Optimal control of PDEs with measures

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Motivation

L^1 control problem

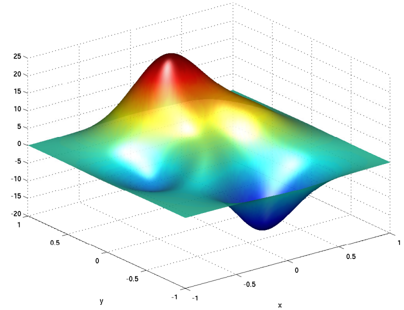
$$(\mathcal{P}_{\mathrm{L}^1}) \qquad \min_{u \in \mathrm{L}^1(\Omega)} \frac{1}{2}\|y - z\|_{L^2}^2 + \alpha \|u\|_{\mathrm{L}^1} \quad \text{subject to} \quad Ay = u$$

(A linear partial differential operator on $\Omega \subset \mathbb{R}^n$, $z \in \mathrm{L}^2(\Omega)$ given)

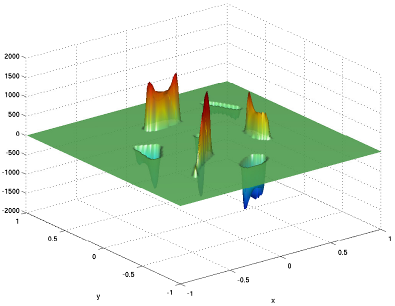
Applications:

- Control cost proportional to magnitude (fuel consumption)
- Sparse controls (optimal placement of discrete actuators)

Motivation: Sparse controls



L^2-type control



L^1-type control

Motivation

- But: $(\mathcal{P}_{\mathrm{L}^1})$ not well-posed (Boundedness does not imply weak subsequence convergence in L^1)
- $\Rightarrow$ Consider space of bounded Borel measures $\mathcal{M}(\Omega)$

$\mathcal{M}$ control problem

$$(\mathcal{P}_{\mathcal{M}}) \qquad \min_{u \in \mathcal{M}} \frac{1}{2}\|y - z\|_{L^2}^2 + \alpha \|u\|_{\mathcal{M}} \quad \text{subject to} \quad Ay = u$$

Motivation

$\mathcal{M}$ control problem

$$(\mathcal{P}_{\mathcal{M}}) \qquad \min_{u \in \mathcal{M}} \frac{1}{2}\|y - z\|_{L^2}^2 + \alpha \|u\|_{\mathcal{M}} \quad \text{subject to} \quad Ay = u$$

Assume A elliptic, second order, homogeneous Dirichlet b.c.:

- PDE is well-posed ($y \in \mathrm{W}_0^{1,p}$, $1 \leq p < \frac{n}{n-1}$)
- Problem has unique solution $(y^*, u^*) \in \mathrm{L}^2 \times \mathcal{M}$
- If $u^* \in \mathrm{L}^1$, u^* is also solution of Problem $(\mathcal{P}_{\mathrm{L}^1})$

$\Rightarrow (\mathcal{P}_{\mathcal{M}})$ well-posed!

Problem formulation

- Problem: $(\mathcal{P}_{\mathcal{M}})$ numerically difficult (discretization of measures)
- But: $\mathcal{M}$ is topological dual of C_0

Idea

- Use Fenchel duality to replace $(\mathcal{P}_{\mathcal{M}})$ with predual problem in C_0
- Use embedding to consider problem in Hilbert space

$$\mathcal{W} := H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow C_0$$

Fenchel duality theorem

- $\mathcal{F} : V \to \overline{\mathbb{R}}, \mathcal{G} : Y \to \overline{\mathbb{R}}$ convex and lower semicontinuous,
- $\Lambda : V \to Y$ linear operator,
- $\exists v_0 \in V$: $\mathcal{F}(v_0) < \infty$, $\mathcal{G}(\Lambda v_0) < \infty$, $\mathcal{G}$ continuous at Λv_0:

$$\text{(FD)} \qquad \inf_{u \in V} \mathcal{F}(u) + \mathcal{G}(\Lambda u) = \sup_{p \in Y^*} -\mathcal{F}^*(\Lambda^* p) - \mathcal{G}^*(-p)$$

Extremality relations: u^*, p^* solutions of (FD) iff

$$\begin{cases} \Lambda^* p^* \in \partial \mathcal{F}(u^*), \\ -p^* \in \partial \mathcal{G}(\Lambda u^*), \end{cases}$$

Predual problem formulation

Formal application of Fenchel duality yields

Predual problem

$$(\mathcal{P}^*_{\mathcal{M}}) \qquad \begin{cases} \min_{p\in\mathcal{W}} \frac{1}{2}\|A^*p+z\|^2_{L^2} - \frac{1}{2}\|z\|^2_{L^2} \\ \text{s.t.} \quad \|p\|_{C_0} \leq \alpha, \end{cases}$$

Theorem

Solution $p^* \in \mathcal{W}$ exists, is unique if $\|A^*\cdot\|_{L^2}$ equivalent norm on $\mathcal{W}$

(main assumption; otherwise add regularization term)

Fenchel duality for predual problem

Define

$$\mathcal{F}: \mathcal{W} \to \overline{\mathbb{R}}, \qquad \mathcal{F}(q) = \frac{1}{2}\|A^* q + z\|_{L^2}^2 - \frac{1}{2}\|z\|_{L^2}^2,$$

$$\mathcal{G}: C_0 \to \overline{\mathbb{R}}, \qquad \mathcal{G}(q) = \begin{cases} 0 & \text{if } \|q\|_{C_0} \le \alpha, \\ \infty & \text{if } \|q\|_{C_0} > \alpha, \end{cases}$$

$$\Lambda: \mathcal{W} \to C_0 \qquad \text{Injection}$$

Fenchel conjugates

$$\mathcal{F}^*: \mathcal{W}^* \to \overline{\mathbb{R}}, \qquad \mathcal{F}^*(v) = \frac{1}{2}\|A^{-1} v - z\|_{L^2}^2,$$

$$\mathcal{G}^*: \mathcal{M} \to \overline{\mathbb{R}}, \qquad \mathcal{G}^*(v) = \alpha \|v\|_{\mathcal{M}},$$

$$(\Lambda^*: \mathcal{M} \to \mathcal{W}^* \qquad \text{Injection})$$

Fenchel duality for predual problem

Application of Fenchel duality theorem yields

Theorem

- *$(\mathcal{P}_{\mathcal{M}})$ is dual problem of $(\mathcal{P}^*_{\mathcal{M}})$*
- *Solutions $u^* \in \mathcal{M}$ of $(\mathcal{P}_{\mathcal{M}})$, $p^* \in \mathcal{W}$ of $(\mathcal{P}^*_{\mathcal{M}})$ satisfy*

$$\begin{cases} u^* = AA^*p^* + Az, \\ 0 \geq \langle -u^*, p - p^* \rangle_{\mathcal{M}, C_0}, \end{cases}$$

for all $p \in \mathcal{W}$ with $\|p\|_{C_0} \leq \alpha$.

Optimality conditions

Theorem

There exists unique pair $(p^*, \lambda^*) \in \mathcal{W} \times \mathcal{M}$*, such that*

(OS) $$\begin{cases} AA^* p^* + Az + \lambda^* = 0, \\ \langle \lambda^*, p - p^* \rangle_{\mathcal{M}, C_0} \leq 0, \end{cases}$$

for all $p \in \mathcal{W}$ *with* $\|p\|_{C_0} \leq \alpha$*.*

Idea of proof

- Set $\lambda^* = -u^* \in \mathcal{M} \subset \mathcal{W}^*$ in extremality relations
- Uniqueness from $\|A^* \cdot\|_{L^2}$ norm on $\mathcal{W}$

Characterization of minimizers

$u^* = u_+^* - u_-^*$, u_+^* and u_-^* positive measures, support:

$$\operatorname{supp}(u_+^*) = \{x \in \Omega :\ p^*(x) = -\alpha\},$$
$$\operatorname{supp}(u_-^*) = \{x \in \Omega :\ p^*(x) = \alpha\}.$$

Interpretation:

- Control non-zero where box constraint on p^* active
- Control has opposite sign of p^*
- Larger $\alpha \Rightarrow$ smaller support of control

Regularization of dual problem

$$(\mathcal{P}^*_{\mathcal{M}}) \qquad \begin{cases} \min_{p\in\mathcal{W}} \frac{1}{2}\|A^*p+z\|^2_{L^2} - \frac{1}{2}\|z\|^2_{L^2} \\ \text{s.t.} \quad \|p\|_{C_0} \leq \alpha, \end{cases}$$

- Non-differentiable problem replaced by smooth box-constrained problem
- Moreau-Yosida regularization for $c > 0 \Rightarrow$ efficient solution by semismooth Newton method
- $(\mathcal{P}^*_c)$ has unique minimizer p_c

Regularization of dual problem

$$(\mathcal{P}_c^*)\quad \begin{cases} \min\limits_{p\in\mathcal{W}} \frac{1}{2}\|A^*p+z\|_{L^2}^2 - \frac{1}{2}\|z\|_{L^2}^2 \\ +\frac{1}{2c}\|\max(0,c(p-\alpha))\|_{L^2}^2 + \frac{1}{2c}\|\min(0,c(p+\alpha))\|_{L^2}^2 \end{cases}$$

- Non-differentiable problem replaced by smooth box-constrained problem
- Moreau-Yosida regularization for $c > 0 \Rightarrow$ efficient solution by semismooth Newton method
- $(\mathcal{P}_c^*)$ has unique minimizer p_c

Optimality system (regularized)

$$(\mathrm{OS}_c) \qquad \begin{cases} AA^* p_c + Az + \lambda_c = 0, \\ \lambda_c = \max(0, c(p_c - \alpha)) + \min(0, c(p_c + \alpha)) \end{cases}$$

Theorem (Convergence)

$$p_c \xrightarrow[c \to \infty]{} p^* \text{ in } \mathcal{W}, \qquad \lambda_c \xrightharpoonup[c \to \infty]{} \lambda^* \text{ in } \mathcal{W}^*$$

Optimality system (regularized)

$$(\mathrm{OS}_c) \qquad \begin{cases} AA^* p_c + Az + \lambda_c = 0, \\ \lambda_c = \max(0, c(p_c - \alpha)) + \min(0, c(p_c + \alpha)) \end{cases}$$

- Nonlinear equation for p_c
- Pointwise max, min semismooth
- $\Rightarrow$ solution by generalized Newton method
- Use continuation method in c

Semismoothness in function space

X, Y Banach spaces, $D \subset X$ open

Definition

$F : D \subset X \to Y$ **Newton differentiable** at $x \in D$, if there is neighborhood $N(x)$, $G : N(x) \to \mathcal{L}(X, Z)$

$$\|F(x+h) - F(x) - G(x+h)h\| = o(\|h\|)$$

Set $\{G(s) : s \in N(x)\}$ **Newton derivative** of F at x.

Definition

F **semismooth** if N-differentiable and $G(s)^{-1}$ uniformly bounded.

F semismooth $\Rightarrow$ generalized Newton method $G(s^k)\delta x = -F(x^k)$, $s^k \in N(x^k)$, converges locally superlinearly.

Application to box constraints

Projection operator

$$P_\alpha(p) := \max(0, (p - \alpha)) + \min(0, (p + \alpha))$$

is semismooth from L^q to L^p, if and only if $q > p$,

Newton derivative

$$D_N P_\alpha(p)h = h\chi_{\{|p|>\alpha\}} := \begin{cases} h(x) & \text{if } |p(x)| > \alpha, \\ 0 & \text{if } |p(x)| \leq \alpha. \end{cases}$$

Application to optimality system

Can be written as $F(p) = 0$ with $F : \mathcal{W} \to \mathcal{W}^*$,

$$F(p) := AA^* p + Az + \max(0, c(p - \alpha)) + \min(0, c(p + \alpha))$$

Sobolev embedding, sum and chain rule for Newton derivatives
$\Rightarrow$ F is semismooth,

Newton derivative

$$D_N F(p)h = AA^* h + ch\chi_{\{|p|>\alpha\}}$$

Semismooth Newton method

Active sets

$$\mathcal{A}_k^+ := \{x : p^k(x) > \alpha\}$$
$$\mathcal{A}_k^- := \{x : p^k(x) < -\alpha\}$$
$$\mathcal{A}_k := \mathcal{A}_k^+ \cup \mathcal{A}_k^-$$

Newton step

Given p^k, solve for p^{k+1}

$$AA^* p^{k+1} + c\chi_{\mathcal{A}_k} p^{k+1} = -Az + c(\chi_{\mathcal{A}_k^+} - \chi_{\mathcal{A}_k^-})$$

Convergence of semismooth Newton method

Theorem

$\|p^0 - p_c\|_{\mathcal{W}}$ *is sufficiently small* $\Rightarrow p^k \xrightarrow[k\to\infty]{} p_c$ *superlinearly in* $\mathcal{W}$

Idea of proof

$(D_N F)^{-1}$ uniformly bounded, since $\|A^* \cdot\|_{L^2}$ norm on $\mathcal{W}$ (Riesz)

Theorem (Termination criterion)

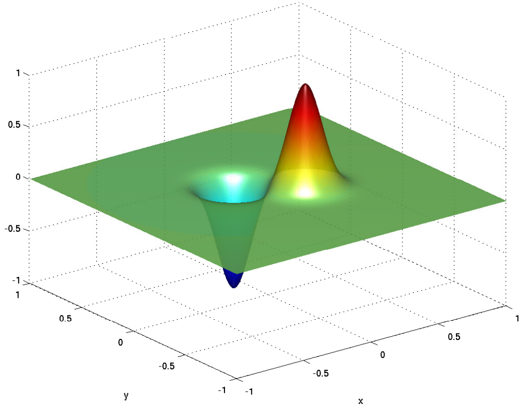
$\mathcal{A}^+_{k+1} = \mathcal{A}^+_k$ *and* $\mathcal{A}^-_{k+1} = \mathcal{A}^-_k \quad \Longrightarrow \quad p^{k+1}$ *satisfies* $F(p^{k+1}) = 0$

Numerical examples

- $\Omega = [-1, 1]^2 \subset \mathbb{R}^2$
- $Ay = -\Delta y$, homogeneous Dirichlet conditions
- Comparison with optimal control in L^2:

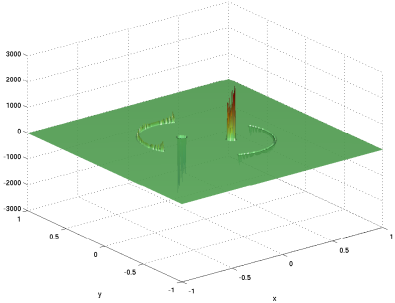
$$(\mathcal{P}_{L^2}) \qquad \begin{cases} \min\limits_{u \in L^2} \frac{1}{2} \|y - z\|_{L^2}^2 + \frac{\alpha}{2} \|u\|_{L^2}^2 \\ \text{s.t.} \quad Ay = u. \end{cases}$$

Target state

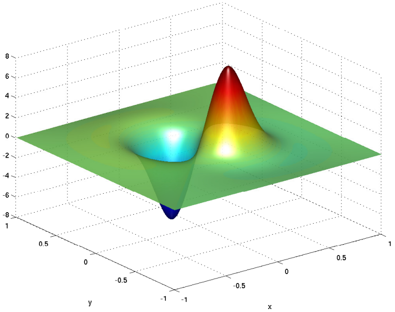


$$z(x, y) = e^{-50[(x-0.2)^2+(y+0.1)^2]} - e^{-50[(y-0.2)^2+(x+0.1)^2]}$$

Optimal control: $\mathcal{M}$ vs. L^2 ($\alpha = 10^{-3}$)

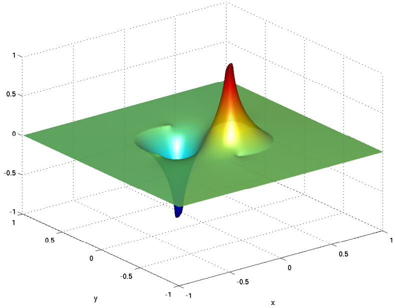


$\mathcal{M}$

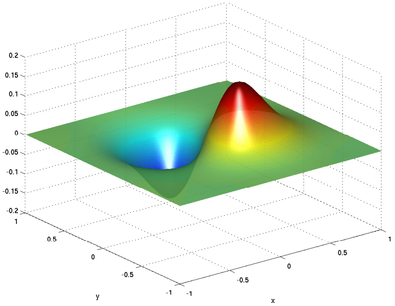


L^2

Optimal state: $\mathcal{M}$ vs. L^2 ($\alpha = 10^{-3}$)

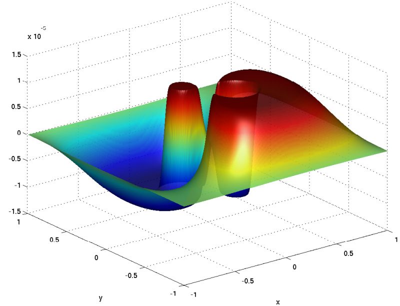


$\mathcal{M}$

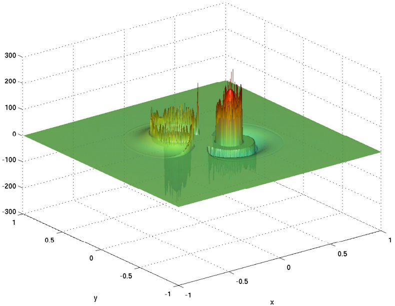


L^2

Sparsity structure ($\alpha = 10^{-5}$)

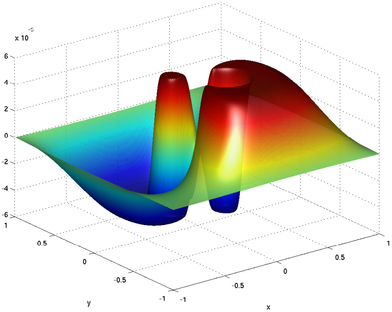


dual solution p^*

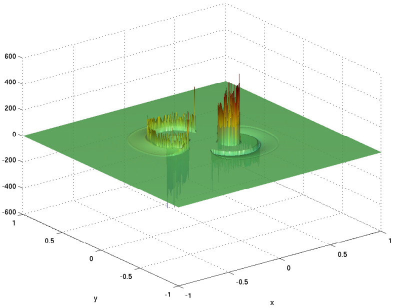


optimal control u^*

Sparsity structure ($\alpha = 5 \cdot 10^{-5}$)

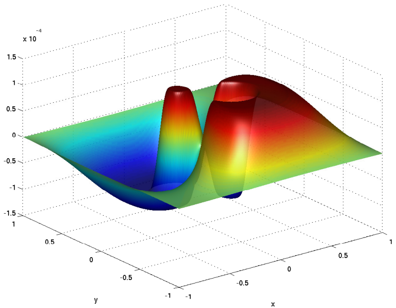


dual solution p^*

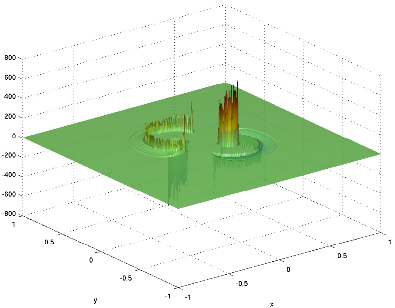


optimal control u^*

Sparsity structure ($\alpha = 10^{-4}$)

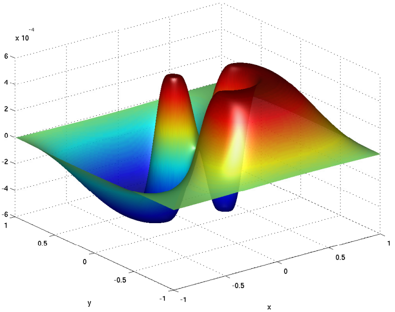


dual solution p^*

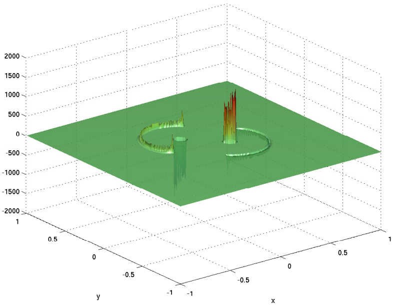


optimal control u^*

Sparsity structure ($\alpha = 5 \cdot 10^{-4}$)

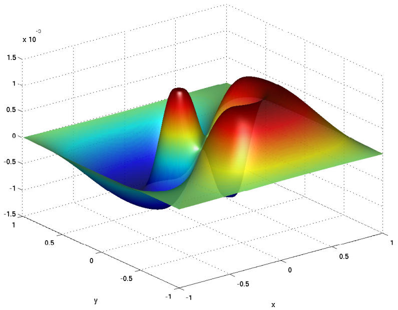


dual solution p^*

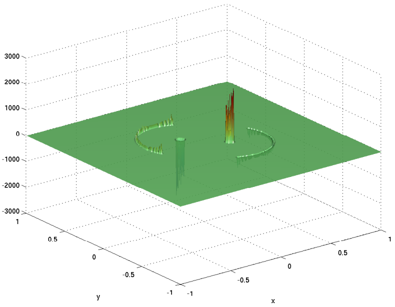


optimal control u^*

Sparsity structure ($\alpha = 10^{-3}$)

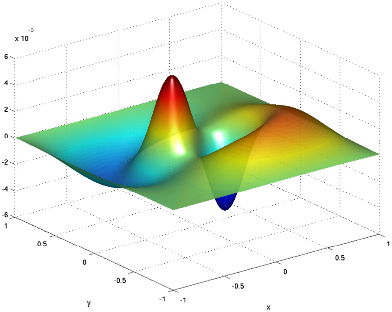


dual solution p^*

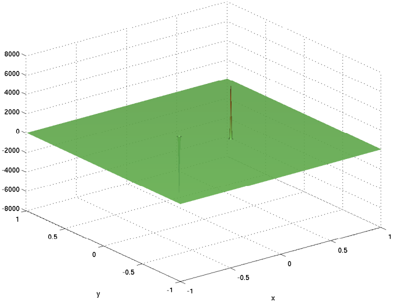


optimal control u^*

Sparsity structure ($\alpha = 5 \cdot 10^{-3}$)

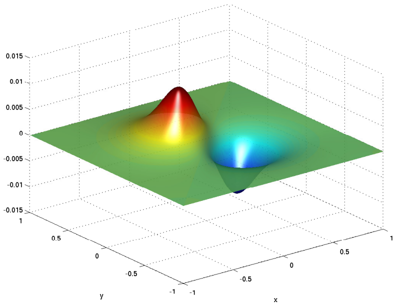


dual solution p^*

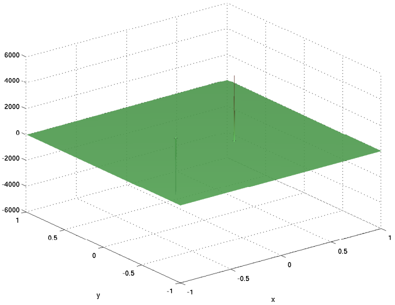


optimal control u^*

Sparsity structure ($\alpha = 10^{-2}$)



dual solution p^*



optimal control u^*

Application to parabolic equations

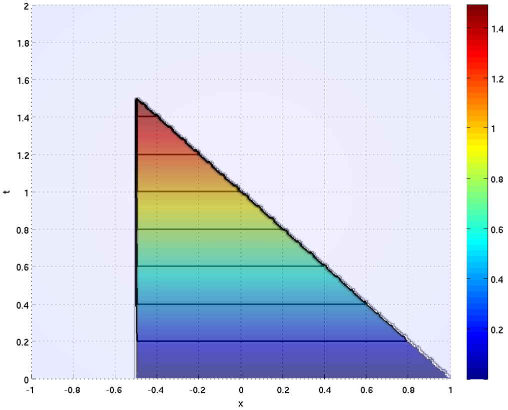
Here:

- Heat equation

$$y_t - \Delta y = u, \qquad y(0) = 0$$

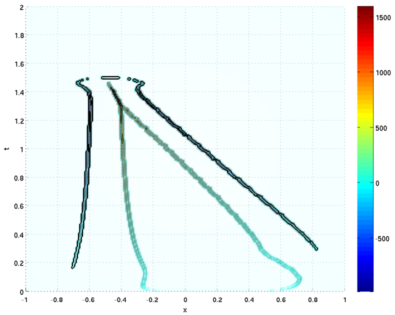
- Control $u \in \mathcal{M}([0, T] \times \Omega)$,
- Set $Ay = y_t - \Delta y \Rightarrow$ formally, approach applicable
- $\Omega = (-1, 1) \subset \mathbb{R}$, $T = 2$, full space-time discretization
- Compare with optimal control in $L^2([0, T] \times \Omega)$

Target state

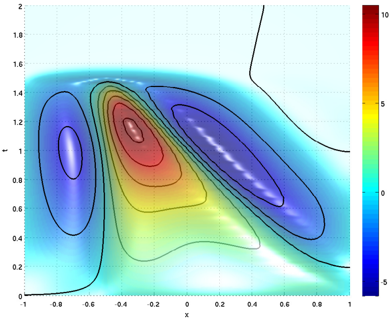


$$z(x,t) = t\chi_{\{-0.5<x<1-t\}}$$

Optimal control: $\mathcal{M}$ vs. L^2 ($\alpha = 10^{-3}$)

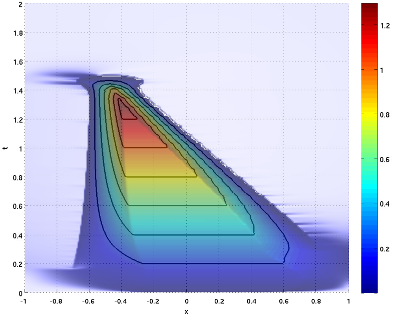


$\mathcal{M}$

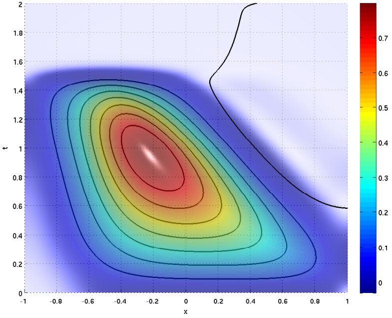


L^2

Optimal state: $\mathcal{M}$ vs. L^2 ($\alpha = 10^{-3}$)

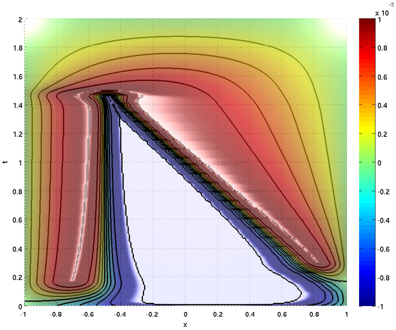


$\mathcal{M}$

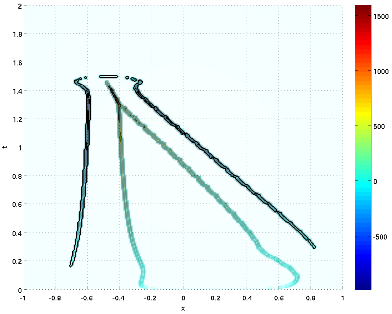


L^2

Sparsity structure ($\alpha = 10^{-3}$)



dual solution p^*



optimal control u^*

Extensions

- Elliptic control, $u \in \mathrm{BV}(\Omega)$
  (done)
- Parabolic control, $u \in \mathcal{M}(0, T)$ vs. $u \in \mathcal{M}(\Omega)$
  (in progress)
- Nonlinear control
  (in progress)