Variational methods for image reconstruction in MR imaging

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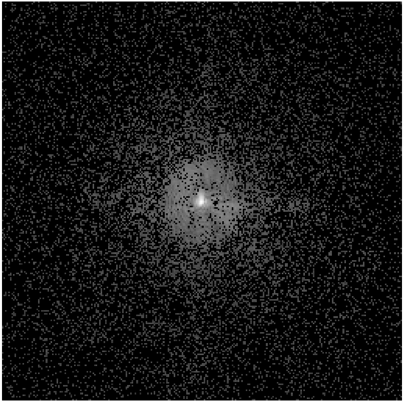
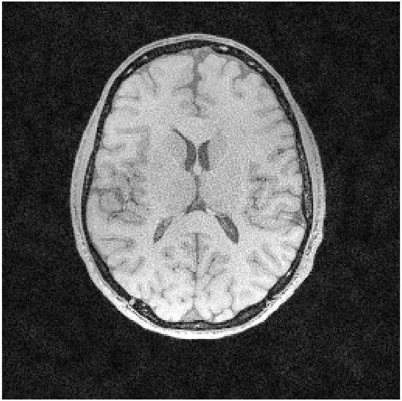
Motivation

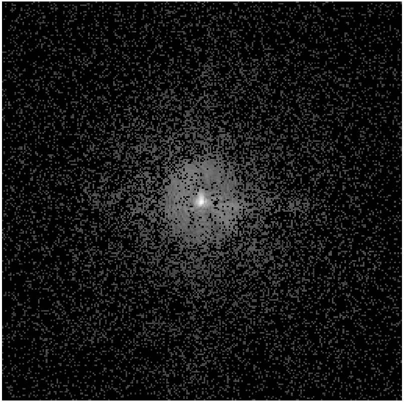


measured data

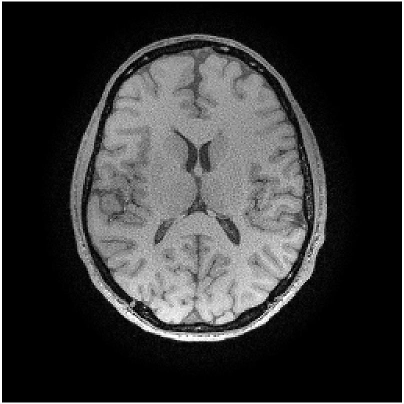


linear reconstruction

Motivation

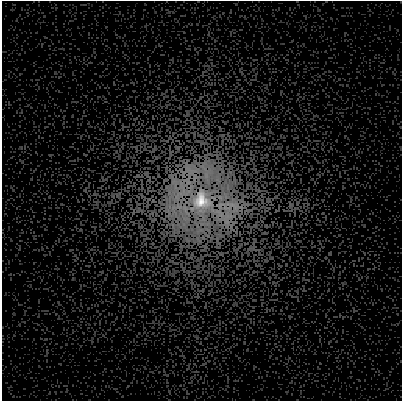


measured data

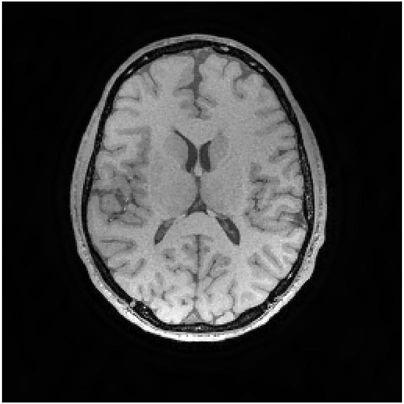


nonlinear reconstruction

Motivation



measured data



nonlinear TV-reconstruction

MR in a nutshell

Basic steps in MR scan:

1. Magnetic field is applied, aligns proton spins
2. Radio pulse at resonance frequency is absorbed by hydrogen nuclei, re-radiated over time at same frequency
3. Decaying time-dependent signal is measured by receiver coil

Fundamental principles of MR:

- Signal **amplitude** proportional to **hydrogen density**
- Signal **frequency** proportional to **magnetic field strength**

Spatial encoding

Problem:

- Measured time-dependent signal is **composite** over whole volume, no spatial information

Solution:

- Use **spatially dependent magnetic fields** to map resonance frequency to spatial location

But: linear superposition of (x, y, z) fields is not unique

$\rightsquigarrow$ **Sequential application** of fields to encode (x, y, z) separately

Spatial encoding

Slice selection (z):

- Use z-proportional magnetic field during RF **excitation**
- $\rightsquigarrow$ Only thin slice has resonance at RF pulse frequency, contributes to measured signal

Frequency encoding (x):

- Use x-proportional magnetic field **during measurement**
- $\rightsquigarrow$ Signal is superposition of resonance **frequencies**
- $\rightsquigarrow$ Each frequency can be identified with x-coordinate
- **Fourier transform** in time $\rightsquigarrow$ amplitude for each frequency $=$ spin density at each x-coordinate (integrated over y)

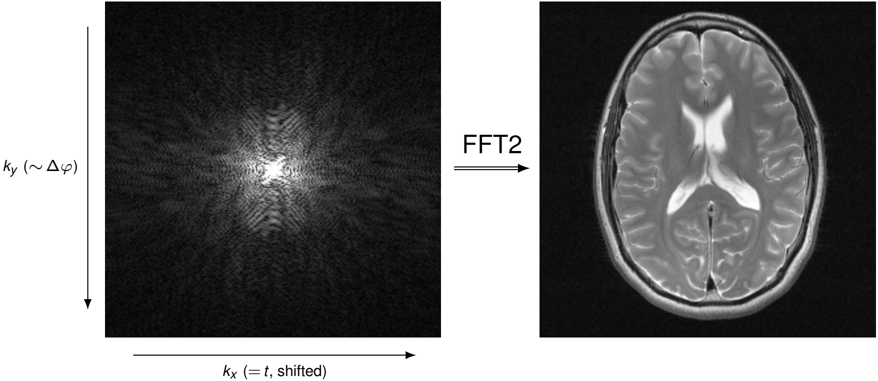
Spatial encoding

Phase encoding (y):

- Use y-proportional magnetic field **before measurement** to change **phase** of signal
- Phase cannot be measured directly, but **phase difference** can
- $\rightsquigarrow$ **Repeat measurements** with linear fields of increasing slope: $G_y = n\, y\, G_0$, $n = -N, \dots, 0, \dots N$
- $\rightsquigarrow$ Phase difference is y-specific: y larger $\Rightarrow$ gradient change, phase change with increasing n larger
- **Fourier transform** in $n \rightsquigarrow$ amplitude for each phase difference ($\sim y$-coordinate)

MRI: reconstruction

stack of measured data ("k-space") image ($\sim$ spin density)



Fourier transform of measured data gives image

Parallel imaging

Problem:

- Data needs to be acquired **sequentially** (for each k_y, z)
- Repetition cannot be faster than **physical limit**
- $\rightsquigarrow$ 4 minutes for single 256×256 slice

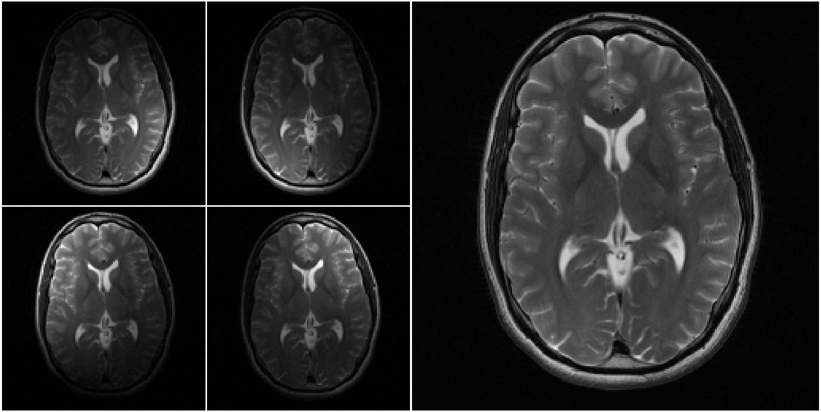
Solution:

- Acquire **fewer k_x lines**, e.g., skip every second line

But: subsampling in k_y $\rightsquigarrow$ **aliasing artifacts**

$\rightsquigarrow$ **Independent measurements** using multiple receiver coils with different **spatial sensitivities**

Example of coil sensitivities



Parallel imaging

Standard approach: if

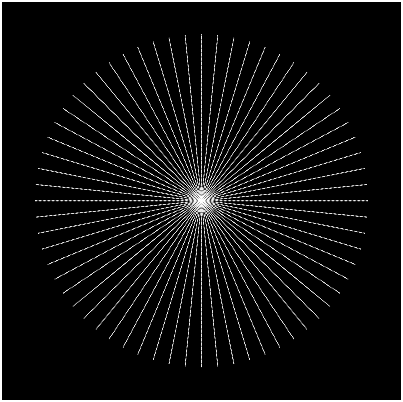
- sufficient number of coils
- coil sensitivities (approximately) known

$\rightsquigarrow$ Solve overdetermined linear system for each pixel

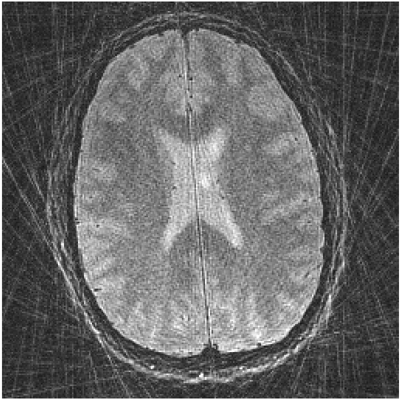
But: reconstruction quality depends on

- accurate estimation of sensitivities
- structure of subsampling artifacts

Subsampling example: Radial

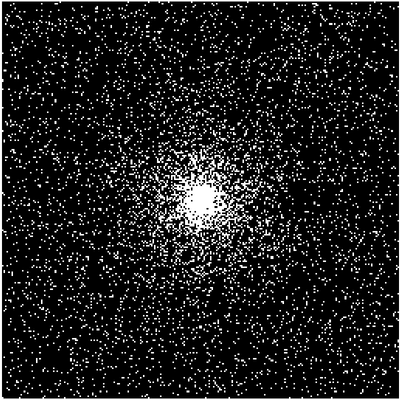


(a) pattern (acceleration factor $R \approx 12$)



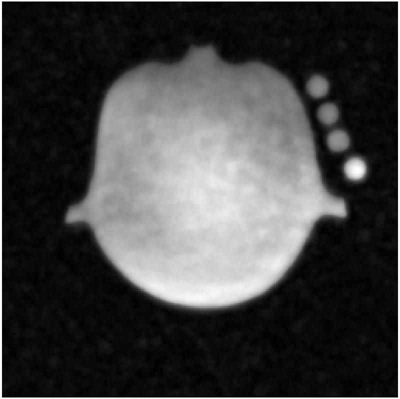
(b) aliasing artifacts

Subsampling example: Random Cartesian



(c) pattern (acceleration factor $R = 10$)

(k_x orthogonal to image plane, phase encoding in y, z)



(d) aliasing artifacts

Parallel imaging

Variational formulation allows including

- coil **sensitivity correction**
- **filtering** (artifact removal)
- arbitrary **sampling strategy**, penalty

in reconstruction

Parallel MRI as inverse problem

Given

- subsampling operator $\mathcal{F}_s$ (IFFT + mask, NFFT)
- acquired k-space coil data $g = (g_1, \dots, g_N)^T$

Find

- image u
- coil sensitivities $c = (c_1, \dots, c_N)^T$

such that

$$F(u, c) := (\mathcal{F}_s(u \cdot c_1), \dots, \mathcal{F}_s(u \cdot c_N))^T = g$$

Nonlinear inverse problem, ill-posed $\rightsquigarrow$ solve using IRGN method

Iteratively regularized Gauß–Newton method

1: choose $x^0 = (u^0, c^0)$, α_0, $q < 1$
2: **repeat**
3: solve for $\delta x = (\delta u, \delta c)$ (e.g., by CG on normal equations)

$$\min_{\delta x} \frac{1}{2}\|F'(x^k)\delta x + F(x^k) - g\|^2 + \frac{\alpha_k}{2}\|W(c^k + \delta c)\|^2 + \frac{\alpha_k}{2}\|u^k + \delta u\|^2$$

4: set $x^{k+1} = x^k + \delta x$, $\alpha_{k+1} = \alpha_k q$, $k = k + 1$
5: **until** $\|F(x^k) - g\| < tol$

W high-order differential operator (enforces smooth sensitivities)

[Uecker/Hohage/Block/Frahm '08]

IRGNTV

Replace L^2 penalty on u^{k+1} with **total variation** $TV(u^{k+1})$:

1: choose $x^0 = (u^0, c^0)$, α_0, β_0, $q < 1$
2: **repeat**
3: solve for $\delta x = (\delta u, \delta c)$

$$\min_{\delta x} \frac{1}{2}\|F'(x^k)\delta x + F(x^k) - g\|^2 + \frac{\alpha_k}{2}\|W(c^k + \delta c)\|^2 + \beta_k TV(u^k + \delta u)$$

4: set $x^{k+1} = x^k + \delta x$, $\alpha_{k+1} = \alpha_k q$, $\beta_{k+1} = \beta_k q$, $k = k + 1$
5: **until** $\|F(x^k) - g\| < tol$
6: **return** u, c

Solution of TV subproblems

Set $J(\delta x) := \frac{1}{2}\|F'(x^k)\delta x + F(x^k) - g\|^2 + \frac{\alpha_k}{2}\|W(c^k + \delta c)\|^2$

Step 3

$$\min_{\delta u, \delta c} J(\delta u, \delta c) + \beta_k TV(u^k + \delta u)$$

Non-smooth, convex optimization problem $\rightsquigarrow$ Use **convex duality**

$$\beta TV(u) = \sup_{\{|p(x)|_2 \leq \beta\}} \langle u, -\operatorname{div} p \rangle$$

Solution of TV subproblems

Saddle point problem

$$\min_{\delta u, \delta c} \max_{p \in C_{\beta_k}} J(\delta u, \delta c) + \langle u^k + \delta u, -\operatorname{div} p \rangle$$

with $C_\beta = \{p : \ |p(x)|_2 \leq \beta \text{ for all } x\}$ convex, J differentiable

$\rightsquigarrow$ Use **projected gradient descent/ascent method**:

- Requires only application of F', F'^* (i.e., $\mathcal{F}_s, \mathcal{F}_s^*$)
- Projection on C_β can be computed pointwise
- Straightforward parallelization (GPU)

Here: based on [Pock/Cremers/Bischof/Chambolle '09]

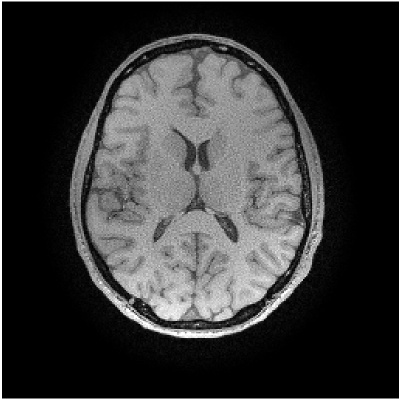
Primal-dual extragradient method

function TVSOLVE$(u, c, \alpha, \beta, \sigma_u, \sigma_c, \tau)$
  $\delta u, \bar{\delta u}, \delta c, \bar{\delta c}, p \leftarrow 0$
  repeat
    $p \leftarrow \text{proj}_\beta(p + \tau\nabla(u + \bar{\delta u}))$
    $\delta u_{old} \leftarrow \delta u, \; \delta c_{old} \leftarrow \delta c$
    $\delta u \leftarrow \delta u - \sigma_u(\partial_u J(u,c)(\bar{\delta u}, \bar{\delta c}) - \text{div}\, p)$
    $\delta c \leftarrow \delta c - \sigma_c(\partial_c J(u,c)(\bar{\delta u}, \bar{\delta c}))$
    $\bar{\delta u} \leftarrow 2\delta u - \delta u_{old}$
    $\bar{\delta c} \leftarrow 2\delta c - \delta c_{old}$
  until convergence
  return $\delta u, \delta c$
end function

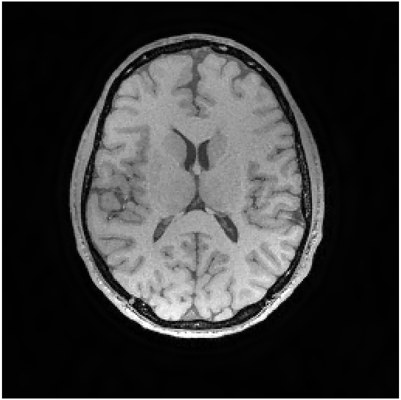
Examples: random sampling

- Raw data fom brain and phantom
- 3D gradient echo sequence, 3T system, 12 channel head coil
- 8 (phantom: 9) virtual channels (SVD) used for reconstruction
- Sequence modified using binary 2D mask for subsampling
- Subsampling $R = 4$ (10)
- Sequence parameters
  - repetition time TR $= 20\,\text{ms}$
  - echo time TE $= 5\,\text{ms}$
  - flip angle FA $= 18^\circ$
  - matrix size $(x, y, z) = 256 \times 256 \times 256$
  - FOV $= 250\,\text{mm}$
  - slice thickness brain 1 mm (phantom 5 mm)

Reconstructions: random ($R = 4$)

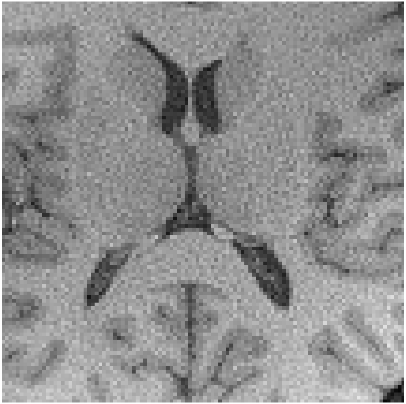


(a) IRGN



(b) IRGNTV

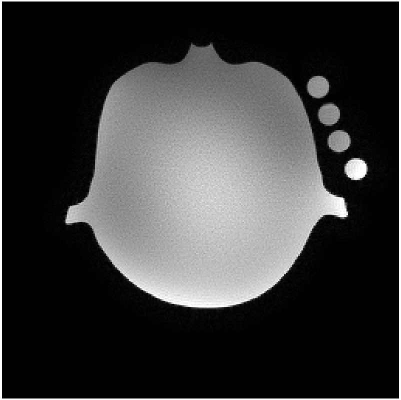
Reconstructions: random ($R = 4$)



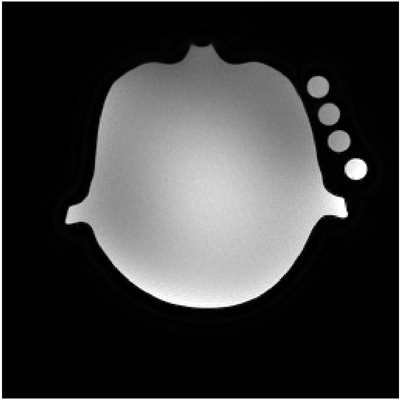
(a) IRGN (detail)

(b) IRGNTV (detail)

Reconstructions: random ($R = 4$)



(a) IRGN



(b) IRGNTV

Effect of TV

Since stopping $\beta_{k^*} > 0$ small, final TV effect is not very strong

Pro: No introduction of typical TV-artifacts (cartooning, stair-casing)

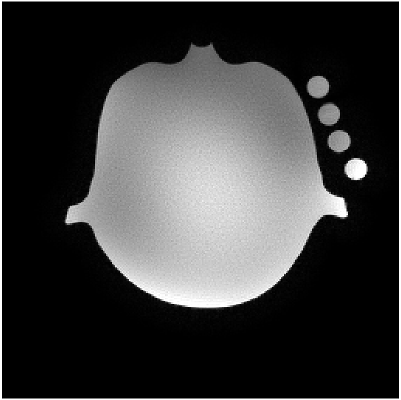
Con: More filtering can be desired for higher acceleration

$\rightsquigarrow$ Stop decreasing TV penalty parameter at desired value:

$$\alpha_{k+1} = \alpha_k q$$
$$\beta_{k+1} = \max(\beta_{\min}, \beta_k q)$$

For illustration: phantom with $\beta_{\min} = 5 \cdot 10^{-3}$

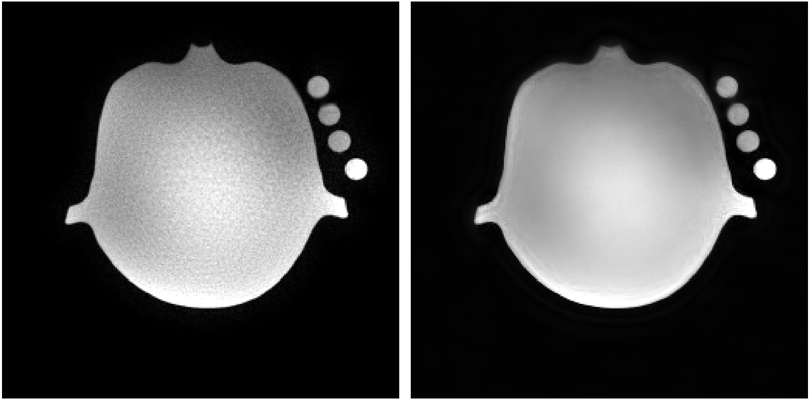
Effect of TV ($R = 4$)



(a) IRGN

(b) IRGNTV

Effect of TV ($R = 10$)



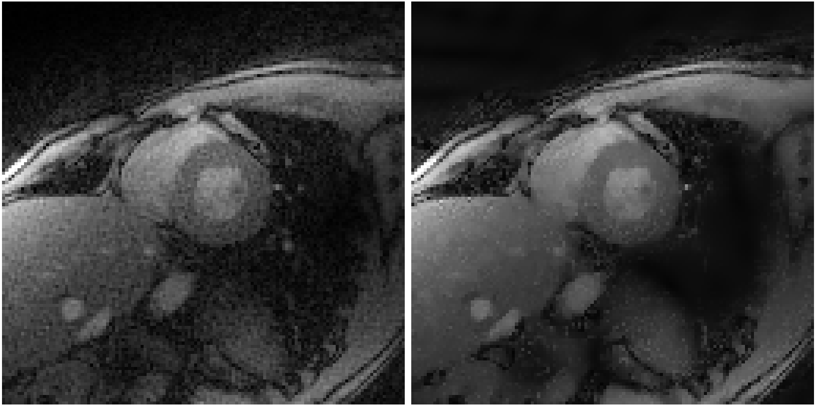
(a) IRGN (b) IRGNTV

Examples: radial sampling

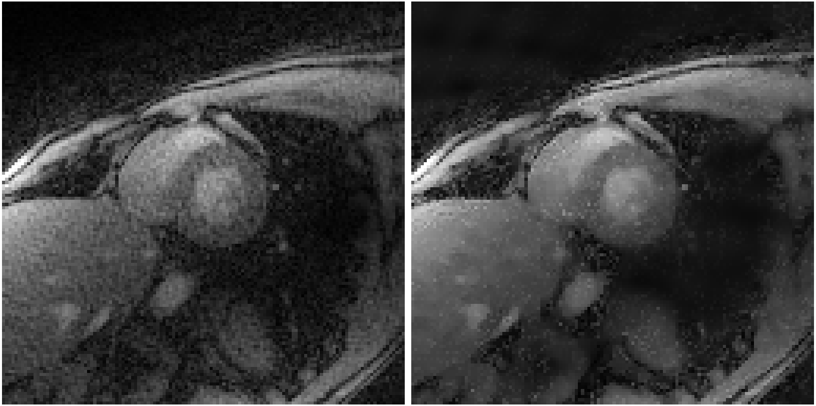
- Raw data of beating heart
- Radial FLASH sequence, 3T System, 32 channel coil
- 12 virtual channels (SVD) used for reconstruction
- 25 (19) projections, $R \approx 8$ (10.5)
- **No postprocessing, temporal view sharing**
- Sequence parameters
  - repetition time $\mathrm{TR} = 2.0\,\mathrm{ms}$
  - echo time $\mathrm{TE} = 1.3\,\mathrm{ms}$
  - flip angle $\mathrm{FA} = 8^\circ$
  - 256 points per proj. (2x oversampling) $\rightsquigarrow$ matrix 128×128
  - slice thickness 8 mm, in plane resolution $2\,\mathrm{mm} \times 2\,\mathrm{mm}$

(data courtesy of Martin Uecker, MPG Göttingen)

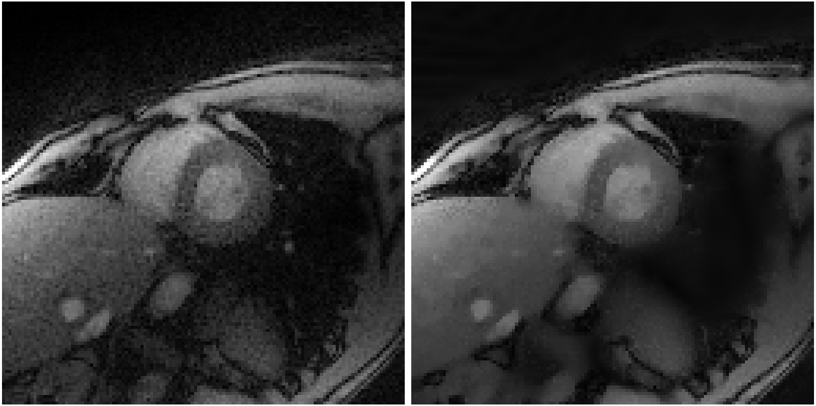
Radial sampling: cardiac (25 proj $\approx$ 20 fps)



Radial sampling: cardiac (19 proj $\approx$ 26 fps)



Radial sampling: temporal median filter



Total generalized variation (TGV)

Reduce stair-casing: use second order **total generalized variation**

$$\beta TGV^2(u) = \sup_{q\in C^2_\beta} \langle u, \operatorname{div}^2 q\rangle$$
$$= \inf_{v\in\mathcal{C}^1(\Omega,\mathbb{C}^d)} \beta\|\nabla u - v\| + 2\beta\|\mathcal{E}v\|$$

with

$$C^2_\beta = \{q \in \mathcal{C}^2_c(\Omega, \mathcal{S}^{d\times d}) : \ \|q\|_\infty \leq 2\beta, \|\operatorname{div} v\|_\infty \leq \beta\}$$
$$\mathcal{E}v = \frac{1}{2}(\nabla v + \nabla v^T) = (-\operatorname{div}^2)^* v$$

(see talk by **Kristian Bredies**)

IRGNTGV

Replace TV penalty on u^{k+1} with $TGV^2(u^{k+1})$:

1: choose $x^0 = (u^0, c^0)$, α_0, β_0, $q < 1$
2: **repeat**
3: solve for $\delta x = (\delta u, \delta c)$

$$\min_{\delta x} \frac{1}{2}\|F'(x^k)\delta x + F(x^k) - g\|^2 + \frac{\alpha_k}{2}\|W(c^k + \delta c)\|^2 + \beta_k TGV^2(u^k + \delta u)$$

4: set $x^{k+1} = x^k + \delta x$, $\alpha_{k+1} = \alpha_k q$, $\beta_{k+1} = \beta_k q$, $k = k+1$
5: **until** $\|F(x^k) - g\| < tol$
6: **return** u, c

Solution of IRGNTGV subproblems

Step 3:

Saddle point problem

$$\min_{\delta u, \delta c, v} \max_{\substack{p \in C_{\beta_k} \\ q \in C^2_{\beta_k}}} J(\delta u, \delta c) + \langle \nabla u^k + \delta u - v, p \rangle + \langle \mathcal{E} v, q \rangle$$

- Solution by projected primal-dual extragradient method
- Projection on C^2_β can be computed pointwise
- Cost dominated by (N)FFT $\rightsquigarrow$ comparable to IRGN(TV)

Primal-dual extragradient method

function TGVSOLVE$(u, c, \alpha, \beta, \sigma_u, \sigma_c, \sigma_v, \tau)$

  $\delta u, \bar{\delta u}, \delta c, \bar{\delta c}, v, \bar{v}, p, q \leftarrow 0$

  repeat

    $p \leftarrow \mathrm{proj}_\beta(p + \tau(\nabla(u + \bar{\delta u}) - v)$

    $q \leftarrow \mathrm{proj}^2_\beta(q + \tau(\mathcal{E} v))$

    $\delta u_{old} \leftarrow \delta u,\ \delta c_{old} \leftarrow \delta c,\ v_{old} \leftarrow v$

    $\delta u \leftarrow \delta u - \sigma_u(\partial_u J(u, c)(\bar{\delta u}, \bar{\delta c}) - \operatorname{div} p)$

    $\delta c \leftarrow \delta c - \sigma_c(\partial_c J(u, c)(\bar{\delta u}, \bar{\delta c}))$

    $v \leftarrow v - \sigma_v(-p - \operatorname{div}^2 q)$

    $\bar{\delta u} \leftarrow 2\delta u - \delta u_{old}$

    $\bar{\delta c} \leftarrow 2\delta c - \delta c_{old}$

    $\bar{v} \leftarrow 2v - v_{old}$

  until convergence

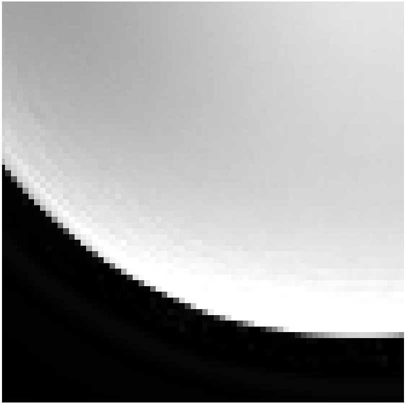
end function

Example: Random ($R = 4$)

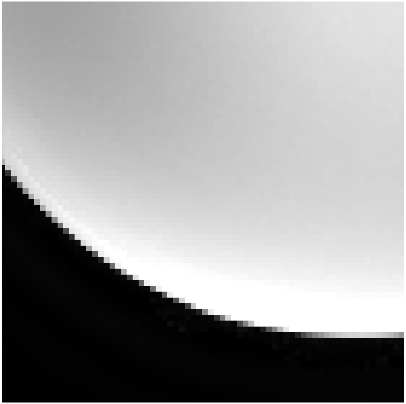
(a) IRGNTV

(b) IRGNTGV

Example: Random ($R = 4$)

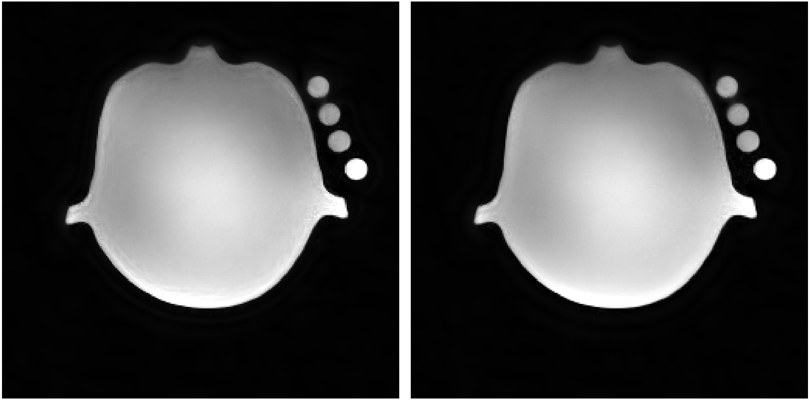


(a) IRGNTV (detail)



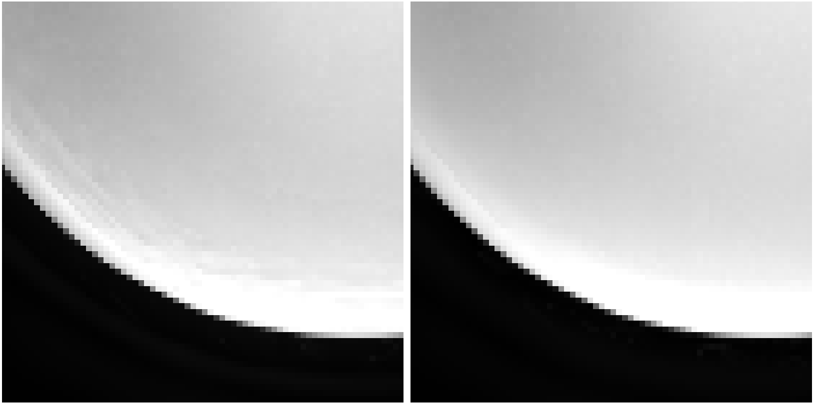
(b) IRGNTGV (detail)

Example: Random ($R = 10$)



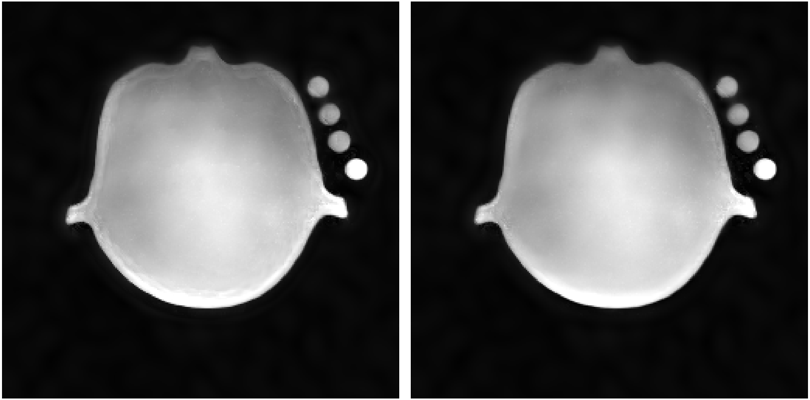
(a) IRGNTV (b) IRGNTGV

Example: Random ($R = 10$)



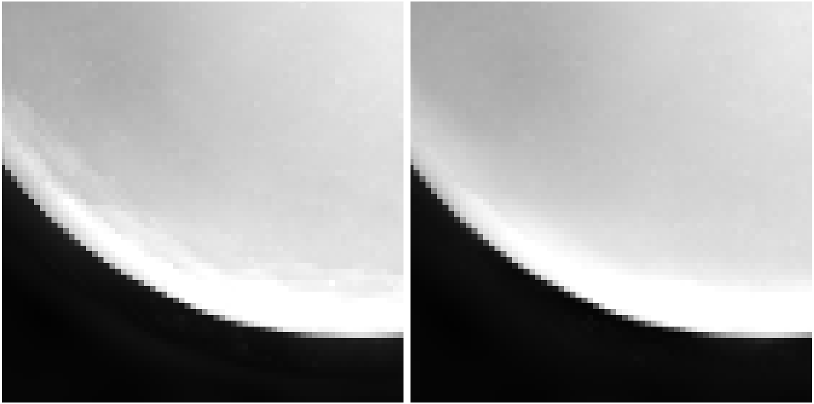
(a) IRGNTV (detail)   (b) IRGNTGV (detail)

Example: Random ($R = 18$)



(a) IRGNTV (b) IRGNTGV

Example: Random ($R = 18$)



(a) IRGNTV (detail) (b) IRGNTGV (detail)

Conclusion

Variational approach gives

- better reconstruction
- more flexibility

Outlook:

- Add constraint on slice/frame differences
- Full 3D T(G)V reconstruction
- Include parameter identification

Preprint, MATLAB code:

`http://www.uni-graz.at/~clason/publications.html`