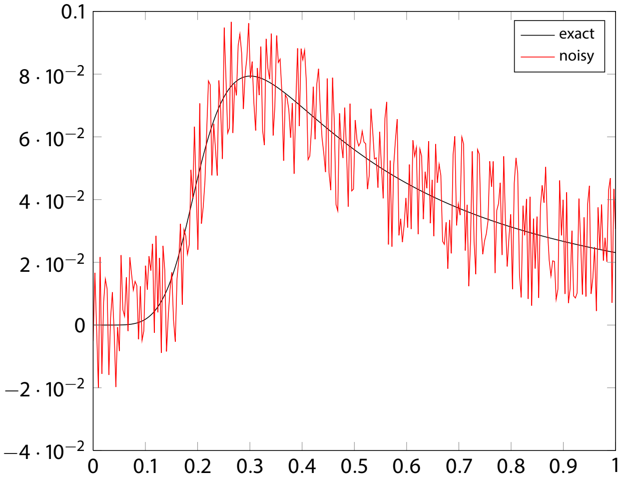
$$y^\delta(t) = (Kx^\dagger)(t) + \xi(t)\|Kx^\dagger\|_{\mathrm{L}^\infty}$$

$\xi(t)$ uniformly distributed between $\left[-\frac{d}{2}, \frac{d}{2}\right]$

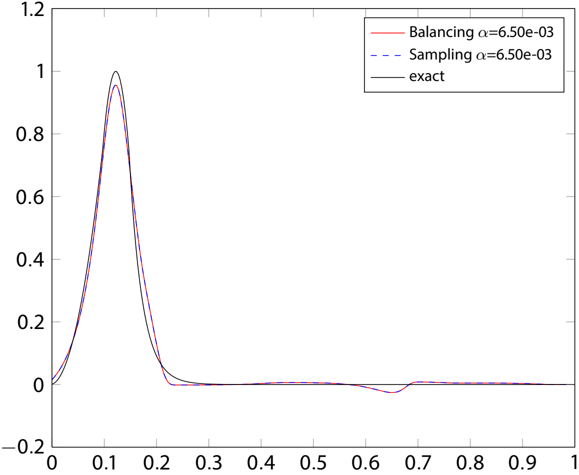
- Comparison with best parameter, L^2 fitting

$$x_{\mathrm{L}^2} = (\alpha I + K^*K)^{-1}(K^*y^\delta)$$

Data ($d = 0.3$)

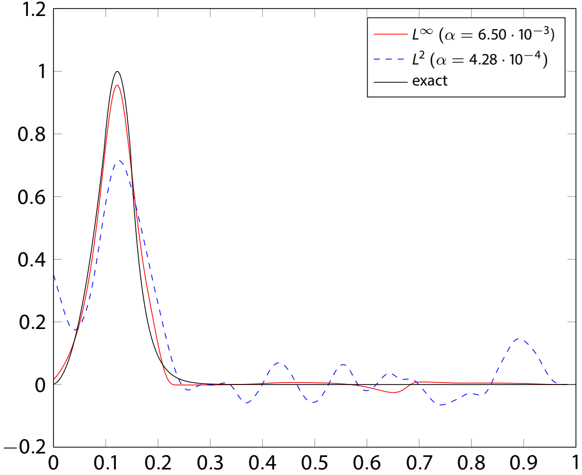

exact
noisy
0.1
$8 \cdot 10^{-2}$
$6 \cdot 10^{-2}$
$4 \cdot 10^{-2}$
$2 \cdot 10^{-2}$
0
$-2 \cdot 10^{-2}$
$-4 \cdot 10^{-2}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Reconstruction ($d = 0.3$)



(a) comparison with sampling

Reconstruction ($d = 0.3$)



(b) comparison with L^2 fitting

Test problem: Inverse source problem

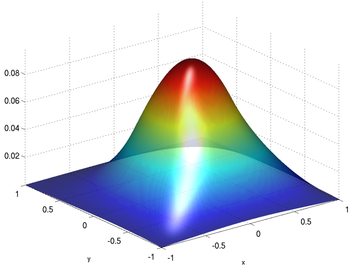
- $X = L^2(\Omega), \Omega = (0,1)^2$,
- $K = A^{-1}$ is solution operator for PDE

$$Ay = -\Delta y - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \nabla y$$

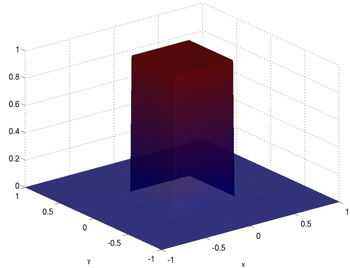
  with homogeneous Dirichlet conditions

- illustrate structure with deterministic uniform "noise":
  1. Quantization (rounding to n_b nearest values)
  2. Checkerboard ("best case noise")

Data

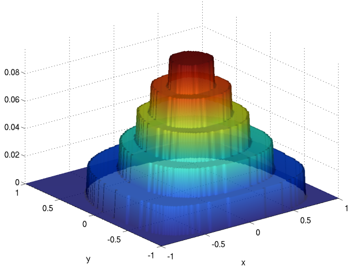


(a) true data $Kx^\dagger$

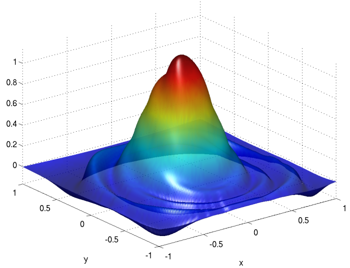


(b) true solution $x^\dagger$

Quantization ($n_b = 5$)

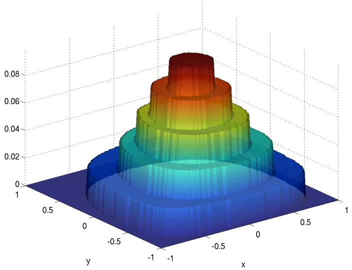


(a) data y^δ

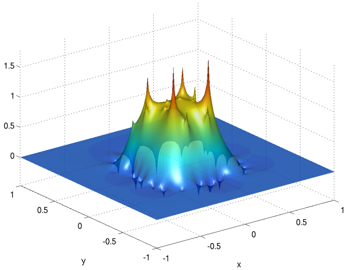


(b) L^2 reconstruction x_α

Quantization ($n_b = 5$)



(a) data y^δ



(b) L^∞ reconstruction x_α

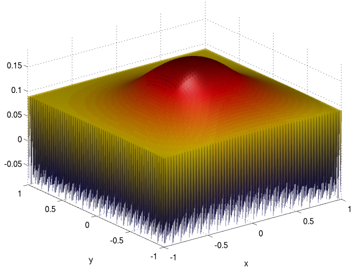
Checkerboard noise

- Observation: Reconstruction error is large in regions where noise does not change sign
- $\rightsquigarrow$ "Best case noise": constant magnitude, alternating sign

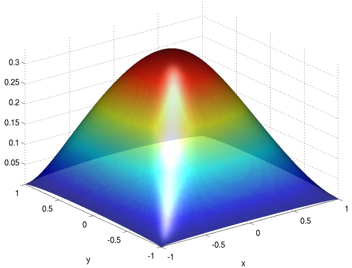
Grid points $t_{ij} = (t_{1,i}, t_{2,j}), 1 \leq i, j \leq N$,

$$y^\delta(t_{ij}) = y^\dagger(t_{ij}) + (-1)^{i+j} d \, \|y^\dagger\|_\infty$$

Checkerboard ($d = 0.9$)

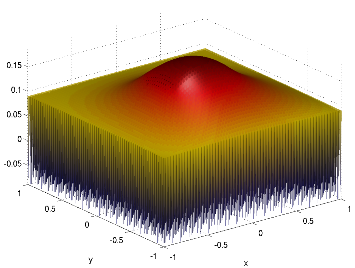


(a) data y^δ

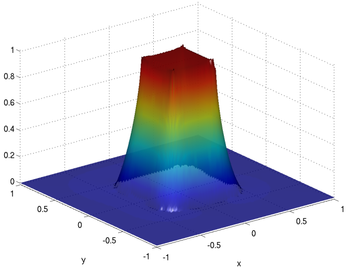


(b) L^2 reconstruction x_α

Checkerboard ($d = 0.9$)



(a) data y^δ



(b) L^∞ reconstruction x_α

Conclusion

- L^∞ fitting more robust for uniform noise
- Solution by semi-smooth Newton method
- For non-Gaussian noise: structure more important than level

Outlook

- Nonlinear parameter identification
- L^∞-L^1 functional ("Dantzig selector")

Preprint, MATLAB/Python code:
`http://www.uni-graz.at/~clason/publications.html`