Minimum effort optimal control of elliptic PDEs

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Minimum effort control problem

$$\begin{cases} \min_{u \in L^\infty} \frac{1}{2}\|y - z\|_{L^2}^2 + \frac{\alpha}{2} \|u\|_{L^\infty}^2 \\ \text{subject to } Ay = u \end{cases}$$

- $\Omega \subset \mathbb{R}^n$
- A linear elliptic operator with boundary conditions
- $z \in L^2(\Omega)$ given target

- Minimize maximal control magnitude (e.g., system design)
- Well-studied for ODEs [Neustadt '62], very little for PDEs

Minimum effort control problem

Reformulation

$$(\mathcal{P}) \qquad \begin{cases} \min\limits_{c\in\mathbb{R},u\in L^\infty} \frac{1}{2}\|y-z\|_{L^2}^2 + \frac{\alpha}{2}c^2 \\ \text{subject to } Ay = u \\ \|u\|_{L^\infty} \leq c \end{cases}$$

- Equivalent formulation
- Different from control constraints
- Related to sup-norm minimization of state
  [Grund/Rösch '01, Prüfert/Schiela '09, C/I/K '10]

Minimum effort control problem

Reformulation

$$(\mathcal{P}) \qquad \begin{cases} \min_{c\in\mathbb{R},u\in L^\infty} \frac{1}{2}\|y-z\|_{L^2}^2 + \frac{\alpha}{2}c^2 \\ \text{subject to } Ay = cu \\ \|u\|_{L^\infty} \le 1 \end{cases}$$

- Equivalent formulation
- Existence of minimizer $(y^*, u^*, c^*) \in H_0^1(\Omega) \times L^\infty(\Omega) \times \mathbb{R}_+$
- Uniqueness iff $c^* \neq 0$ (iff $J(y^*, c^*) < \frac{1}{2}\|z\|_{L^2}^2$)

Optimality system

Standard subdifferential calculus yields existence of $p^* \in H_0^1(\Omega)$,

$$
\text{(OS)} \qquad \begin{cases} \langle -p^*, u - u^* \rangle_{L^2} \geq 0 \quad \text{for all } \|u\|_{L^\infty} \leq 1, \\ \alpha c^* - \langle u^*, p^* \rangle_{L^2} = 0, \\ y^* - z + A^* p^* = 0, \\ A y^* - c^* u^* = 0. \end{cases}
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$$

pointwise inspection:

$$
u^*(x) = \operatorname{sign}(p^*(x)) = \begin{cases} 1 & \text{if } p^*(x) > 0, \\ -1 & \text{if } p^*(x) < 0, \\ t \in [-1, 1] & \text{if } p^*(x) = 0 \end{cases}
$$

Reduced optimality system

(OS')
$$\begin{cases} AA^*p^* + c^*\,\mathrm{sign}(p^*) = Az, \\ \alpha c^* - \|p^*\|_{L^1} = 0. \end{cases}$$

$$\mathrm{sign}(p)(x) = \begin{cases} 1 & \text{if } p(x) > 0, \\ -1 & \text{if } p(x) < 0, \\ t \in [-1, 1] & \text{if } p(x) = 0 \end{cases}$$

$\rightsquigarrow$ not differentiable (even in generalized sense)

Regularized problem

$$(\mathcal{P}_\beta) \qquad \begin{cases} \min_{c\in\mathbb{R}_+, u\in L^\infty} \frac{1}{2}\|y-z\|_{L^2}^2 + \frac{\beta c}{2}\|u\|_{L^2}^2 + \frac{\alpha}{2}c^2 \\ \text{subject to } Ay = cu \quad \text{in } \Omega, \\ \|u\|_{L^\infty} \leq 1. \end{cases}$$

- Existence of minimizer $(y_\beta, u_\beta, c_\beta) \in H_0^1(\Omega) \times L^\infty(\Omega) \times \mathbb{R}_+$ from standard arguments ($c_\beta > 0$ assumed)
- Uniqueness for α sufficiently large; uniqueness for any α if u_β or c_β given (bilinear structure)
- Motivation: Huber-type smoothing of reduced optimality system

Regularized optimality system

$$
(\mathrm{OS}_\beta) \quad \begin{cases} \langle \beta u_\beta - p_\beta, u - u_\beta \rangle_{L^2} \geq 0 \quad \text{for all } \|u\|_{L^\infty} \leq 1, \\ \alpha c_\beta + \frac{\beta}{2}\|u_\beta\|_{L^2}^2 - \langle u_\beta, p_\beta \rangle_{L^2} = 0, \\ y_\beta - z + A^* p_\beta = 0, \\ A y_\beta - c_\beta u_\beta = 0. \end{cases}
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Regularized optimality system

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pointwise inspection:

$$u_\beta(x) = \operatorname{sign}_\beta(p_\beta(x)) := \begin{cases} 1 & \text{if } p_\beta(x) > \beta, \\ -1 & \text{if } p_\beta(x) < -\beta, \\ \frac{1}{\beta} p_\beta(x) & \text{if } |p_\beta(x)| \leq \beta \end{cases}$$

Reduced optimality system

(OS')
$$\begin{cases} AA^* p_\beta + c_\beta \operatorname{sign}_\beta(p_\beta) = Az, \\ \alpha c_\beta - \|p_\beta\|_{L^1_\beta} = 0. \end{cases}$$

$$\operatorname{sign}_\beta(p)(x) := \begin{cases} 1 & \text{if } p(x) > \beta, \\ -1 & \text{if } p(x) < -\beta, \\ \frac{1}{\beta} p(x) & \text{if } |p(x)| \leq \beta \end{cases}$$

$$\|p\|_{L^1_\beta} := \int_\Omega |p(x)|_\beta \, dx, \quad |p(x)|_\beta := \begin{cases} p(x) - \frac{\beta}{2} & \text{if } p(x) > \beta, \\ -p(x) - \frac{\beta}{2} & \text{if } p(x) < -\beta, \\ \frac{1}{2\beta} p(x)^2 & \text{if } |p(x)| \leq \beta. \end{cases}$$

Convergence

Lemma (Monotonicity)

For $\beta < \beta'$ *and* $J(y,c) = \frac{1}{2}\|y - z\|_{L^2}^2 + \frac{\alpha}{2}c^2$, *we have*

$$c_{\beta'}\|u_{\beta'}\|_{L^2}^2 \leq c_\beta\|u_\beta\|_{L^2}^2$$

$$J(y_\beta, c_\beta) \leq J(y_{\beta'}, c_{\beta'})$$

Theorem (Convergence)

As $\beta \to 0$, *subsequence of* $(y_\beta, u_\beta, c_\beta) \to (y^*, u^*, c^*)$

- *weakly-*$\star$ *in* $H_0^1(\Omega) \times L^\infty(\Omega) \times \mathbb{R}_+$
- *strongly in* $H_0^1(\Omega) \times L^q(\Omega) \times \mathbb{R}_+$ *for any* $q \in [1, \infty)$

Convergence

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Corollary (Convergence rate)

For $\beta \to 0$ *and* $J(y, c) = \frac{1}{2}\|y - z\|_{\mathrm{L}^2}^2 + \frac{\alpha}{2}c^2$*, we have*

$$J(y_\beta, c_\beta) - J(y^*, c^*) = o(\beta)$$

Semi-smooth Newton method

Consider (OS'_β) as $T(p,c) = 0$ for $T : H^2_0 \times \mathbb{R}_+ \to H^{-2}(\Omega) \times \mathbb{R}$,

$$T(p,c) = \begin{pmatrix} AA^*p + c\,\mathrm{sign}_\beta(p) - Az \\ \alpha c - \|p\|_{L^1_\beta} \end{pmatrix}$$

- T differentiable with respect to c
- $t \mapsto |t|_\beta$ differentiable, globally Lipschitz derivative $t \mapsto \mathrm{sign}_\beta(t)$

  $\Rightarrow$ Nemytskii operators
  - $p \mapsto \|p\|_{L^1_\beta}$ differentiable on $L^p(\Omega)$, $p \geq 4$
  - $p \mapsto \mathrm{sign}_\beta(p)$ semi-smooth from $L^p(\Omega)$ to $L^q(\Omega)$, $p > q \geq 1$

$\Rightarrow$ T semi-smooth [Ulbrich '02, Schiela '08]

Semi-smooth Newton method

Newton derivatives

$$D_N(\|p\|_{L^1_\beta})h = \langle \mathrm{sign}_\beta(p), h\rangle_{L^2}$$

$$\big(D_N\, \mathrm{sign}_\beta(p)h\big)(x) = \begin{cases} 0, & \text{if } |p(x)| > \beta \\ \frac{1}{\beta}h(x), & \text{if } |p(x)| \le \beta \end{cases}$$

Semi-smooth Newton step

$$\begin{pmatrix} AA^*\delta p + c^k \frac{1}{\beta}\chi_{\{|p^k(x)|\le\beta\}}\delta p + \mathrm{sign}_\beta(p^k)\delta c \\ \alpha\delta c - \langle \mathrm{sign}_\beta(p^k), \delta p\rangle \end{pmatrix} = -T(p^k, c^k)$$

Semi-smooth Newton method

Semi-smooth Newton step

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Theorem

If A^* is positive definite, semi-smooth Newton method converges locally superlinearly to solution of (OS'_β) for every $\alpha, \beta > 0$.

Regularization parameter choice

Local convergence only, radius shrinks with β

$\rightsquigarrow$ Continuation scheme:

1. Set β_0 large, q_0 small, $t < 1$
2. If Newton method converges (no change in active sets):
   - Accept solution, $n \leftarrow n + 1$
   - Decrease $\beta_n \leftarrow \beta_{n-1} q_m$

   else
   - Reject solution, $m \leftarrow m + 1$
   - Increase $q_m \leftarrow q_{m-1}^t$, $\beta_n \leftarrow q_m \beta_{n-1}$

Stopping criterion?

Stopping criterion

- Observation: $\beta \mapsto c_\beta$ saturates
- Lemma: $\beta \mapsto c_\beta \|u_\beta\|^2_{L^2}$ monotonically increasing for $\beta \to 0$

$\rightsquigarrow$ Choose efficiency level $0 \ll \mu < 1$, stop at β_{n^*} with

$$c_{\beta_{n^*}} \|u_{\beta_{n^*}}\|^2_{L^2} > \mu c^* \|u^*\|^2_{L^2}$$

u^*, c^* unknown $\rightsquigarrow$ model function approach

Stopping criterion

Model function

$$m(\beta) = \frac{K_1}{(K_2 + \beta)^2} \approx c_\beta \|u_\beta\|^2_{L^2}$$

- Motivated by simplified ODE model of optimality conditions
- K_1, K_2 from interpolation of two successive iterates β_n, β_{n-1}

$\rightsquigarrow$ Choose $0 < \mu < 1$, stop at n^* with

$$c_{\beta_{n^*}} \|u_{\beta_{n^*}}\|^2_{L^2} > \mu m_{n^*}(0)$$

Numerical examples

- Model problem: $\Omega = [-1, 1]^2$,

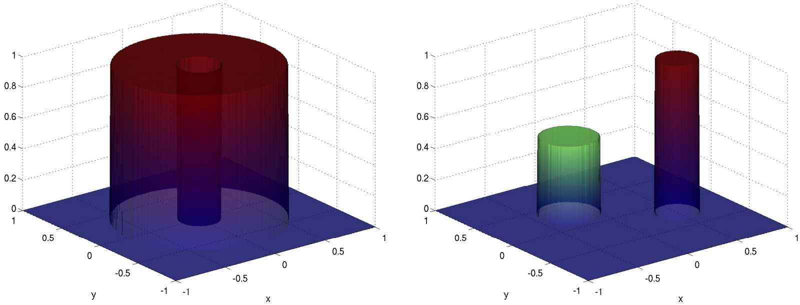
$$\begin{cases} -\nu\Delta y + \langle b, \nabla y\rangle = u, \\ y|_{\partial\Omega} = 0, \end{cases}$$

$\nu = 10^{-1}$, $b = (-1, 0)^T$

- Finite differences on uniform grid, $N = 256$ nodes

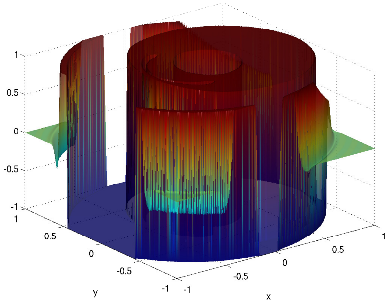
- $\beta_0 = 1$, $q_0 = 10^{-1}$, $t = 0.5$, $\mu = 0.99$, max. 10 SSN iterations
- Termination at $\beta_{n^*} \approx 2 \cdot 10^{-7}$

Examples: Targets



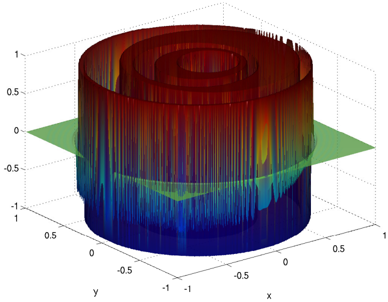
(a) z_1 (b) z_2

Examples: Controls ($\alpha = 5 \cdot 10^{-3}$)

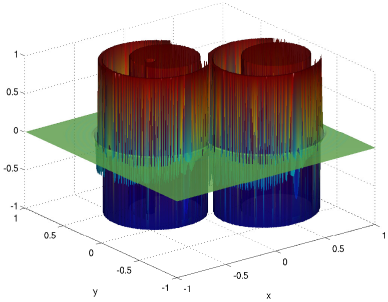


(a) $u_{1,\alpha}$ ($c_{1,\alpha} = 1.9622$) (b) $u_{2,\alpha}$ ($c_{2,\alpha} = 0.8788$)

Examples: Controls ($\alpha = 5 \cdot 10^{-4}$)

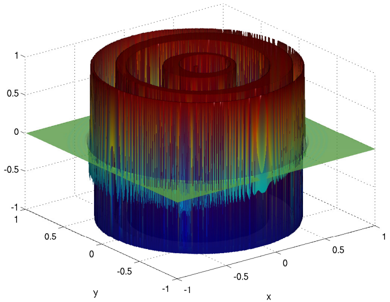


(a) $u_{1,\alpha}$ ($c_{1,\alpha} = 4.4236$)

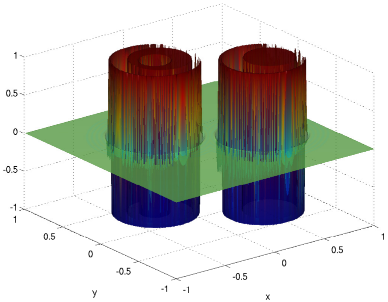


(b) $u_{2,\alpha}$ ($c_{2,\alpha} = 2.6066$)

Examples: Controls ($\alpha = 5 \cdot 10^{-5}$)

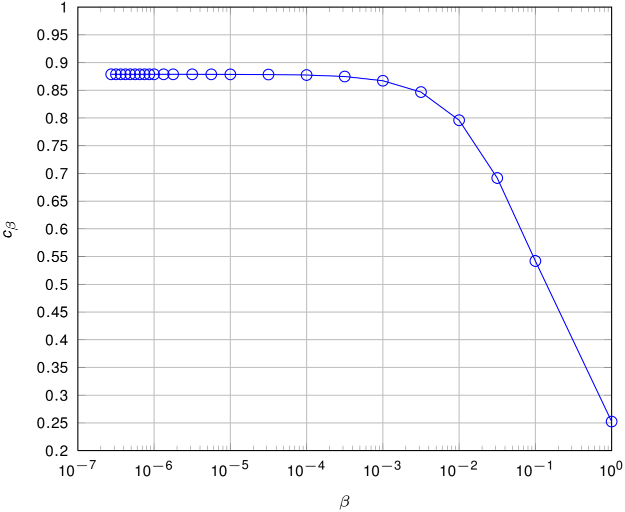


(a) $u_{1,\alpha}$ ($c_{1,\alpha} = 9.8518$)

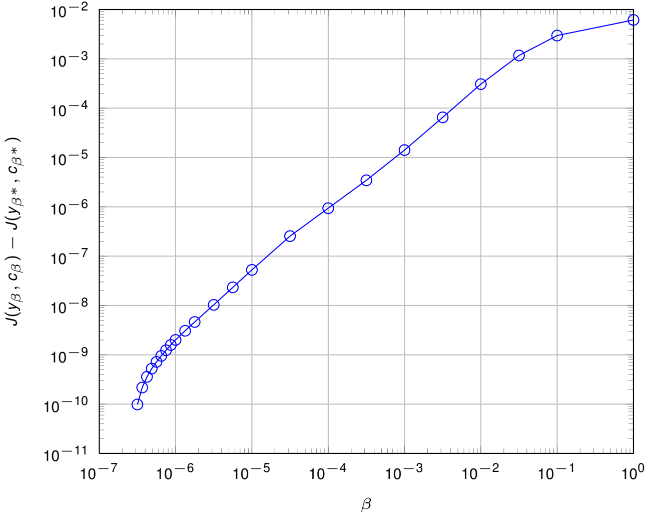


(b) $u_{2,\alpha}$ ($c_{2,\alpha} = 6.8161$)

Examples: Convergence c_β



Examples: Convergence $J(y_\beta, c_\beta)$



Conclusion

Minimum effort problem for PDEs:

- Non-differentiable (in generalized sense)
- Efficient approximation via semi-smooth Newton method with path following
- Regularized controls are "bang-zero-bang": $\text{sign}_\beta(0) = 0$
- (Limit controls are true "bang-bang": $\text{sign}(0) \neq 0$)

Preprint, MATLAB code:

`http://www.uni-graz.at/~clason/publications.html`