Parameter identification problems with uniform noise

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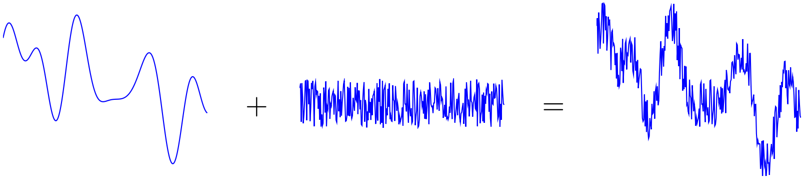
Motivation

Inverse problems with uniform noise:



exact data noise measurement

- Appears in digital acquisition, processing (quantization errors)
- Maximum likelihood estimate $\rightsquigarrow$ L^∞ data fitting

L^∞ data fitting

$$\min_{u\in X} \|S(u) - y^\delta\|_{L^\infty} + \frac{\alpha}{2}\|u\|_X^2$$

- $S : U \subset X \to Y$ (nonlinear)
- $y^\delta \in L^\infty(\Omega)$ data with uniform noise
- X Hilbert space (e.g. $L^2(D)$, $H^1(D)$)
- Y Banach space, $S(U) \hookrightarrow L^\infty(\Omega)$

- Well-posedness, regularization properties:
  [Hofmann/Kaltenbacher/Pöschl/Scherzer 2007]
  [Scherzer/Grasmair/Grossauer/Haltmeier/Lenzen 2009]

Model problems

1. Potential problem: $S : U \subset L^2(\Omega) \to H^1(\Omega), u \mapsto y,$

$$\langle \nabla y, \nabla v \rangle_{L^2} + \langle uy, v \rangle_{L^2} = \langle f, v \rangle_{L^2} \qquad \text{for all } v \in H^1(\Omega)$$

2. Robin problem: $S : U \subset L^2(\Gamma_i) \to H^{1/2}(\Gamma_c), u \mapsto y|_{\Gamma_c},$

$$\langle \nabla y, \nabla v \rangle_{L^2} + \langle uy, v \rangle_{L^2(\Gamma_i)} = \langle f, v \rangle_{L^2(\Gamma_c)} \qquad \text{for all } v \in H^1(\Omega)$$

3. Conductivity problem: $S : U \subset H^1(\Omega) \to H^1_0(\Omega), u \mapsto y,$

$$\langle u \nabla y, \nabla v \rangle_{L^2} = \langle f, v \rangle_{L^2} \qquad \text{for all } v \in H^1_0(\Omega)$$

Assumptions

- $S(U) \hookrightarrow L^\infty(\Omega)$ (domain, data sufficiently smooth)
- S uniformly bounded in $U \subset X$; $u_n \rightharpoonup u$ in X implies

$$S(u_n) \to S(u) \quad \text{in } L^\infty(\Omega)$$

- S twice Fréchet differentiable with uniformly bounded derivatives

  (Directional derivatives given by solution of linearized equations, computable using formal Lagrangian technique)

  $\rightsquigarrow$ Newton method?

Parameter identification

Goal:

Numerical method for solution of non-differentiable parameter identification problem

Approach:

1. Apply nonsmooth calculus to obtain optimality conditions
2. Apply smoothing to obtain (generalized) differentiable optimality system
3. Apply semi-smooth Newton method together with continuation in smoothing to compute approximation

Reformulation

$$\min_{(u,c)\in X\times \mathbb{R}} c + \frac{\alpha}{2}\|u\|_X^2 \quad \text{subject to} \quad \|S(u) - y^\delta\|_{L^\infty(\Omega)} \leq c$$

- equivalent reformulation
  [Grund/Rösch '01, Prüfert/Schiela '09, C/Ito/Kunisch '10, C '12]
- "augmented Morozov"
- existence of minimizer $(u_\alpha, c_\alpha) \in X \times \mathbb{R}$
- optimality conditions (Maurer–Zowe regular point condition)
  but: Lagrange multipliers are in $L^\infty(\Omega)^*$

Approximation

(Generalized) Moreau–Yosida approximation

$$\min_{(u,c)\in X\times \mathbb{R}} c + \frac{\alpha}{2}\|u\|_X^2 + \frac{\gamma}{2}\|\max(0, S(u) - y^\delta - c)\|_{L^2(\Omega)}^2 + \frac{\gamma}{2}\|\min(0, S(u) - y^\delta + c)\|_{L^2(\Omega)}^2$$

- existence of minimizers $(u_\gamma, c_\gamma) \in X \times \mathbb{R}$
- strong convergence to (u_α, c_α) as $\gamma \to \infty$

Optimality conditions

Optimality system

$$\begin{cases} \alpha j_X(u_\gamma) + \gamma S'(u)^* \left((S(u) - y^\delta - c)^+ + (S(u) - y^\delta + c)^- \right) = 0, \\ 1 + \gamma \int_\Omega -(S(u) - y^\delta - c)^+ + (S(u) - y^\delta + c)^- \, dx = 0. \end{cases}$$

with $(\cdot)^+ = \max(0, \cdot), \quad (\cdot)^- = \min(0, \cdot)$

$\rightsquigarrow$ consider as $F(u, c) = 0$ for $F : X \times \mathbb{R} \to X^* \times \mathbb{R}$

Semi-smooth Newton method

Smoothing properties of S, embedding $c \in \mathbb{R} \hookrightarrow L^\infty(\Omega)$
$\rightsquigarrow F(u,c)$ semi-smooth in u and c

Newton derivatives

$$D_{N,u}(S(u) - y^\delta - c)^+ \delta u = \chi_{\mathcal{A}}(S'(u)\delta u)$$

$$= \begin{cases} (S'(u)\delta u)(x) & \text{if } (S(u) - y^\delta)(x) \geq c \\ 0 & \text{else} \end{cases}$$

$$D_{N,c}(S(u) - y^\delta - c)^+ \delta c = -\delta c \int_\Omega \chi_{\mathcal{A}}(x)\, dx$$

Semi-smooth Newton method

Semi-smooth Newton step

$$\begin{pmatrix} D_{N,u}F_1(u^k, c^k) & D_{N,c}F_1(u^k, c^k) \\ D_{N,u}F_2(u^k, c^k) & D_{N,c}F_2(u^k, c^k) \end{pmatrix} \begin{pmatrix} \delta u \\ \delta c \end{pmatrix} = - \begin{pmatrix} F_1(u^k, c^k) \\ F_2(u^k, c^k) \end{pmatrix}$$

Action on given $\delta u, \delta c$ can be calculated by solving linearized state, adjoint equation (obtained by formal Lagrange approach)

$\rightsquigarrow$ solve using matrix-free Krylov-method (GMRES, BiCGStab)

Semi-smooth Newton method

Local coercivity condition

$$\gamma\langle S''(u_\gamma)(h,h), (S(u_\gamma) - y^\delta - c_\gamma)^+ + (S(u_\gamma) - y^\delta + c_\gamma)^-\rangle_{L^2} + \alpha\|h\|_X^2 \geq c\|h\|_X^2 \quad \text{for all } h \in X$$

Here: satisfied for

- large α (for large noise)
- small γ or small residual (for small noise)

- Implies regularity condition, superlinear convergence
- Continuation in $\gamma \to \infty$ for globalization

Automatic parameter choice

Noise level δ unknown: choose α^* such that

Balancing principle

$$\sigma \|S(u^\delta_{\alpha^*}) - y^\delta\|_{L^\infty} = \frac{\alpha^*}{2}\|u^\delta_{\alpha^*}\|^2_X$$

is satisfied (σ fixed, depends on S, X, not noise)

Fixed point iteration

$$\alpha_{k+1} = \sigma \frac{\|S(u^\delta_{\alpha_k}) - y^\delta\|_{L^\infty}}{\frac{1}{2}\|u^\delta_{\alpha_k}\|^2_X}$$

Automatic parameter choice

Theorem

If initial guess α_0 satisfies

$$\sigma \|S(u^\delta_{\alpha_0}) - y^\delta\|_{L^\infty} - \frac{\alpha_0}{2}\|u^\delta_{\alpha_0}\|_X^2 < 0,$$

sequence $\{\alpha_k\}$

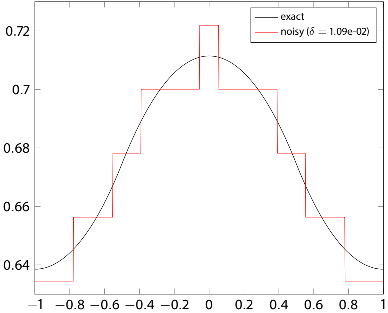
- *is monotonically decreasing*
- *converges to solution of balancing equation*

Constructive: Fix α_0, choose σ sufficiently small

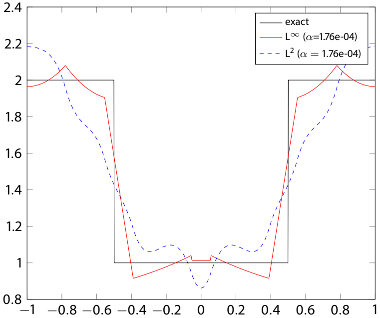
Numerical examples

- Model problems, discretization with linear finite elements
- Quantization noise: round $y^\dagger = S(u^\dagger)$ to n_b equidistant values
- Choice of α by fixed point iteration (4–7 iterations)
- Termination of continuation at $\gamma \approx 10^9$
- Comparison with L^2 fitting (optimal choice of α by sampling)

Results: potential problem ($n_b = 5$)

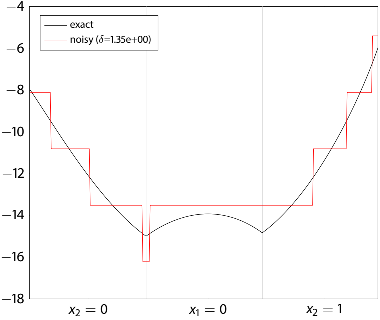


(a) data

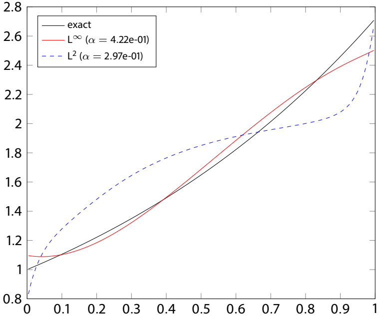


(b) reconstruction

Results: Robin problem ($n_b = 5$)

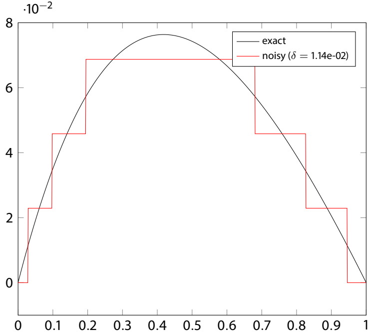


(a) data

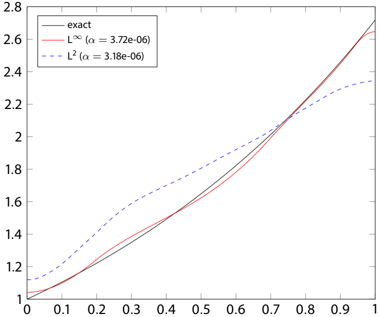


(b) reconstruction

Results: Diffusion problem ($n_b = 4$)



(a) data



(b) reconstruction

Conclusion

For non-Gaussian noise models:

- Noise structure more important than noise level
- Nonsmooth optimization methods allow efficient solution
- Parameter choice by balancing principle

Outlook:

- Stochastic inverse problems with non-Gaussian noise
- Dantzig selector (L^∞-L^1), L^∞ regularization
- Applications