Multibang regularization of a coefficient inverse problem for the wave equation

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joint work with Thi Bich Tram Do, Florian Kruse, Karl Kunisch

Mathematical and Numerical Approaches for Multi-Wave Inverse Problems
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Motivation: hybrid discrete optimization

$$\min_{u \in U} \mathcal{F}(u) + \frac{\alpha}{2}\|u\|^2$$

- $\mathcal{F}$ discrepancy term (involving PDE)
- U discrete set

$$U = \left\{u \in L^p(\Omega) : u(x) \in \{u_1, \dots, u_d\} \text{ a.e.}\right\}$$

- $u_1, \dots, u_d$ given voltages, velocities, materials, …
  (assumed here: ranking by magnitude possible!)
- motivation: topology optimization, medical imaging

Motivation: penalty

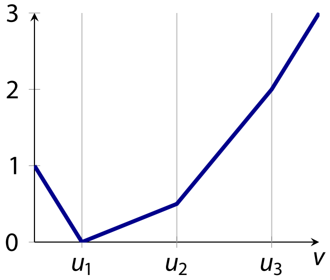
- convex relaxation: replace U by convex hull $u(x) \in [u_1, u_d]$
- works only for $d = 2$, cf. bang-bang control ($\alpha = 0$)
- $\rightsquigarrow$ promote $u(x) \in \{u_1, \dots, u_d\}$ by convex pointwise penalty

$$\mathcal{G}(u) = \int_\Omega g(u(x))\,dx$$

- generalize L^1 norm: polyhedral epigraph with vertices $u_1, \dots, u_d$
- not exact relaxation/penalization (in general)!

Motivation: penalty

- generalize L^1 norm: polyhedral epigraph with vertices $u_1, \ldots, u_d$



- motivation: convex envelope of $\frac{1}{2}\|u\|^2 + \delta_U$
- multibang (generalized bang-bang) control
- $\rightsquigarrow$ non-smooth optimization in function spaces

Fenchel duality

$\mathcal{F} : V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ convex, V Banach space, V^* dual space

- subdifferential

$$\partial\mathcal{F}(\bar{v}) = \left\{ v^* \in V^* : \langle v^*, v - \bar{v}\rangle_{V^*,V} \leqslant \mathcal{F}(v) - \mathcal{F}(\bar{v}) \quad \text{for all } v \in V \right\}$$

- Fenchel conjugate (always convex)

$$\mathcal{F}^* : V^* \to \overline{\mathbb{R}}, \qquad \mathcal{F}^*(v^*) = \sup_{v \in V} \langle v^*, v\rangle_{V^*,V} - \mathcal{F}(v)$$

- "convex inverse function theorem":

$$v^* \in \partial\mathcal{F}(v) \quad \Leftrightarrow \quad v \in \partial\mathcal{F}^*(v^*)$$

Fenchel duality: application

$$\mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u}) = \min_u \mathcal{F}(u) + \mathcal{G}(u)$$

1. Fermat principle: $0 \in \partial\left(\mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u})\right)$
2. sum rule: $0 \in \partial\mathcal{F}(\bar{u}) + \partial\mathcal{G}(\bar{u})$, i.e., there is $\bar{p} \in V^*$ with

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{p} \in \partial\mathcal{G}(\bar{u}) \end{cases}$$

3. Fenchel duality:

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{u} \in \partial\mathcal{G}^*(\bar{p}) \end{cases}$$

Proximal mapping

$\mathcal{G}$ non-smooth $\rightsquigarrow$ subdifferentials set-valued $\rightsquigarrow$ regularize

$u, p \in L^2(\Omega)$ Hilbert space $\rightsquigarrow$ consider for $\gamma > 0$

Proximal mapping

$$\mathrm{prox}_{\gamma\mathcal{G}}(v) = \arg\min_w \mathcal{G}(w) + \frac{1}{2\gamma}\|w - v\|^2$$

- single-valued, Lipschitz continuous
- coincides with resolvent $(\mathrm{Id} + \gamma\partial\mathcal{G})^{-1}(v)$
- $\mathcal{G}^*$ integral functional $\rightsquigarrow$ can be computed pointwise

Proximal mapping

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Proximal mapping

$$\mathrm{prox}_{\gamma\mathcal{G}}(v) = \arg\min_{w} \mathcal{G}(w) + \frac{1}{2\gamma}\|w - v\|^2$$

Complementarity formulation of $p \in \partial\mathcal{G}(u)$

$$p = \mathrm{prox}_{\gamma\mathcal{G}}(u + \gamma p)$$

- equivalent for every $\gamma > 0$
- single-valued, Lipschitz continuous, implicit
  $\rightsquigarrow$ proximal splitting method

Primal-dual proximal splitting

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$$
\begin{aligned}
u^{k+1} &= \mathrm{prox}_{\tau\mathcal{G}}\left(u^k - \tau K'(u^k)^* p^k\right)\\
\bar{u}^{k+1} &= 2u^{k+1} - u^k\\
p^{k+1} &= \mathrm{prox}_{\sigma\mathcal{F}^*}\left(p^k + \sigma K(\bar{u}^{k+1})\right)
\end{aligned}
$$

- nonlinear variant of Chambolle–Pock
  [Valkonen '14, C./Mazurenko/Valkonen '18]
- $\tau, \sigma > 0$ step sizes
- local convergence in Hilbert space under
  1. second-order type condition on K
  2. τ, σ sufficiently small
- can be accelerated if $\mathcal{F}$ and/or $\mathcal{G}$ strongly convex

Moreau–Yosida regularization

Alternative: regularize $\mathcal{G}^*$

Proximal mapping

$$\mathrm{prox}_{\gamma\mathcal{G}^*}(p) = \arg\min_w \mathcal{G}^*(w) + \frac{1}{2\gamma}\|w - p\|^2$$

Complementarity formulation of $u \in \partial\mathcal{G}^*(p)$

$$u = \frac{1}{\gamma}\left((p + \gamma u) - \mathrm{prox}_{\gamma\mathcal{G}^*}(p + \gamma u)\right)$$

- equivalent for every $\gamma > 0$
- single-valued, Lipschitz continuous, implicit

Moreau–Yosida regularization

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Proximal mapping

$$\mathrm{prox}_{\gamma \mathcal{G}^*}(p) = \arg\min_{w} \mathcal{G}^*(w) + \frac{1}{2\gamma}\|w - p\|^2$$

Moreau–Yosida regularization of $u \in \partial \mathcal{G}^*(p)$

$$u = \frac{1}{\gamma}\left(p - \mathrm{prox}_{\gamma \mathcal{G}^*}(p)\right) =: \partial \mathcal{G}^*_\gamma(p)$$

- $\partial \mathcal{G}^*_\gamma = \partial \left(\mathcal{G} + \frac{\gamma}{2}\|\cdot\|^2\right)^* \to \partial \mathcal{G}^*$ as $\gamma \to 0$ (no smoothing of $\mathcal{G}$!)
- single-valued, Lipschitz continuous, explicit
  $\rightsquigarrow$ nonsmooth operator equation, Newton method

Semismooth Newton method

f locally Lipschitz, piecewise C^1:

$$f(v) = 0, \qquad f : \mathbb{R}^n \to \mathbb{R}$$

Newton derivative

$$D_N f(v)\delta v \in \partial_C f(v)\delta v$$

Clarke generalized gradient: convex hull of piecewise derivatives

semismooth Newton method

$$D_N f(v^k)\delta v = -f(v^k), \qquad v^{k+1} = v^k + \delta v$$

converges locally superlinearly

Semismooth Newton method

f locally Lipschitz, piecewise C^1:

$$F(u) = 0, \qquad F : L^r(\Omega) \to L^s(\Omega), \quad [F(u)](x) = f(u(x))$$

Newton derivative

$$[D_N F(u)\delta u](x) \in \partial_C f(\delta u(x))\delta u(x)$$

any measurable selection of Clarke generalized gradient

semismooth Newton method

$$D_N F(u^k)\delta u = -F(u^k), \qquad u^{k+1} = u^k + \delta u$$

converges locally superlinearly if $r > s$

Multibang penalty

$$g : \mathbb{R} \to \overline{\mathbb{R}}, \qquad v \mapsto \begin{cases} \frac{1}{2}\left((u_i + u_{i+1})v - u_i u_{i+1}\right) & v \in [u_i, u_{i+1}] \\ \infty & \text{else} \end{cases}$$

piecewise differentiable $\rightsquigarrow$ subdifferential convex hull of derivatives

$$\partial g(v) = \begin{cases} \left(-\infty, \frac{1}{2}(u_1 + u_2)\right] & v = u_1 \\ \left\{\frac{1}{2}(u_i + u_{i+1})\right\} & v \in (u_i, u_{i+1}) \quad 1 \leqslant i < d \\ \left[\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})\right] & v = u_i \qquad 1 < i < d \\ \left[\frac{1}{2}(u_{d-1} + u_d), \infty\right) & v = u_d \end{cases}$$

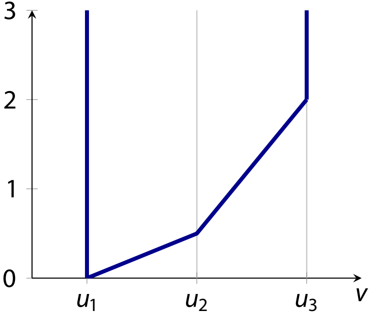
Multibang penalty

$$\partial g(v) = \begin{cases} \left(-\infty, \frac{1}{2}(u_1 + u_2)\right] & v = u_1 \\ \left\{\frac{1}{2}(u_i + u_{i+1})\right\} & v \in (u_i, u_{i+1}) \quad 1 \leqslant i < d \\ \left[\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})\right] & v = u_i \quad 1 < i < d \\ \left[\frac{1}{2}(u_{d-1} + u_d), \infty\right) & v = u_d \end{cases}$$

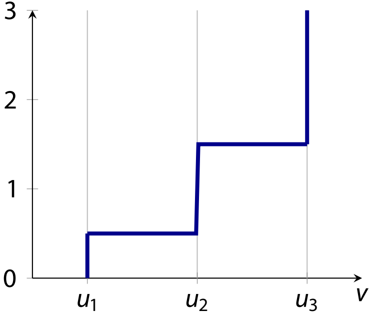
convex inverse function theorem:

$$\partial g^*(q) \in \begin{cases} \{u_1\} & q \in \left(-\infty, \frac{1}{2}(u_1 + u_2)\right) \\ [u_i, u_{i+1}] & q = \frac{1}{2}(u_i + u_{i+1}), \quad 1 \leqslant i < d \\ \{u_i\} & q \in \left(\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})\right) \quad 1 < i < d, \\ \{u_d\} & q \in \left(\frac{1}{2}(u_{d-1} + u_d), \infty\right) \end{cases}$$

Multibang penalty: sketch

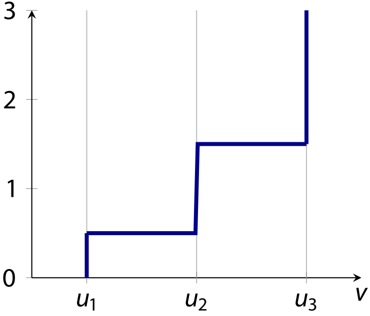


(a) g ($u_1 = 0, u_2 = 1, u_3 = 2$)

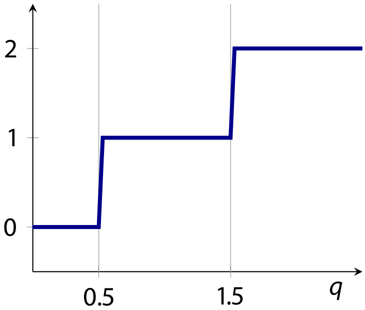


(b) ∂g ($u_1 = 0, u_2 = 1, u_3 = 2$)

Multibang penalty: sketch



(c) ∂g $(u_1 = 0, u_2 = 1, u_3 = 2)$



(d) ∂g^* $(u_1 = 0, u_2 = 1, u_3 = 2)$

Proximal point mapping

Proximal mapping $\quad \operatorname{prox}_{\gamma(\alpha g)}(v) = w \text{ iff } v \in \{w\} + \gamma\alpha\partial g(w)$

case-wise inspection of subdifferential:

$$\operatorname{prox}_{\gamma(\alpha g)}(v) = \begin{cases} u_i & \text{if } v \in P_i^\gamma \\ v - \frac{\alpha\gamma}{2}(u_i + u_{i+1}) & \text{if } v \in P_{i,i+1}^\gamma \end{cases}$$

$$P_i^\gamma = \left[\left(1 + \frac{\alpha\gamma}{2}\right) u_{i-1} + \frac{\alpha\gamma}{2} u_i, \left(1 + \frac{\alpha\gamma}{2}\right) u_i + \frac{\alpha\gamma}{2} u_{i+1}\right]$$
$$P_{i,i+1}^\gamma = \left(\left(1 + \frac{\alpha\gamma}{2}\right) u_i + \frac{\alpha\gamma}{2} u_{i+1}, \left(1 + \frac{\alpha\gamma}{2}\right) u_{i+1} + \frac{\alpha\gamma}{2} u_i\right)$$

$\rightsquigarrow$ generalized soft thresholding

Moreau–Yosida regularization

Proximal mapping $\quad \text{prox}_{\gamma g^*}(q) = w \text{ iff } q \in \{w\} + \gamma\partial g^*(w)$

case-wise inspection of subdifferential:

$$\partial g_\gamma^*(q) = \frac{1}{\gamma}\left(q - \text{prox}_{\gamma g^*}(q)\right) = \begin{cases} u_i & q \in Q_i^\gamma \\ \frac{1}{\gamma}\left(q - \frac{1}{2}(u_i + u_{i+1})\right) & q \in Q_{i,i+1}^\gamma \end{cases}$$

$$Q_i^\gamma = \left(\tfrac{1}{2}(u_{i-1} + u_i) + \gamma u_i, \tfrac{1}{2}(u_i + u_{i+1}) + \gamma u_i\right)$$
$$Q_{i,i+1}^\gamma = \left[\tfrac{1}{2}(u_i + u_{i+1}) + \gamma u_i, \tfrac{1}{2}(u_i + u_{i+1}) + \gamma u_{i+1}\right]$$

Multibang regularization

$$\min_{u\in L^2(\Omega)} \frac{1}{2}\|K(u) - y^\delta\|_Y^2 + \alpha\, \mathcal{G}(u)$$

- $K : L^2(\Omega) \to Y$ (nonlinear) forward mapping, weakly closed
- $y^\delta \in L^2(\Omega)$ noisy data with $\|y - y^\delta\|_Y \leqslant \delta$
- $u_1 < \cdots < u_d$ given parameter values ($d > 2$)
- $\mathcal{G}$ multibang penalty

Multibang regularization

$$\min_{u \in L^2(\Omega)} \frac{1}{2} \|K(u) - y^\delta\|_Y^2 + \alpha\, \mathcal{G}(u)$$

- $\mathcal{G}$ multibang penalty convex:
  1. existence of solution u_α^δ for every $\alpha > 0$
  2. $\delta \to 0$ implies $u_\alpha^\delta \rightharpoonup u_\alpha$ for every $\alpha > 0$
  3. $\delta \to 0$, $\alpha \to 0$, $\delta\alpha^{-2} \to 0$ implies $u_\alpha^\delta \rightharpoonup u^\dagger$

(standard arguments, e.g. [Burger/Osher 04, Ito/Jin 14])

Multibang regularization

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|K(u) - y^\delta\|_Y^2 + \alpha\, \mathcal{G}(u)$$

- standard source condition: $p^\dagger := K'(u^\dagger)^* w \in \partial\mathcal{G}(u^\dagger)$ for $w \in Y$,
  1. a priori choice $\alpha(\delta) = c\delta$
  2. a posteriori choice $\|K(u^\delta_{\alpha(\delta)}) - y^\delta\|_Y \leqslant \tau\delta, \tau > 1$

  $\rightsquigarrow$ convergence rate

$$d_{\mathcal{G}}^{p^\dagger}(u_\alpha^\delta, u^\dagger) \leqslant C\delta$$

  in Bregman distance

$$d_{\mathcal{G}}^{p_1}(u_2, u_1) = \mathcal{G}(u_2) - \mathcal{G}(u_1) - \langle p_1, u_2 - u_1\rangle_X, \qquad p_1 \in \partial\mathcal{G}(u_1)$$

Multibang regularization

Pointwise definition of Bregman distance, ∂g:

- $u^\dagger(x) = u_i$ and $p^\dagger \notin \left\{\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})\right\}$ implies

$$d_g^{p^\dagger(x)}(u_{\alpha(\delta)}^\delta(x), u^\dagger(x)) \to 0 \qquad \text{for } \delta \to 0$$

- $u^\dagger(x) \in (u_i, u_{i+1})$ implies

$$d_g^{p^\dagger(x)}(u(x), u^\dagger(x)) = 0 \qquad \text{for any } u(x) \in [u_i, u_{i+1}]$$

- $\leadsto$ $u_{\alpha(\delta)}^\delta \to u^\dagger$ pointwise a.e. iff $u^\dagger(x) \in \{u_1, \ldots, u_d\}$ a.e.

- (convergence not uniform $\leadsto$ no pointwise rates)

Optimality system

$$\bar{p} = \frac{1}{\alpha}K'(\bar{u})^*(y^\delta - K\bar{u})$$

$$\bar{u} \in \partial \mathcal{G}^*(\bar{p}) = \begin{cases} \{u_i\} & \bar{p}(x) \in Q_i \\ [u_i, u_{i+1}] & \bar{p}(x) \in \overline{Q}_i \cap \overline{Q}_{i+1} \end{cases}$$

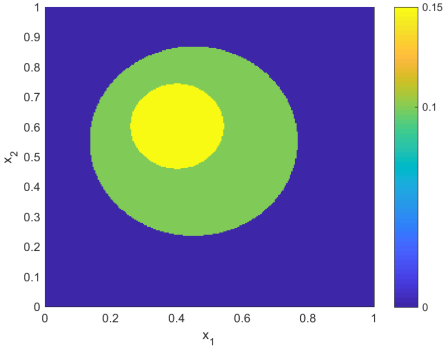
- $\rightsquigarrow$ unique solution $(\bar{u}, \bar{p}) \in L^2(\Omega) \times L^2(\Omega)$
- singular set $\mathcal{S} = \{x : \bar{u}(x) \neq u_i\} \subset \{x : \bar{p}(x) = \frac{1}{2}(u_i + u_{i+1})\}$
- for suitable K, $\bar{p}(x)$ constant implies $[y^\delta - K\bar{u}](x) = 0$ (e.g., $K = A^{-1}$ for A pure second-order elliptic)

$\rightsquigarrow$ $|\{x : K(\bar{u}(x)) = y^\delta(x)\}| = 0 \;\Rightarrow\; \bar{u} \in \{u_1, \ldots u_d\}$ a. e. (true multibang)

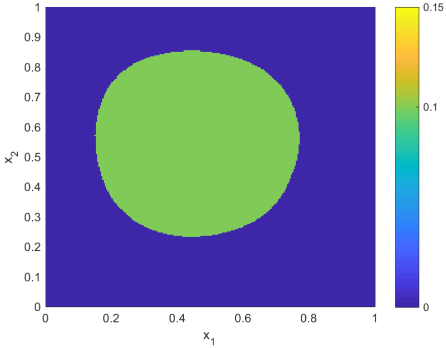
Example: linear inverse problem

- $\Omega = [0,1]^2, \quad K = A^{-1}, \quad A = -\Delta$
- $$u^\dagger(x) = u_1 + u_2\,\chi_{\{x:(x_1-0.45)^2+(x_2-0.55)^2<0.1\}}(x) + (u_3 - u_2)\,\chi_{\{x:(x_1-0.4)^2+(x_2-0.6)^2<0.02\}}(x)$$
- $d = 3, \quad u_1 = 0, \quad u_2 = 0.1, \quad u_3 \in \{0.15, 0.11\}$
- $y^\delta = y^\dagger + \xi, \quad \xi \in \mathcal{N}\left(y^\dagger, \delta\|y^\dagger\|_\infty\right)$
- finite element discretization: uniform grid, 256×256 nodes
- solution by (regularized) semismooth Newton method ($\gamma < 10^{-12}$)
- $\alpha = \alpha(\delta)$ by Morozov discrepancy principle

Numerical example: $u_3 = 0.15$

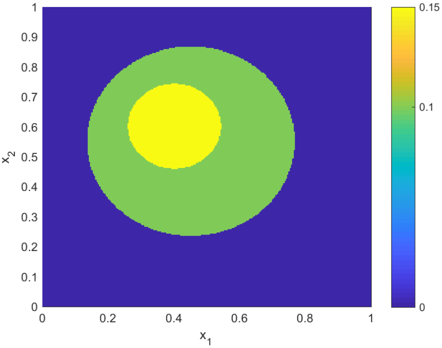


(a) $u^\dagger$

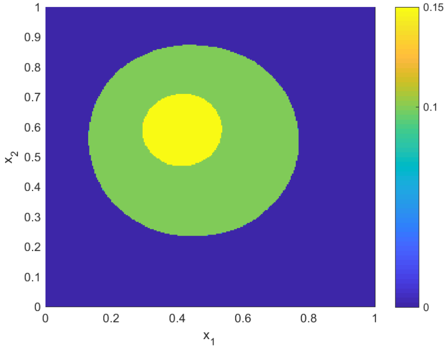


(b) u_α^δ, $\delta \approx 1.89 \cdot 10^{-1}$

Numerical example: $u_3 = 0.15$

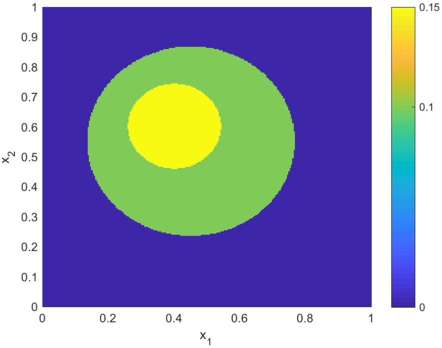


(c) $u^\dagger$

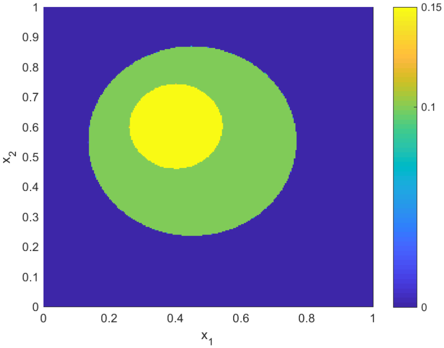


(d) u_α^δ, $\delta \approx 2.37 \cdot 10^{-2}$

Numerical example: $u_3 = 0.15$

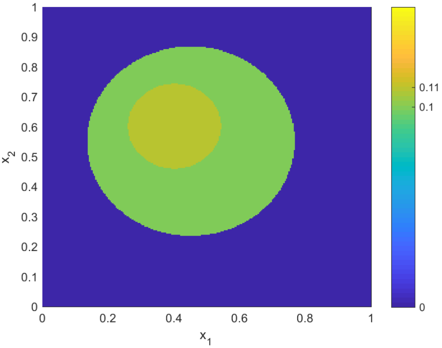


(e) $u^\dagger$

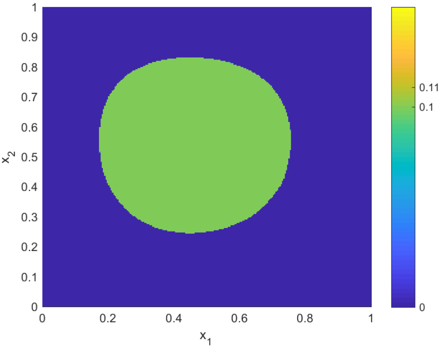


(f) $u_\alpha^\delta, \delta \approx 3.69 \cdot 10^{-4}$

Numerical example: $u_3 = 0.11$

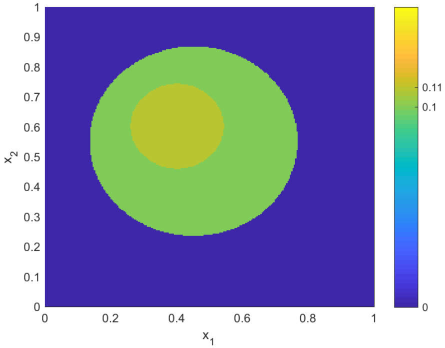


(a) $u^\dagger$

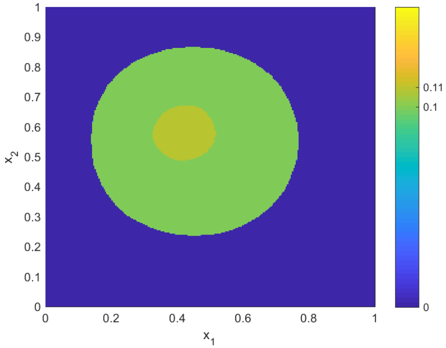


(b) u_α^δ, $\delta \approx 1.68 \cdot 10^{-1}$

Numerical example: $u_3 = 0.11$

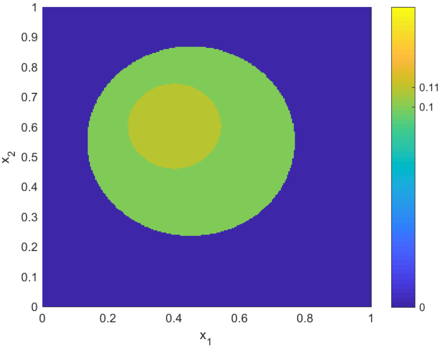


(c) $u^\dagger$

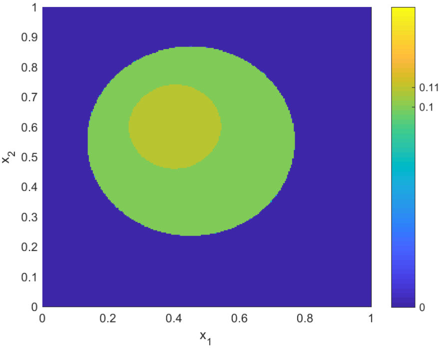


(d) u_α^δ, $\delta \approx 2.17 \cdot 10^{-2}$

Numerical example: $u_3 = 0.11$

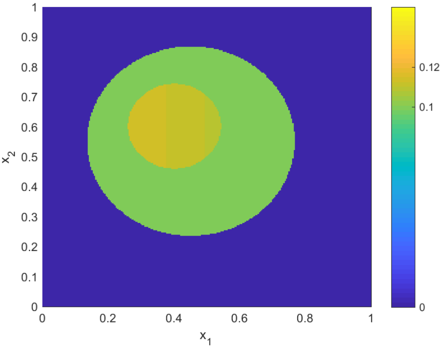


(e) $u^\dagger$

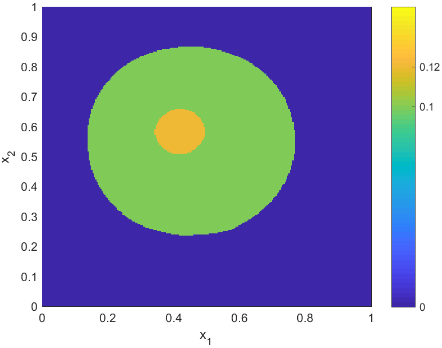


(f) $u_\alpha^\delta, \delta \approx 3.29 \cdot 10^{-4}$

Numerical example: $u_3(x) = 0.12(1 - x_1)$

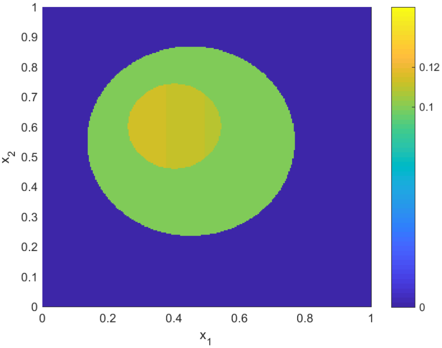


(a) $u^\dagger$

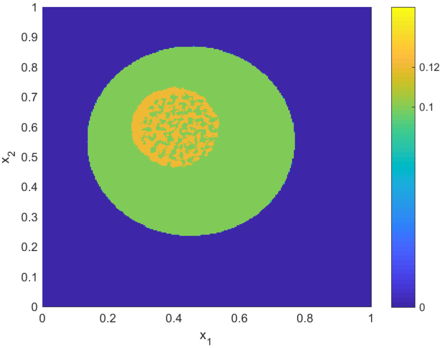


(b) u_α^δ, $\delta \approx 2.11 \cdot 10^{-2}$

Numerical example: $u_3(x) = 0.12(1 - x_1)$

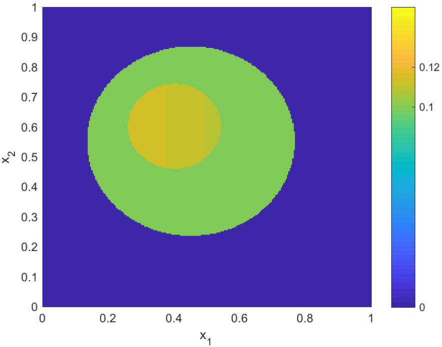


(c) $u^\dagger$

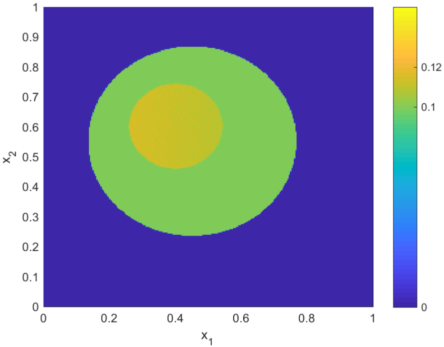


(d) u_α^δ, $\delta \approx 3.29 \cdot 10^{-4}$

Numerical example: $u_3(x) = 0.12(1 - x_1)$



(e) $u^\dagger$



(f) u_α^δ, $\delta \approx 1.29 \cdot 10^{-6}$

1 Overview

2 Approach
- Convex analysis
- Algorithms
- Multibang penalty

3 Multibang regularization
- Regularization properties
- Structure and numerical solution

4 Wave equation

Nonlinear forward mapping

Goal: application to coefficient inverse problem for wave equation

- $K : u \mapsto y$ solving

$$\begin{cases} y_{tt} - \nabla \cdot (u \nabla y) = f \\ y(0) = y_0, \quad y_t(0) = y_1 \end{cases}$$

- difficulty: $\bar{u} \in L^\infty(\Omega) \quad \rightsquigarrow \quad K$ not weakly-$*$ closed
  1. lack of existence of minimizer ($\bar{y} \neq K(\bar{u})$, cf. homogenization)
  2. lack of convergence $\gamma \to 0$
  3. lack of Newton differentiability of H_γ (no norm gap)

- $\rightsquigarrow$ TV regularization: add $TV(u) := \|Du\|_{\mathcal{M}}$
- $\rightsquigarrow u \in BV(\Omega) \cap L^\infty(\Omega) \hookrightarrow_c L^p(\Omega)$

TV regularization

Difficulty:

- existence requires box constraints $\rightsquigarrow$ use penalty

$$\big(G(u) + \delta_{[u_1,u_d]}(u)\big) + TV(u)$$

(here: G multibang penalty with $\operatorname{dom} G = L^1(\Omega)$)

- but: $TV(u) + \delta_{[u_1,u_d]}(u)$ not continuous on $L^p(\Omega)$, $p < \infty$
- but: multipliers $\xi \in \partial TV(u)$, $q \in \partial G(u)$ not pointwise on BV, L^∞
- $\rightsquigarrow$ replace box constraints by ($C^{1,1}$) projection of $u \in L^1(\Omega)$

$$[\Phi_\varepsilon(u)](x) = \operatorname{proj}^\varepsilon_{[u_1,u_d]}(u(x)) \qquad \text{a.e. } x \in \Omega$$

- $\rightsquigarrow$ use higher regularity of solution to wave equation

Wave equation

$$\begin{cases} \int_0^T -(\partial_t y, \partial_t v) + (u\nabla y(t), \nabla v(t))\,dt = \int_0^T (f(t), v(t))\,dt + (y_1, v(0)) \\ y(0) = y_0 \end{cases}$$

for all $v \in W := L^2(0,T;H^1) \cap H^1(0,T;L^2)$ with $v(T) = 0$ [Lions/Magenes '72]

solution mapping $S : u \mapsto y$ on $U := \{u \in L^\infty(\Omega) : u_1 \leqslant u \leqslant u_d \text{ a.e.}\}$

- $S(u)$ uniformly bounded in $W \cap H^2(0,T;H^{-1}) := Z$
- S Lipschitz continuous from L^∞ to $L^2(0,T;L^2)$
- $S(u_n) \rightharpoonup S(u)$ in Z if $u_n \to u$ in $L^r(\Omega)$, $r \in [1,\infty]$
- $S(u_n) \to S(u)$ in W if $u_n \to u$ in $L^\infty(\Omega)$

Wave equation

$$\begin{cases} \int_0^T -(\partial_t y, \partial_t v) + (u\nabla y(t), \nabla v(t))\,dt = \int_0^T (f(t), v(t))\,dt + (y_1, v(0)) \\ y(0) = y_0 \end{cases}$$

for all $v \in W := L^2(0,T;H^1) \cap H^1(0,T;L^2)$ with $v(T) = 0$ [Lions/Magenes '72]

Assumption:

- $f \in H^2(0,T;L^2), \quad y_0 \in H^3(\Omega), \partial_\nu y_0 = 0, \quad y_1 \in H^2(\Omega)$
- there is $\omega_c \subset \Omega$ with u constant on $\Omega \setminus \omega_c$, y_0 constant on ω_c

then

- $S(u)$ uniformly bounded in $L^\infty(0,T;W^{1,s})$ for some $s > 2$ (proof: combination of higher hyperbolic and maximal elliptic regularity [Wloka '87, Gröger '89])

TV regularization: existence

$$\begin{cases} \displaystyle\min_{u \in BV(\Omega)} \frac{1}{2}\|y - z\|^2_{L^2(\Omega)} + \alpha\, G(u) + \beta\, TV(u) \\ \text{s.t.} \quad y_{tt} - \nabla \cdot (\Phi_\varepsilon(u)\nabla y) = f \\ \qquad y(0) = y_0, \quad y_t(0) = y_1 \end{cases}$$

- existence of optimal $\bar{u} \in BV(\Omega) \cap U$ for $\varepsilon \geqslant 0$
- tracking term Fréchet differentiable in $\Phi_\varepsilon(u) \in L^\infty$ for $\varepsilon > 0$
- improved regularity of state $\rightsquigarrow$ derivative in $L^r(\Omega)$, $r > 1$ (instead of $L^\infty(\Omega)^*$)
- $\rightsquigarrow$ sum rule applicable, subgradients in $L^r(\Omega)$, $r > 1$

TV regularization: optimality conditions

$$\begin{cases} 0 = F'(\Phi(\bar{u}))\Phi'_\varepsilon(\bar{u}) + \alpha\bar{q} + \beta\bar{\xi} \\ \bar{u} \in \partial G^*(\bar{q}) \\ \bar{\xi} \in \partial TV(\bar{u}) \end{cases}$$

- $F'(\Phi_\varepsilon(\bar{u})) = \int_0^T \nabla\bar{y} \cdot \nabla\bar{p}\, dt \in L^r(\Omega), r > 1$ (optimal state, adjoint)
- $\bar{q} \in L^r(\Omega), r > 1 \rightsquigarrow$ pointwise multibang
- $\bar{\xi} \in L^r(\Omega), r > 1 \rightsquigarrow$ characterization via *full trace* [Bredies/Holler '12]
- $\rightsquigarrow$ pointwise optimality conditions
- proximal splitting/semi-smooth Newton
  (after discretization, regularization)

Proximal splitting algorithm

$$
\begin{aligned}
u^{k+1} &= \operatorname{prox}_{\tau\alpha\mathcal{G}}\left(u^k - \tau S'(u^k)^*(r^k) - \tau\nabla^*\psi^k\right)\\
\bar{u}^{k+1} &= 2u^{k+1} - u^k\\
r^{k+1} &= \frac{1}{1+\frac{\sigma}{\nu_1}}\left(r_1^k + \sigma(S(\bar{u}^{k+1}) - y^d)\right)\\
q^{k+1} &= \psi^k + \sigma\nabla\bar{u}^{k+1}\\
\psi^{k+1} &= \frac{\beta q^{k+1}}{\max\{\beta, |q^{k+1}|_2\}}
\end{aligned}
$$

- $S(u)$ solution of wave equation
- $S'(u)^*r$ solution of wave, adjoint equation (with rhs r), integration
- proximal mappings pointwise ($\mathcal{G}$ includes projection)

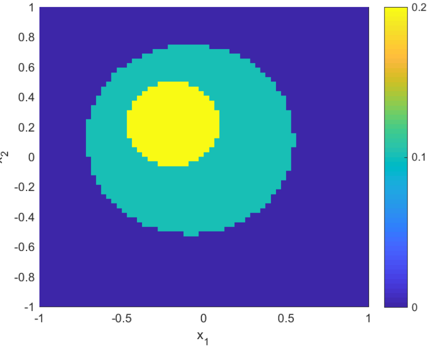
Numerical example: stationary

Example: elliptic problem ($\partial_{tt} y \equiv 0$)

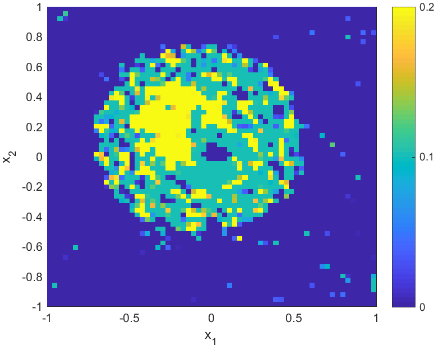
approach: discretize before optimize

- consider finite element discretization of problem (p.w. linear)
- include projection in multi-bang penalty, eliminate Φ_ε
- apply sum rule, chain rule for $\partial TV(u_h) = -\text{div}_h \partial(\|\cdot\|_1)(\nabla_h u_h)$
- apply Moreau–Yosida regularization to $\partial \mathcal{G}^*$, $\partial(\|\cdot\|_1)^*$
- $\rightsquigarrow$ semi-smooth Newton-type method (modified Newton step to avoid kernel of div_h; line search)
- local convergence: path-following with extrapolation

Numerical example: stationary

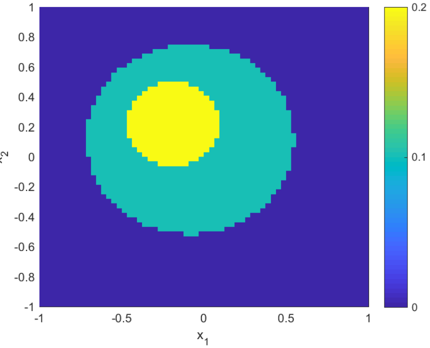


(a) $u^\dagger$

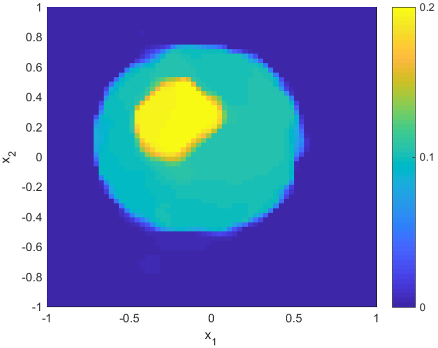


(b) $\alpha = 5 \cdot 10^{-4}, \beta = 0$

Numerical example: stationary



(c) $u^\dagger$



(d) $\alpha = 5 \cdot 10^{-4}, \beta = 10^{-5}$

Conclusion

Convex relaxation of discrete regularization:

- well-posed regularization method
- pointwise convergence under general assumptions
- strong structural regularization
- efficient numerical solution

Outlook:

- (heuristic) multi-parameter choice
- boundary observation
- vector-valued coefficient
- (block) acceleration of proximal splitting

Preprint, MATLAB/Python codes: (when ready …)

`http://www.uni-due.de/mathematik/agclason/clason_pub.php`