Convex regularization of discrete-valued inverse problems

Christian Clason

Faculty of Mathematics, Universität Duisburg-Essen

joint work with Thi Bich Tram Do, Florian Kruse, Karl Kunisch

Workshop on Computational Imaging and Tomography
Innsbruck, August 18, 2017

Motivation: discrete optimization

$$\min_{u \in U} \mathcal{F}(u) + \frac{\alpha}{2}\|u\|^2$$

- $\mathcal{F}$ discrepancy term (involving PDEs)
- U discrete set,

$$U = \left\{u \in L^p(\Omega) : u(x) \in \{u_1, \dots, u_d\} \text{ a.e.}\right\}$$

- $u_1, \dots, u_d$ given voltages, velocities, materials, …
  (assumed here: ranking by magnitude possible!)
- motivation: topology optimization, medical imaging

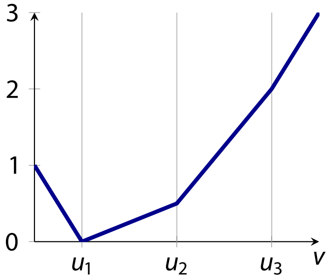
Motivation: penalty

- convex relaxation: replace U by convex hull
- works only for $d = 2$, cf. bang-bang control ($\alpha = 0$)
- $\rightsquigarrow$ promote $u(x) \in \{u_1, \ldots, u_d\}$ by convex pointwise penalty

$$\mathcal{G}(u) = \int_\Omega g(u(x))\, dx$$

- generalize L^1 norm: polyhedral epigraph with vertices $u_1, \ldots, u_d$
- not exact relaxation/penalization (in general)!

- generalize L^1 norm: polyhedral epigraph with vertices $u_1, \dots, u_d$



- motivation: convex envelope of $\frac{1}{2}\|u\|^2 + \delta_U$
- multi-bang (generalized bang-bang) control
- $\rightsquigarrow$ non-smooth optimization in function spaces

Fenchel duality

$\mathcal{F} : V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ convex, V Banach space, V^* dual space

- subdifferential

$$\partial\mathcal{F}(\bar{v}) = \left\{ v^* \in V^* : \langle v^*, v - \bar{v}\rangle_{V^*,V} \leqslant \mathcal{F}(v) - \mathcal{F}(\bar{v}) \quad \text{for all } v \in V \right\}$$

- Fenchel conjugate (always convex)

$$\mathcal{F}^* : V^* \to \overline{\mathbb{R}}, \qquad \mathcal{F}^*(v^*) = \sup_{v\in V} \langle v^*, v\rangle_{V^*,V} - \mathcal{F}(v)$$

- "convex inverse function theorem":

$$v^* \in \partial\mathcal{F}(v) \quad \Leftrightarrow \quad v \in \partial\mathcal{F}^*(v^*)$$

Fenchel duality: application

$$\mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u}) = \min_u \mathcal{F}(u) + \mathcal{G}(u)$$

1. Fermat principle: $0 \in \partial\left(\mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u})\right)$
2. sum rule: $0 \in \partial\mathcal{F}(\bar{u}) + \partial\mathcal{G}(\bar{u})$, i.e., there is $\bar{p} \in V^*$ with

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{p} \in \partial\mathcal{G}(\bar{u}) \end{cases}$$

3. Fenchel duality:

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{u} \in \partial\mathcal{G}^*(\bar{p}) \end{cases}$$

Regularization

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$\mathcal{G}$ non-smooth $\rightsquigarrow$ subdifferential $\partial\mathcal{G}^*$ set-valued $\rightsquigarrow$ regularize

$u, p \in L^2(\Omega)$ Hilbert space $\rightsquigarrow$ consider for $\gamma > 0$

Proximal mapping

$$\mathrm{prox}_{\gamma\mathcal{G}^*}(p) = \arg\min_w \mathcal{G}^*(w) + \frac{1}{2\gamma}\|w - p\|^2$$

- single-valued, Lipschitz continuous
- coincides with resolvent $(\mathrm{Id} + \gamma\partial\mathcal{G}^*)^{-1}(p)$
- (also required for primal-dual first-order methods)

Regularization

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Proximal mapping

$$\mathrm{prox}_{\gamma \mathcal{G}^*}(p) = \arg\min_{w} \mathcal{G}^*(w) + \frac{1}{2\gamma}\|w - p\|^2$$

Complementarity formulation of $u \in \partial \mathcal{G}^*(p)$

$$u = \frac{1}{\gamma}\left((p + \gamma u) - \mathrm{prox}_{\gamma \mathcal{G}^*}(p + \gamma u)\right)$$

- equivalent for every $\gamma > 0$
- single-valued, Lipschitz continuous, implicit

Regularization

Proximal mapping

$$\operatorname{prox}_{\gamma\mathcal{G}^*}(p) = \arg\min_w \mathcal{G}^*(w) + \frac{1}{2\gamma}\|w-p\|^2$$

Moreau–Yosida regularization of $u \in \partial\mathcal{G}^*(p)$

$$u = \frac{1}{\gamma}\left(p - \operatorname{prox}_{\gamma\mathcal{G}^*}(p)\right) =: \partial\mathcal{G}^*_\gamma(p)$$

- $\partial\mathcal{G}^*_\gamma = \partial\left(\mathcal{G} + \frac{\gamma}{2}\|\cdot\|^2\right)^* \to \partial\mathcal{G}^*$ as $\gamma \to 0$
- single-valued, Lipschitz continuous, explicit
  $\rightsquigarrow$ nonsmooth operator equation, Newton method

Semismooth Newton method

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f locally Lipschitz, piecewise C^1:

$$f(v) = 0, \qquad f : \mathbb{R}^n \to \mathbb{R}$$

Newton derivative

$$D_N f(v)\delta v \in \partial_C f(v)\delta v$$

Clarke generalized gradient: convex hull of piecewise derivatives

semismooth Newton method

$$D_N f(v^k)\delta v = -f(v^k), \qquad v^{k+1} = v^k + \delta v$$

converges locally superlinearly

Semismooth Newton method

f locally Lipschitz, piecewise C^1:

$$F(u) = 0, \qquad F : L^r(\Omega) \to L^s(\Omega), \quad [F(u)](x) = f(u(x))$$

Newton derivative

$$[D_N F(u)\delta u](x) \in \partial_C f(\delta u(x))\delta u(x)$$

any measurable selection of Clarke generalized gradient

semismooth Newton method

$$D_N F(u^k)\delta u = -F(u^k), \qquad u^{k+1} = u^k + \delta u$$

converges locally superlinearly if $r > s$

Numerical solution: summary

For (non)convex $\mathcal{G} : L^2(\Omega) \to \mathbb{R}$, $\quad \mathcal{G}(u) = \int_\Omega g(u(x))\,dx$,

Approach: pointwise

1. compute subdifferential ∂g (or Fenchel conjugate g^*)
2. compute subdifferential ∂g^*
3. compute proximal mapping $\mathrm{prox}_{\gamma g^*}$
4. compute Moreau–Yosida regularization ∂g^*_γ
5. compute Newton derivative $D_N \partial g^*_\gamma$

⇝ semismooth Newton method, continuation in γ for

superposition operator $\quad [\partial \mathcal{G}^*_\gamma(p)](x) = \partial g^*_\gamma(p(x))$

Multi-bang penalty

$$g : \mathbb{R} \to \overline{\mathbb{R}}, \qquad v \mapsto \begin{cases} \frac{1}{2}\left((u_i + u_{i+1})v - u_i u_{i+1}\right) & v \in [u_i, u_{i+1}] \\ \infty & \text{else} \end{cases}$$

piecewise differentiable $\rightsquigarrow$ subdifferential convex hull of derivatives

$$\partial g(v) = \begin{cases} \left(-\infty, \frac{1}{2}(u_1 + u_2)\right] & v = u_1 \\ \left\{\frac{1}{2}(u_i + u_{i+1})\right\} & v \in (u_i, u_{i+1}) \quad 1 \leqslant i < d \\ \left[\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})\right] & v = u_i \qquad\quad 1 < i < d \\ \left[\frac{1}{2}(u_{d-1} + u_d), \infty\right) & v = u_d \end{cases}$$

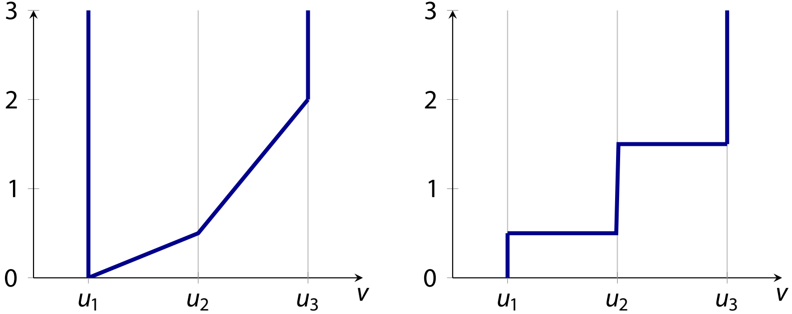
Multi-bang penalty

$$\partial g(v) = \begin{cases} \left(-\infty, \frac{1}{2}(u_1 + u_2)\right] & v = u_1 \\ \left\{\frac{1}{2}(u_i + u_{i+1})\right\} & v \in (u_i, u_{i+1}) \quad 1 \leqslant i < d \\ \left[\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})\right] & v = u_i \qquad\qquad 1 < i < d \\ \left[\frac{1}{2}(u_{d-1} + u_d), \infty\right) & v = u_d \end{cases}$$

convex inverse function theorem:

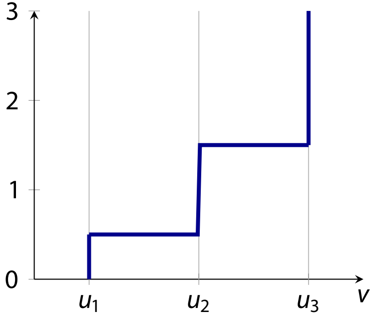
$$\partial g^*(q) \in \begin{cases} \{u_1\} & q \in \left(-\infty, \frac{1}{2}(u_1 + u_2)\right) \\ [u_i, u_{i+1}] & q = \frac{1}{2}(u_i + u_{i+1}), \qquad 1 \leqslant i < d \\ \{u_i\} & q \in \left(\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})\right) \quad 1 < i < d, \\ \{u_d\} & q \in \left(\frac{1}{2}(u_{d-1} + u_d), \infty\right) \end{cases}$$

Multi-bang penalty: sketch

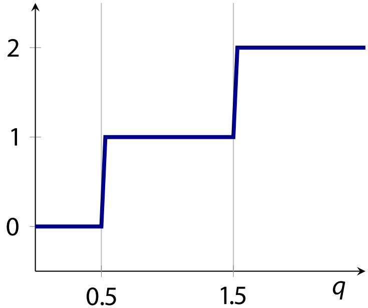


(a) g $(u_1 = 0, u_2 = 1, u_3 = 2)$ (b) ∂g $(u_1 = 0, u_2 = 1, u_3 = 2)$

Multi-bang penalty: sketch



(c) ∂g ($u_1 = 0, u_2 = 1, u_3 = 2$)



(d) ∂g^* ($u_1 = 0, u_2 = 1, u_3 = 2$)

Moreau–Yosida regularization

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Proximal mapping $\quad \mathrm{prox}_{\gamma g^*}(q) = w \text{ iff } q \in \{w\} + \gamma \partial g^*(w)$

case-wise inspection of subdifferential:

$$\partial g^*_\gamma(q) = \frac{1}{\gamma}\left(q - \mathrm{prox}_{\gamma g^*}(q)\right) = \begin{cases} u_i & q \in Q_i^\gamma \\ \frac{1}{\gamma}\left(q - \frac{1}{2}(u_i + u_{i+1})\right) & q \in Q_{i,i+1}^\gamma \end{cases}$$

$$Q_i^\gamma = \left(\tfrac{1}{2}(u_{i-1} + u_i) + \gamma u_i, \tfrac{1}{2}(u_i + u_{i+1}) + \gamma u_i\right)$$
$$Q_{i,i+1}^\gamma = \left[\tfrac{1}{2}(u_i + u_{i+1}) + \gamma u_i, \tfrac{1}{2}(u_i + u_{i+1}) + \gamma u_{i+1}\right]$$

Multi-bang regularization

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$$\min_{u\in L^2(\Omega)} \frac{1}{2}\|Ku - y^\delta\|_Y^2 + \alpha\, \mathcal{G}(u)$$

- $K : L^2(\Omega) \to Y$ (linear) forward mapping, weakly closed
- $y^\delta \in L^2(\Omega)$ noisy data with $\|y - y^\delta\|_Y \leqslant \delta$
- $u_1 < \cdots < u_d$ given parameter values $(d > 2)$
- $\mathcal{G}$ multi-bang penalty

Multi-bang regularization

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|Ku - y^\delta\|_Y^2 + \alpha\, \mathcal{G}(u)$$

- $\mathcal{G}$ multi-bang penalty convex:
  1. existence of solution u_α^δ for every $\alpha > 0$
  2. $\delta \to 0$ implies $u_\alpha^\delta \rightharpoonup u_\alpha$ for every $\alpha > 0$
  3. $\delta \to 0$, $\alpha \to 0$, $\delta\alpha^{-2} \to 0$ implies $u_\alpha^\delta \rightharpoonup u^\dagger$

(standard arguments, e.g. [Burger/Osher 04, Ito/Jin 14])

Multi-bang regularization

$$\min_{u\in L^2(\Omega)} \frac{1}{2}\|Ku - y^\delta\|_Y^2 + \alpha\,\mathcal{G}(u)$$

- standard source condition: $p^\dagger := K^* w \in \partial\mathcal{G}(u^\dagger)$ for $w \in Y$,
  1. a priori choice $\alpha(\delta) = c\delta$
  2. a posteriori choice $\|Ku^\delta_{\alpha(\delta)} - y^\delta\|_Y \leqslant \tau\delta$

$\rightsquigarrow$ convergence rate

$$d_{\mathcal{G}}^{p^\dagger}(u_\alpha^\delta, u^\dagger) \leqslant C\delta$$

in Bregman distance

$$d_{\mathcal{G}}^{p_1}(u_2, u_1) = \mathcal{G}(u_2) - \mathcal{G}(u_1) - \langle p_1, u_2 - u_1\rangle_X, \qquad p_1 \in \partial\mathcal{G}(u_1)$$

Multi-bang regularization

Pointwise definition of Bregman distance, ∂g:

- $u^\dagger(x) = u_i$ and $p^\dagger \notin \left\{\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i, u_{i+1})\right\}$ implies

$$d_g^{p^\dagger(x)}(u_{\alpha(\delta)}^\delta(x), u^\dagger(x)) \to 0 \qquad \text{for } \delta \to 0$$

- $u^\dagger(x) \in (u_i, u_{i+1})$ implies

$$d_g^{p^\dagger(x)}(u(x), u^\dagger(x)) = 0 \qquad \text{for any } u(x) \in [u_i, u_{i+1}]$$

- $\rightsquigarrow$ $u_{\alpha(\delta)}^\delta \to u^\dagger$ pointwise a.e. iff $u^\dagger(x) \in \{u_1, \dots, u_d\}$ a.e.
- (convergence not uniform $\rightsquigarrow$ no pointwise rates)

Optimality system

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$$\bar{p} = \frac{1}{\alpha} K^*(y^\delta - K\bar{u})$$

$$\bar{u} \in \partial \mathcal{G}^*(\bar{p}) = \begin{cases} \{u_i\} & \bar{p}(x) \in Q_i \\ [u_i, u_{i+1}] & \bar{p}(x) \in \overline{Q}_i \cap \overline{Q}_{i+1} \end{cases}$$

- $\rightsquigarrow$ unique solution $(\bar{u}, \bar{p}) \in L^2(\Omega) \times L^2(\Omega)$
- singular arc $\mathcal{S} = \{x : \bar{u}(x) \notin \{u_i\}\} \subset \{x : \bar{p}(x) = \frac{1}{2}(u_i + u_{i+1})\}$
- for suitable K, $\bar{p}(x)$ constant implies $[y^\delta - K\bar{u}](x) = 0$ (e.g., $K = A^{-1}$ for A pure second-order elliptic)

$\rightsquigarrow$ $|\{x : K\bar{u}(x) = y^\delta(x)\}| = 0 \;\Rightarrow\; \bar{u} \in \{u_1, \dots u_d\}$ a. e. (true multi-bang)

Regularized optimality system

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$$\begin{cases} p_\gamma = \frac{1}{\alpha} K^*(y^\delta - K u_\gamma) \\ u_\gamma = \partial \mathcal{G}_\gamma^*(p_\gamma) \end{cases}$$

- optimality conditions for $\mathcal{F}(u) + \alpha\, \mathcal{G}(u) + \frac{\gamma}{2} \|u\|^2$
- $\rightsquigarrow$ unique solution (u_γ, p_γ)
- $(u_\gamma, p_\gamma) \rightharpoonup (\bar{u}, \bar{p})$ as $\gamma \to 0$
- ∂g_γ^* Lipschitz continuous, piecewise C^1, norm gap $V \hookrightarrow L^2(\Omega)$
- $\rightsquigarrow$ semismooth Newton method

Regularized optimality system

$$\begin{cases} p_\gamma = \frac{1}{\alpha} K^*(y^\delta - Ku_\gamma) \\ u_\gamma = \partial \mathcal{G}_\gamma^*(p_\gamma) \end{cases}$$

- $\rightsquigarrow$ semismooth Newton method
- inverse source problem: $K = A^{-1}$, A elliptic differential operator
- introduce $y_\gamma = Ku_\gamma$, eliminate $u_\gamma = \mathcal{G}_\gamma^*(p_\gamma)$

$$\begin{cases} A^* p_\gamma = \frac{1}{\alpha}(y^\delta - y_\gamma) \\ A y_\gamma = \mathcal{G}_\gamma^*(p_\gamma) \end{cases}$$

Semismooth Newton method

$$\begin{pmatrix} \frac{1}{\alpha}\,\mathrm{Id} & A^* \\ A & -D_N\mathcal{G}_\gamma^*(p) \end{pmatrix} \begin{pmatrix} \delta y \\ \delta p \end{pmatrix} = -\begin{pmatrix} A^*p + \frac{1}{\alpha}(y - y^\delta) \\ Ay - \mathcal{G}_\gamma^*(p) \end{pmatrix}$$

$$[D_N\mathcal{G}_\gamma^*(p)\delta p](x) = \begin{cases} \frac{1}{\gamma}\delta p(x) & p(x) \in Q_{i,i+1}^\gamma \\ 0 & \text{else} \end{cases}$$

- symmetric, but: local convergence
- $\rightsquigarrow$ continuation in $\gamma \to 0$
- $\rightsquigarrow$ backtracking line search based on residual norm
- only number of sets Q_i^γ depends on d $\rightsquigarrow$ linear complexity

Example: linear inverse problem

- $\Omega = [0,1]^2, \quad A = -\Delta$

- $$u^\dagger(x) = u_1 + u_2\,\chi_{\{x:(x_1-0.45)^2+(x_2-0.55)^2<0.1\}}(x) + (u_3 - u_2)\,\chi_{\{x:(x_1-0.4)^2+(x_2-0.6)^2<0.02\}}(x)$$

- $d = 3, \quad u_1 = 0, \quad u_2 = 0.1, \quad u_3 \in \{0.15, 0.12\}$

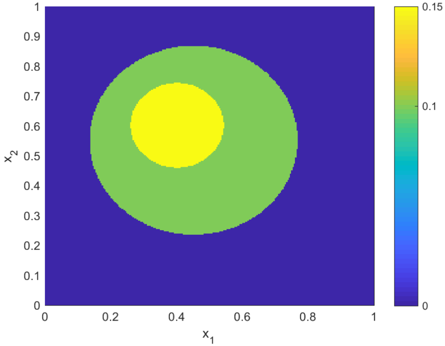
- $y^\delta = y^\dagger + \xi, \quad \xi \in \mathcal{N}\left(y^\dagger, \delta\|y^\dagger\|_\infty\right)$

- finite element discretization: uniform grid, 256×256 nodes

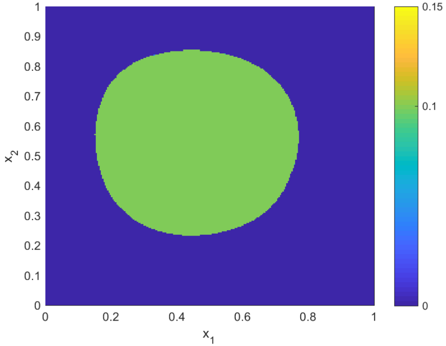
- $\alpha = \alpha(\delta)$ by Morozov discrepancy principle

- terminate at $\gamma < 10^{-12}$

Numerical example: $u_3 = 0.15$

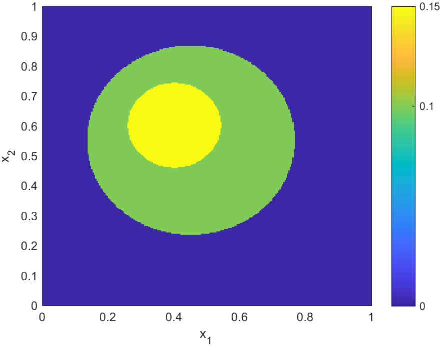


(a) $u^\dagger$

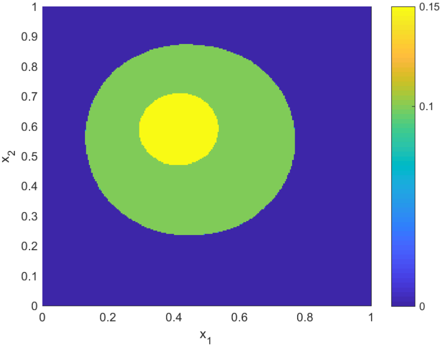


(b) u_α^δ, $\delta \approx 1.89 \cdot 10^{-1}$

Numerical example: $u_3 = 0.15$

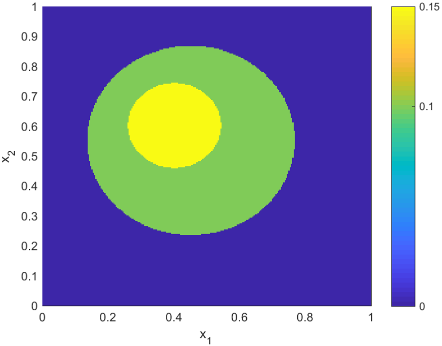


(c) $u^\dagger$

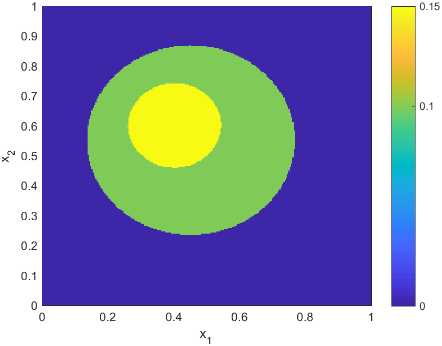


(d) $u_\alpha^\delta, \delta \approx 2.37 \cdot 10^{-2}$

Numerical example: $u_3 = 0.15$

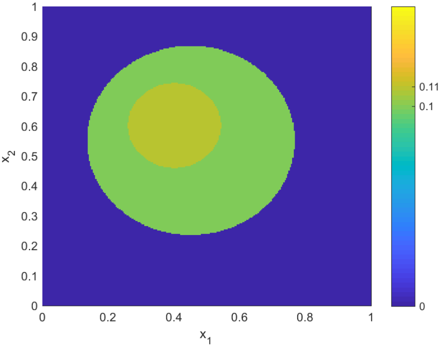


(e) $u^\dagger$

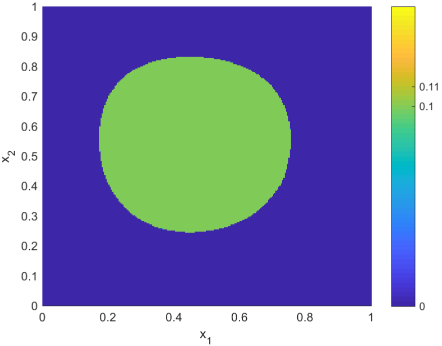


(f) u_α^δ, $\delta \approx 3.69 \cdot 10^{-4}$

Numerical example: $u_3 = 0.11$

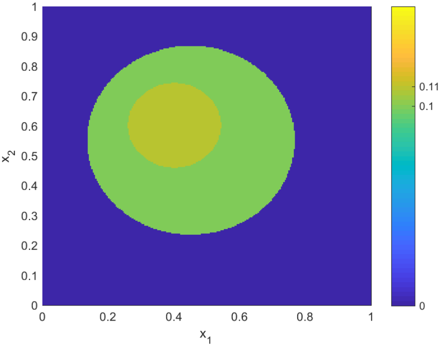


(a) $u^\dagger$

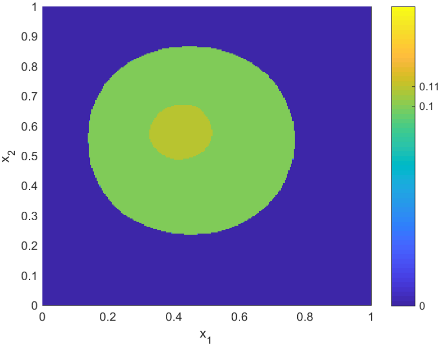


(b) u_α^δ, $\delta \approx 1.68 \cdot 10^{-1}$

Numerical example: $u_3 = 0.11$

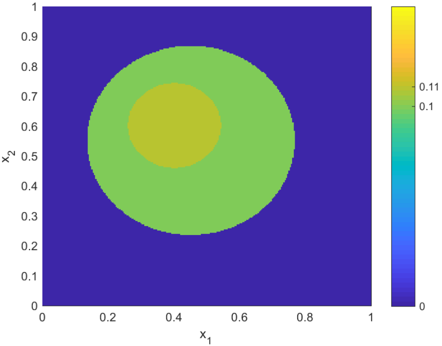


(c) $u^{\dagger}$

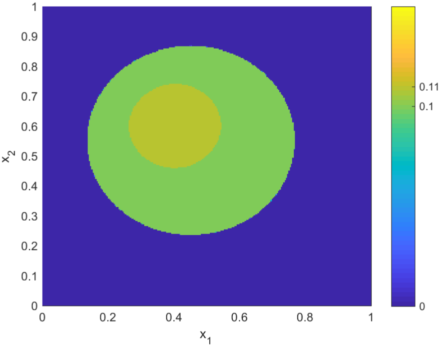


(d) $u_{\alpha}^{\delta}, \delta \approx 2.17 \cdot 10^{-2}$

Numerical example: $u_3 = 0.11$

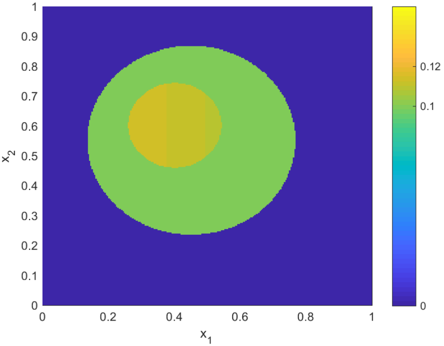


(e) $u^{\dagger}$

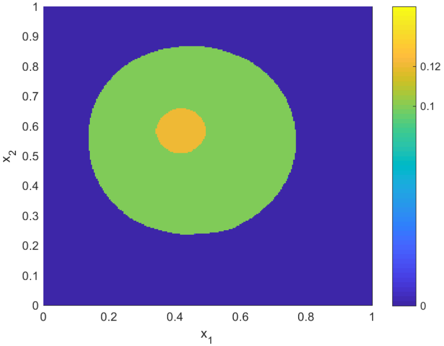


(f) u_{α}^{δ}, $\delta \approx 3.29 \cdot 10^{-4}$

Numerical example: $u_3(x) = 0.12(1 - x_1)$

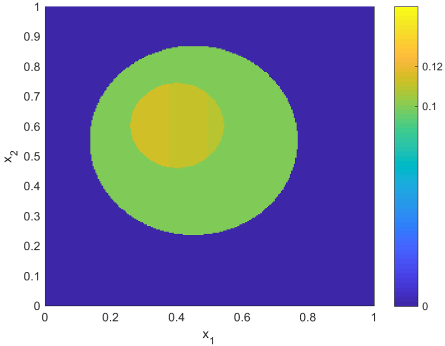


(a) $u^\dagger$

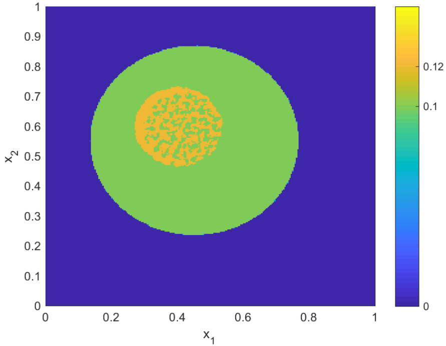


(b) u_α^δ, $\delta \approx 2.11 \cdot 10^{-2}$

Numerical example: $u_3(x) = 0.12(1 - x_1)$

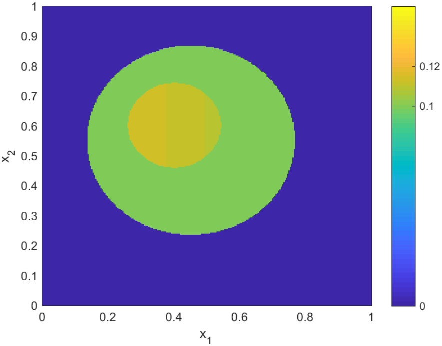


(c) $u^\dagger$

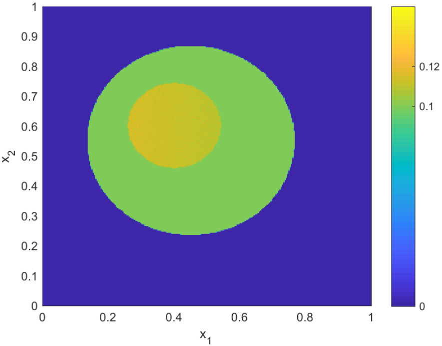


(d) u_α^δ, $\delta \approx 3.29 \cdot 10^{-4}$

Numerical example: $u_3(x) = 0.12(1 - x_1)$



(e) $u^\dagger$



(f) u_α^δ, $\delta \approx 1.29 \cdot 10^{-6}$

Nonlinear forward mapping

Forward mapping $S : u \mapsto y$ nonlinear:

- approach applicable if S
  1. weak-to-weak continuous
  2. twice Fréchet-differentiable
- example: $u \mapsto y$ solving $\quad -\Delta y + uy = f$
- existence, optimality conditions

$$\begin{cases} -\bar{p} = S'(\bar{u})^*(S(\bar{u}) - y^\delta) \\ \bar{u} \in \partial \mathcal{G}^*(\bar{p}) \end{cases}$$

- semismooth Newton method (regularity condition technical)

Example: nonlinear inverse problem

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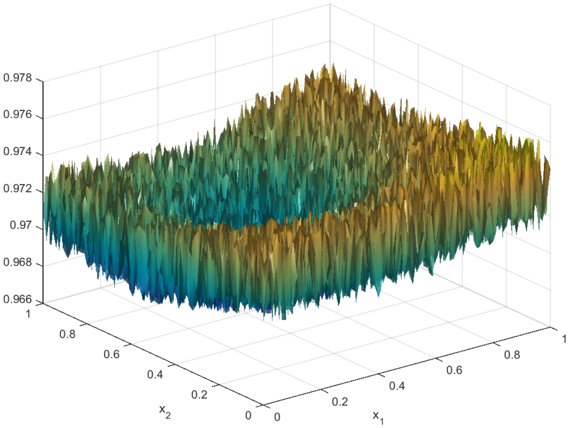
- $S : u \mapsto y$ solving

$$-\Delta y + u y = f$$

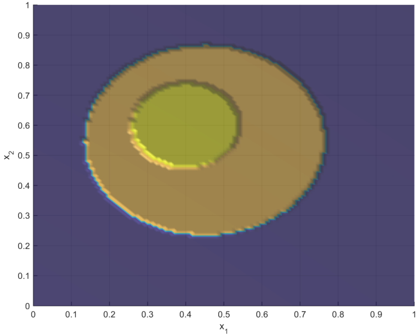
- approach applicable, but $\mathcal{F}$ nonconvex
- numerical example: $\Omega = [0,1]^2, \quad f \equiv 1$
- $u^\dagger(x) = u_1 + (u_2 - u_1)\,\chi_{\{x:(x_1-0.45)^2+(x_2-0.55)^2<0.1\}}(x) + (u_3 - u_2 - u_1)\,\chi_{\{x:(x_1-0.4)^2+(x_2-0.6)^2<0.02\}}(x)$
- $y^\delta = S(u^\dagger) + \xi$
- $\alpha = 3 \cdot 10^{-5}, \quad \gamma \to 10^{-12}$

Numerical example: nonlinear problem

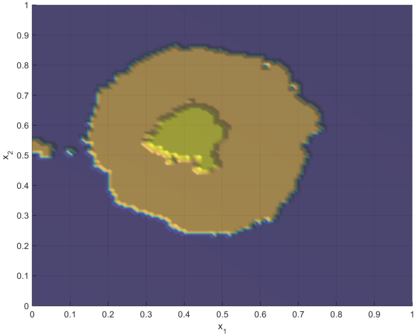


(a) noisy data y^δ

Numerical example: nonlinear problem



(b) $u^\dagger$



(c) u_α^δ

Nonlinear forward mapping

Goal: application to EIT

- $S : u \mapsto y$ solving
$$-\nabla \cdot (u\nabla y) = f$$
- difficulty: $\bar{u} \in L^\infty(\Omega) \quad \rightsquigarrow \quad S$ not weakly-$*$ closed
  1. lack of existence of minimizer ($\bar{y} \neq S(\bar{u})$, cf. homogenization)
  2. lack of convergence $\gamma \to 0$
  3. lack of Newton differentiability of H_γ (no norm gap)
- remedies: higher regularity of y or u by
  1. local smoothing: consider $\quad -\nabla \cdot \left(\int_{B_\varepsilon(x)} u(s)\, ds \nabla y\right)$
  2. TV regularization: add $\|Du\|_{\mathcal{M}} \quad \rightsquigarrow u \in BV(\Omega) \cap L^\infty(\Omega) \hookrightarrow_c L^p(\Omega)$

TV regularization

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Difficulty:

- existence requires box constraints $\rightsquigarrow$ use penalty

$$G(u) + TV(u) + \delta_{[u_1,u_d]}(u)$$

- but: $TV(u) + \delta_{[u_1,u_d]}(u)$ not continuous on $L^p(\Omega)$, $p < \infty$
- but: multipliers $\xi \in \partial TV(u)$, $q \in \partial G(u)$ not pointwise on BV, L^∞

- $\rightsquigarrow$ replace box constraints by ($C^{1,1}$) projection of $u \in L^1(\Omega)$

$$[\Phi(u)](x) = \operatorname{proj}^{\varepsilon}_{[u_1,u_d]}(u(x)) \qquad \text{a.e. } x \in \Omega$$

TV regularization: existence

$$\begin{cases} \min_{u \in BV(\Omega)} \frac{1}{2}\|y - z\|^2_{L^2(\Omega)} + \alpha\, G(u) + \beta\, TV(u) \\ \text{s.t.} \quad -\nabla \cdot (\Phi(u)\nabla y) = f \text{ in } \Omega \\ \qquad\qquad\qquad y = 0 \text{ on } \partial\Omega \end{cases}$$

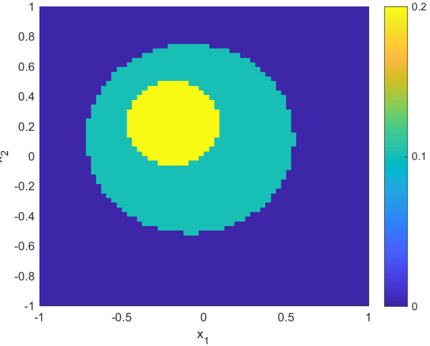
- existence of optimal $\bar{u} \in BV(\Omega) \cap L^\infty(\Omega)$ for $\varepsilon \geqslant 0$
- tracking term Fréchet differentiable in $\Phi(u) \in L^\infty$ for $\varepsilon > 0$
- regularity of state, adjoint $\rightsquigarrow$ derivative in $L^r(\Omega)$, $r > 1$ (instead of $L^\infty(\Omega)^*$)
- $\rightsquigarrow$ sum rule applicable, subgradients in $L^r(\Omega)$, $r > 1$

TV regularization: optimality conditions

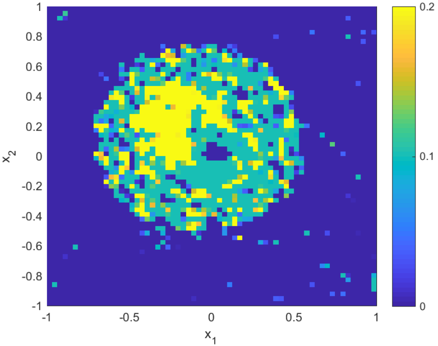
$$\begin{cases} 0 = F'(\Phi(\bar{u}))\Phi'(\bar{u}) + \alpha\bar{q} + \beta\bar{\xi} \\ \bar{u} \in \partial G^*(\bar{q}) \\ \bar{\xi} \in \partial TV(\bar{u}) \end{cases}$$

- $F'(\Phi(\bar{u})) = (\nabla\bar{y} \cdot \nabla\bar{p}) \in L^r(\Omega)$ (optimal state, adjoint)
- $\bar{q} \in L^r(\Omega)$, $r > 1 \rightsquigarrow$ pointwise multi-bang
- $\bar{\xi} \in L^r(\Omega)$, $r > 1 \rightsquigarrow$ characterization via *full trace* [Bredies/Holler '12]
- $\rightsquigarrow$ pointwise optimality conditions
- semi-smooth Newton (after discretization, regularization)

Numerical example: total variation

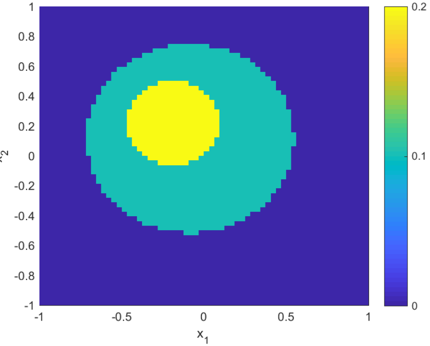


(a) $u^\dagger$

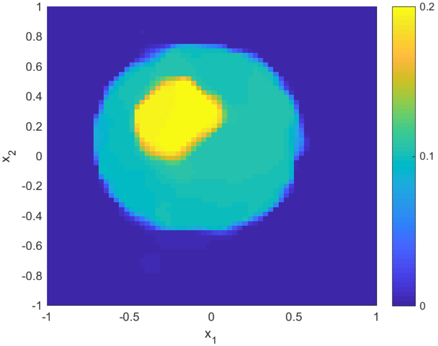


(b) $\alpha = 5 \cdot 10^{-4}, \beta = 0$

Numerical example: total variation



(c) $u^\dagger$



(d) $\alpha = 5 \cdot 10^{-4}, \beta = 10^{-5}$

Conclusion

Convex relaxation of discrete regularization:

- well-posed regularization method
- pointwise convergence under general assumptions
- strong structural regularization
- efficient numerical solution (superlinear convergence)

Outlook:

- regularization properties, parameter choice
- nonlinear inverse problems: EIT
- combination with TV regularization
- other hybrid discrete–continuous problems

Preprint, MATLAB/Python codes:

`http://www.uni-due.de/mathematik/agclason/clason_pub.php`