Inverse source problems with L^1-type functionals

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Inverse Problems: Modeling and Simulation
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Inverse source problems

Tikhonov functional

$$\min_{u \in X} \|Su - y^\delta\|_Y + \alpha \|u\|_X$$

- $S : X \to Y$, $u \mapsto A^{-1}u$, A elliptic partial differential operator
- y^δ noisy data
- Use L^1 norm to **promote sparsity**
  ($\|u\|_{L^1}$ is convex relaxation of $\|u\|_{L^0} := \lambda(\operatorname{supp} u)$)

Inverse source problems

Tikhonov functional

$$\min_{u\in X} \|Su - y^\delta\|_Y + \alpha \|u\|_X$$

1. **Penalty term** $\|\cdot\|_X$ is L^1-norm ⇒ sparse **solution**
2. **Data fit term** $\|\cdot\|_Y$ is L^1-norm ⇒ sparse **residual** (for impulse noise: salt & pepper, random-valued; large outliers)

Here: other term is $\frac{1}{2}\|\cdot\|_{L^2}^2$

General approach

- **Fenchel duality**
  allows transformation of non-smooth problem into smooth problem with box constraints
- **Semismooth Newton method**
  allows superlinearly convergent algorithm for solution of (regularized) problem

Fenchel duality theorem

- $\mathcal{F} : V \to \overline{\mathbb{R}}$, $\mathcal{G} : Y \to \overline{\mathbb{R}}$ **convex** and lower semicontinuous,
- $\Lambda : V \to Y$ **linear** operator,
- $\exists v_0 \in V$: $\mathcal{F}(v_0) < \infty$, $\mathcal{G}(\Lambda v_0) < \infty$, $\mathcal{G}$ continuous at Λv_0:

$$(*) \qquad \inf_{u \in V} \mathcal{F}(u) + \mathcal{G}(\Lambda u) = \sup_{p \in Y^*} -\mathcal{F}^*(\Lambda^* p) - \mathcal{G}^*(-p)$$

Extremality relations: u^*, p^* satisfy $(*)$ iff

$$\begin{cases} \Lambda^* p^* \in \partial \mathcal{F}(u^*), \\ -p^* \in \partial \mathcal{G}(\Lambda u^*), \end{cases}$$

Fenchel duality

Fenchel conjugate of $\mathcal{F} : V \to \overline{\mathbb{R}}$:

$$\mathcal{F}^* : V^* \to \overline{\mathbb{R}}$$
$$\mathcal{F}^*(p) = \sup_{u \in V} \langle p, u \rangle_{V^*,V} - \mathcal{F}(u)$$

In particular:

$$\mathcal{F}(u) = \alpha \|u\|_{L^1} \quad \Rightarrow \quad \mathcal{F}^*(p) = I_{\{\|p\|_{L^\infty} \le \alpha\}} := \begin{cases} 0 & \|p\|_{L^\infty} \le \alpha \\ \infty & \text{else} \end{cases}$$

$$\mathcal{F}(u) = \frac{\alpha}{2} \|u\|_{L^2}^2 \quad \Rightarrow \quad \mathcal{F}^*(p) = \frac{1}{2\alpha} \|p\|_{L^2}^2$$

Solution of dual problem

Dual problem: J is smooth,

$$\min_p J(p) \quad \text{subject to } \|p\|_{L^\infty} \leq \alpha$$

Moreau-Yosida regularization

$$\min_{p_c} J(p_c) + \frac{1}{2c}\|\max(0, c(p_c - \alpha))\|_{L^2}^2 + \frac{1}{2c}\|\min(0, c(p_c + \alpha))\|_{L^2}^2$$

Optimality system can be solved by **semismooth Newton method**

Convergence $p_c \to p$ for $c \to \infty \Rightarrow$ Continuation strategy

Semismooth Newton method

Pointwise projection operator

$$P_\alpha(p) := \max(0, c(p-\alpha)) + \min(0, c(p+\alpha))$$

is **Newton-differentiable** from L^{q_1} to L^{q_2}, if and only if $q_1 > q_2$,

Newton derivative

$$D_N P_\alpha(p)h = ch\chi_{\{|p|>\alpha\}} := \begin{cases} ch(x) & \text{if } |p(x)| > \alpha, \\ 0 & \text{if } |p(x)| \leq \alpha. \end{cases}$$

Semismooth Newton method

Optimality system

$$F(p) := \nabla J(p) + \max(0, c(p-\alpha)) + \min(0, c(p+\alpha)) = 0$$

Newton method

$$\nabla^2 J(p^k)\delta p + c\chi_{\{|p^k|>\alpha\}}\delta p = -F(p^k)$$

If $\nabla^2 J$ has smoothing properties,

- converges locally superlinearly
- $F(p^{k+1}) = 0$ if $\operatorname{sgn}(p^{k+1})\chi_{\{|p^{k+1}|>\alpha\}} = \operatorname{sgn}(p^k)\chi_{\{|p^k|>\alpha\}}$

L^1 penalty

Tikhonov functional

$$\min_{u\in\mathcal{M}} \frac{1}{2}\|Su - y^\delta\|_{L^2}^2 + \alpha\|u\|_{\mathcal{M}}$$

- Minimization problem not well-posed in L^1 (Boundedness does not imply weak subsequential convergence in L^1)
- Need to consider space of **bounded Borel measures** $\mathcal{M}(\Omega)$ (L^1 can be isometrically identified with subspace of $\mathcal{M}$)
- Use embedding of H^2 into (pre)dual C_0

Fenchel duality

Dual problem

$$\begin{cases} \min_{p \in \mathrm{H}^2} \frac{1}{2}\|A^* p\|_{\mathrm{L}^2}^2 - \langle p, y^\delta \rangle_{\mathrm{L}^2} \\ \text{s.t.} \quad \|p\|_{\mathrm{C}_0} \leq \alpha, \end{cases}$$

Extremality relations: solutions u^*, p^* satisfy

$$\begin{cases} u^* = AA^* p^* + Ay^\delta, \\ 0 \geq \langle -u^*, p - p^* \rangle_{\mathcal{M}, \mathrm{C}_0}, \end{cases}$$

for all $p \in \mathrm{H}^2$ with $\|p\|_{\mathrm{C}_0} \leq \alpha$.

Characterization of solution

$u^* = u^*_+ - u^*_-$, u^*_+ and u^*_- positive measures, support:

$$\text{supp}(u^*_+) = \{x \in \Omega : p^*(x) = -\alpha\},$$
$$\text{supp}(u^*_-) = \{x \in \Omega : p^*(x) = \alpha\}.$$

Interpretation:

- Control non-zero only where box constraint on p^* active
- Control has opposite sign of p^*
- Larger $\alpha \Rightarrow$ smaller support of control

Semismooth Newton method

Write regularized **optimality system** as $F(p) = 0$,

$$F(p) := AA^*p + Ay^\delta + \max(0, c(p-\alpha)) + \min(0, c(p+\alpha))$$

$p \in \mathrm{H}^2 \subset \mathrm{L}^q$ for all $q \quad \Rightarrow \quad F$ is Newton-differentiable,

Newton derivative

$$D_N F(p)h = AA^*h + ch\chi_{\{|p|>\alpha\}}$$

Semismooth Newton method

Given p^k, $k = 0$:

1. Define **active sets**:

$$\begin{aligned} \mathcal{A}_k^+ &:= \{x : p^k(x) > \alpha\} \\ \mathcal{A}_k^- &:= \{x : p^k(x) < -\alpha\} \end{aligned}, \qquad \mathcal{A}_k := \mathcal{A}_k^+ \cup \mathcal{A}_k^-$$

2. Solve Newton step for p^{k+1}:

$$AA^* p^{k+1} + c\chi_{\mathcal{A}_k} p^{k+1} = -Ay^\delta + c(\chi_{\mathcal{A}_k^+} - \chi_{\mathcal{A}_k^-})$$

3. Set $k = k + 1$, repeat

L^1 fitting

Tikhonov functional

$$\min_{u \in L^2} \|Su - y^\delta\|_{L^1} + \frac{\alpha}{2}\|u\|_{L^2}^2$$

- More robust in the presence of outliers, impulse noise
- Applicable in imaging (single pixel failure)
- Regularization assumes smooth solution; alternative: TV

Fenchel duality

Dual problem

$$\begin{cases} \min_{p \in L^2} \frac{1}{2\alpha} \|S^* p\|_{L^2}^2 - \langle p, y^\delta \rangle_{L^2} \\ \text{s.t.} \quad \|p\|_{L^\infty} \leq 1, \end{cases}$$

Extremality relations: solutions u^*, p^* satisfy

$$\begin{cases} S^* p^* = \alpha u^*, \\ 0 \leq \langle S u^* - y^\delta, p - p^* \rangle_{L^2}, \end{cases}$$

for all $p \in L^2$ with $\|p\|_{L^\infty} \leq 1$.

Characterization of minimizer

For all $p \in L^2$, $p \geq 0$:

$$\left\{x \in \Omega : (Su^* - y^\delta)(x) > 0\right\} = \{x \in \Omega : p^*(x) = 1\},$$
$$\left\{x \in \Omega : (Su^* - y^\delta)(x) < 0\right\} = \{x \in \Omega : p^*(x) = -1\}.$$

Interpretation:

- Box constraint on p^* active where data is not attained by u^*
- Sign of p^* gives sign of noise
- $\Rightarrow p^*$ is **noise indicator**

Semismooth Newton method

Write regularized **optimality system** as $F(p) = 0$,

$$F(p) := \frac{1}{\alpha} SS^* p - \beta \Delta p + \max(0, c(p-1)) + \min(0, c(p+1)) - y^\delta$$

Smoothing ensures $p \in L^q$, $q > 2 \;\Rightarrow\; F$ Newton-differentiable,

Newton derivative

$$D_N F(p)h = \frac{1}{\alpha} SS^* h - \beta \Delta h + ch\chi_{\{|p|>1\}}$$

Semismooth Newton method

Given p^k, $k = 0$, $\beta > 0$:

1. Define **active sets**:

$$\begin{aligned} \mathcal{A}_k^+ &:= \{x : p^k(x) > 1\} \\ \mathcal{A}_k^- &:= \{x : p^k(x) < -1\} \end{aligned}, \qquad \mathcal{A}_k := \mathcal{A}_k^+ \cup \mathcal{A}_k^-$$

2. Solve Newton step for p^{k+1} using Krylov method:

$$\frac{1}{\alpha} SS^* p^{k+1} - \beta \Delta p^{k+1} + c\chi_{\mathcal{A}_k} p^{k+1} = y^\delta + c(\chi_{\mathcal{A}_k^+} - \chi_{\mathcal{A}_k^-})$$

3. Set $k = k + 1$, repeat till convergence
4. Decrease β, repeat with $p^0 = p^k$

Numerical Examples

1. L^1 penalty: $\frac{1}{2}\|Su - y_1^\delta\|_{L^2}^2 + \alpha \|u\|_{\mathcal{M}}$,

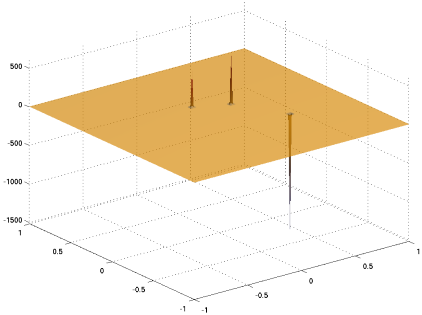
$$y_1^\delta = y_1 + \sigma \|y_1\|_\infty \xi,$$

2. L^1 fitting: $\|Su - y_2^\delta\|_{L^1} + \frac{\alpha}{2} \|u\|_{L^2}^2$

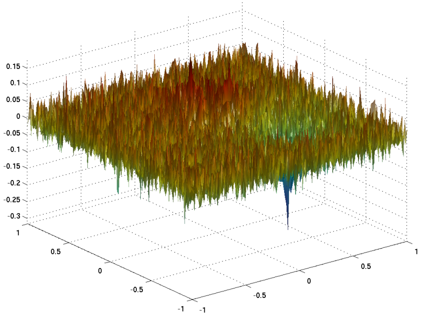
$$y_2^\delta(x) = \begin{cases} y_2(x) + \|y_2\|_\infty \xi & \text{with probability } \sigma, \\ y_2(x), & \text{otherwise.} \end{cases}$$

- $Su = (-\Delta)^{-1} u$ + zero Dirichlet b.c.
- ξ Gaussian random variable: mean 0, standard deviation 1

Results (L^1 penalty)

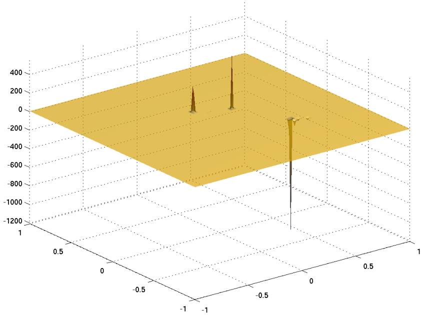


(a) true solution $u^\dagger$

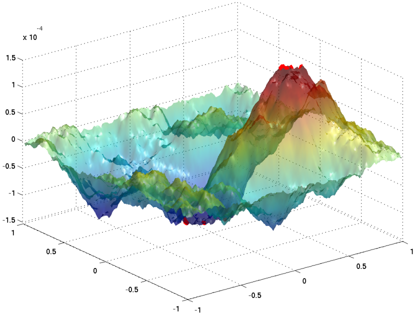


(b) data y_1^δ ($\sigma = 0.1$)

Results (L^1 penalty)

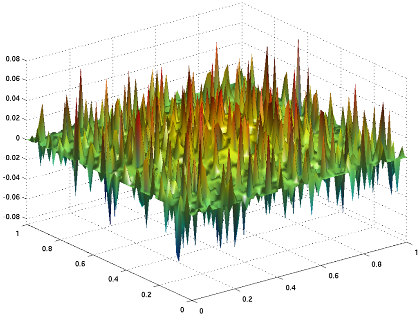


(c) reconstruction u^* ($\alpha = 1.5 \cdot 10^{-4}$)

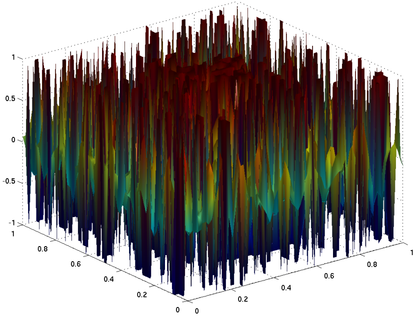


(d) dual solution p^*

Results (L^1 fitting)

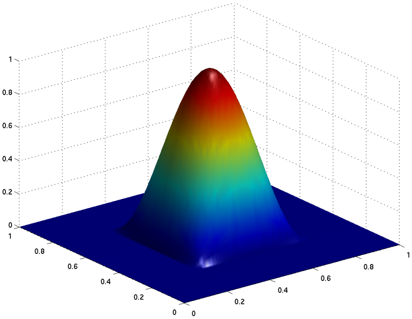


(a) data y_2^δ ($\sigma = 0.3$)

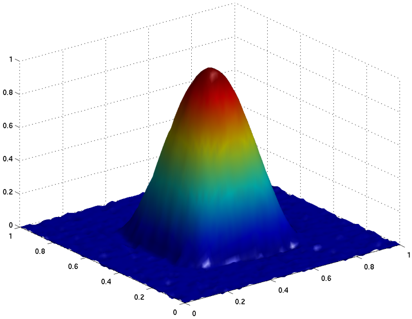


(b) dual solution p^*

Results (L^1 fitting)



(c) true solution $u^\dagger$



(d) reconstruction u^* ($\alpha = 8.8 \cdot 10^{-3}$)

Conclusion

Fenchel duality and semismooth Newton method allow efficient solution of L^1-type problems

Current work:

- Nonlinear problems
- Partial observation, time dependent problems
- Sparsity in transform space (gradient, Fourier transform)

Thank you for your attention!

References

- CC, K. Kunisch: *A duality-based approach to elliptic control problems in non-reflexive Banach spaces*, ESAIM: COCV, 2010
- CC, B. Jin, K. Kunisch: *A semismooth Newton method for L^1 data fitting with automatic choice of regularization parameters and noise calibration*, SIAM J Imag Sci, 2010
- CC, B. Jin, K. Kunisch: *A duality-based splitting method for l1-TV image restoration with automatic regularization parameter choice*, SIAM J Sci Comp, 2010

Papers, MATLAB code:

`http://www.uni-graz.at/~clason/publications.html`

Model function approach (for L^1 fitting)

u_α minimizer for given α:

Value function

$$F(\alpha) = \|Su_\alpha - y^\delta\|_{L^1} + \frac{\alpha}{2}\|u_\alpha\|_{L^2}^2$$

$F(\alpha)$ continuous, increasing, differentiable,

Derivative

$$F'(\alpha) = \frac{1}{2}\|u_\alpha\|_{L^2}^2$$

Balancing principle

Balance data residual (increasing in α)

$$F(\alpha) - \alpha F'(\alpha) = \|Su_\alpha - y^\delta\|_{L^1}$$

with penalty term (decreasing in α)

$$\alpha F'(\alpha) = \frac{\alpha}{2}\|u\|_{L^2}^2$$

Choose α^* such that

$$(\sigma - 1)(F(\alpha^*) - \alpha^* F'(\alpha^*)) = \alpha^* F'(\alpha^*)$$

$\sigma > 1$ controls relative weight

Model function

Padé approximation of value function

$$m(\alpha) = b + \frac{c}{t+\alpha}.$$

- c, t from interpolation conditions:

$$m(\alpha) = F(\alpha), \quad m'(\alpha) = F'(\alpha),$$

- b from asymptotic behavior:

$$\lim_{\alpha\to\infty} m(\alpha) = b := \|y^\delta\|_{L^1} = \lim_{\alpha\to\infty} F(\alpha)$$

Fixed point iteration

1. Given α_k, compute minimizer u_{α_k}
2. Evaluate $F(\alpha_k)$ and $F'(\alpha_k)$
3. Construct model function $m_k(\alpha) = b + \frac{c_k}{t_k+\alpha}$
4. Solve for α_{k+1} in

$$m_k(\alpha_{k+1}) = \sigma(F(\alpha_k) - \alpha_k F'(\alpha_k))$$

5. repeat

Fixed point α^* satisfies balancing equation:

$$F(\alpha^*) = \sigma(F(\alpha^*) - \alpha^* F'(\alpha^*))$$

Semismoothness in function space

X, Y Banach spaces, $D \subset X$ open

Definition

$F : D \subset X \to Y$ **Newton differentiable** at $x \in D$, if there is neighborhood $N(x)$, $G : N(x) \to \mathcal{L}(X, Z)$

$$\|F(x+h) - F(x) - G(x+h)h\| = o(\|h\|)$$

Set $\{G(s) : s \in N(x)\}$ **Newton derivative** of F at x.

Definition

F **semismooth** if N-differentiable and $G(s)^{-1}$ uniformly bounded.

Generalized Newton method $G(s^k)\delta x = -F(x^k)$, $s^k \in N(x^k)$, converges locally superlinearly.

back