Adaptive discretization and first-order methods for nonsmooth inverse problems for PDEs

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Tikhonov regularization

$$\min_{u \in U} F(S(u) - y^\delta) + G_\alpha(u)$$

- $F : Y \to \overline{\mathbb{R}}$ discrepancy term, y^δ noisy data
- $G_\alpha : X \to \overline{\mathbb{R}}$ regularization term, $\alpha > 0$
- $S : U \supset X \to Y$ forward operator, involves solution to PDE

Adaptive regularization and discretization:

- regularization: $u^\delta_{\alpha(\delta)} \to u^\dagger$ with $S(u^\dagger) = y$ for $\delta \to 0$
- discretization: FE approximation $u^\delta_{\alpha,h} \to u^\delta_\alpha$ for $h \to 0$
- **goal**: combined choice $(\alpha, h)(\delta)$ for nonsmooth functionals

Motivation

Primal-dual extragradient method:

- first-order algorithm for nonsmooth convex problems with linear operators [Chambolle/Pock 2011]
- *very* popular in imaging (TV denoising, deblurring, ...)
- acceleration (Nesterov, $\mathcal{O}(1/k^2)$ convergence)
- version for nonlinear operators [Valkonen 2014]

Here:

- application to parameter identification for PDEs
- $\rightsquigarrow$ function space algorithm

Difficulty:

- convergence proof requires set-valued analysis in infinite-dimensional spaces

Model problems

L¹-fitting

$$\min_u \|S(u) - y^\delta\|_{L^1} + \frac{\alpha}{2}\|u\|_{L^2}^2$$

L∞-fitting

$$\min_u \frac{1}{2}\|u\|_{L^2}^2 \quad \text{s.t.} \quad |S(u)(x) - y^\delta(x)| \leqslant \delta \quad \text{a.e. in } \Omega$$

$S : U \subset L^2(\Omega) \to L^2(\Omega), \quad S(u) =: y$ satisfies

$$\begin{cases} -\Delta y + uy = f \\ \partial_\nu y = 0 \end{cases}$$

Adaptive discretization

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Tikhonov functional for $\alpha > 0$

$$u_\alpha^\delta = \arg\min_{u \in U} J_\alpha(u) := F(S(u) - y^\delta) + G_\alpha(u)$$

Discretized Tikhonov functional for $\alpha > 0, h > 0$

$$u_{\alpha,h}^\delta = \arg\min_{u \in U} J_{\alpha,h}(u) := F(S_h(u) - y^\delta) + G_\alpha(u)$$

- S_h (conforming) finite element approximation of S
- $S_h(u) \to S(u)$ for $h \to 0$ for any $u \in U$

Choose for given $\delta > 0$:

1. $\alpha = \alpha(\delta)$ such that

$$\delta \leqslant F(S(u_\alpha^\delta) - y^\delta) \leqslant \tau\delta$$

2. $h = h(\alpha, \delta)$ such that

$$\begin{aligned} \left|F(S(u_{\alpha,h}^\delta - y^\delta)) - F(S(u_\alpha^\delta) - y^\delta)\right| &\leqslant c_1\delta \\ J_{\alpha,h}(u_{\alpha,h}^\delta) - J_\alpha(u_\alpha^\delta) &\leqslant c_2\delta \end{aligned}$$

Then, for all $\delta > 0$:

$$G_\alpha(u_{\alpha,h}^\delta) \leqslant G_\alpha(u^\dagger), \qquad F(S_h(u_{\alpha,h}^\delta) - y^\delta) \leqslant \tau\delta$$

- no direct control of $\|u_{\alpha,h}^\delta - u_\alpha^\delta\|_X$ needed!
- $\rightsquigarrow$ convergence, rates for $\|u_{\alpha,h}^\delta - u^\dagger\|_X \to 0$ for $\delta \to 0$ (standard)

Functional error estimator

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Goal: a posteriori error estimator for discrepancy, functional accuracy
Here: linear operator $S = A^{-1}$ for elliptic differential operator A

Functional error estimator:

If there exist φ_α, ψ with

1. $\lambda(1-\lambda)\psi(y_1, y_2) \leqslant \lambda F(y_1) + (1-\lambda)F(y_2) - F(\lambda y_1 + (1-\lambda)y_2)$
2. $\lambda(1-\lambda)\varphi_\alpha(u_1, u_2) \leqslant \lambda G_\alpha(u_1) + (1-\lambda)G_\alpha(u_2) - G_\alpha(\lambda u_1 + (1-\lambda)u_2)$

Then, for all $u \in U, y \in Y$:

$$\psi(Su_\alpha^\delta, Su) + \varphi_\alpha(u_\alpha^\delta, u) \leqslant F(Su) + F^*(-y) + G_\alpha(v) + G_\alpha^*(S^* y)$$

- F^*, G_α^* Fenchel conjugates, S^* adjoint
- proof by strong convexity, weak duality
- $\rightsquigarrow$ apply for $u = u_{\alpha,h}^\delta, y = Su_{\alpha,h}^\delta$

Application to quadratic penalty

X, Y Hilbert space:

$$F(y) = \frac{1}{2}\|y - y^\delta\|_Y^2 \qquad G_\alpha(u) = \frac{\alpha}{2}\|u\|_X^2$$

$$\psi(y_1, y_2) = \frac{1}{2}\|y_1 - y_2\|_Y^2 \qquad \varphi_\alpha(u_1, u_2) = \frac{\alpha}{2}\|u_1 - u_2\|_X^2$$

$$F^*(y) = \frac{1}{2}\|y\|_Y^2 - (y, y^\delta) \qquad (G_\alpha)^*(u) = \frac{1}{2\alpha}\|u\|_X^2$$

Functional error estimate for $\bar{u} = u_\alpha^\delta, \bar{u}_h = u_{\alpha,h}^\delta$

$$\alpha\|\bar{u} - \bar{u}_h\|_X + \|S\bar{u} - S\bar{u}_h\| \leqslant \frac{1}{\alpha}\|\alpha\bar{u}_h + S^*(S\bar{u}_h - y^\delta)\|_X^2 + \|S\bar{u} - S\bar{u}_h\|_Y^2$$

Application to quadratic penalty

Residual estimate for right-hand side for $\bar{y}_h = S_h\bar{u}_h, \bar{y} = S\bar{u}$

$$\alpha\|\bar{u} - \bar{u}_h\|_X + \|\bar{y} - \bar{y}_h\| \leqslant \frac{2}{\alpha}\|S^*\rho_w + \rho_u\|_X^2 + 4\|S\rho_y\|_Y^2$$

$$J_\alpha(\bar{u}_h) - J_\alpha(\bar{u}) \leqslant \frac{1}{2\alpha}\|S^*\rho_w + \rho_u\|_X^2 + (S\bar{u}_h - y^\delta, S\rho_y)_Y$$

$$\begin{aligned}
\rho_w &:= A^*\bar{w}_h + \bar{y}_h - y^\delta \\
\rho_u &:= \alpha\bar{u}_h - \bar{w}_h = \alpha\bar{u}_h - S_h^*(\bar{y}_h - y^\delta) \\
\rho_y &:= A\bar{y}_h + \bar{u}_h
\end{aligned}$$

- residuals of optimality conditions
- standard element-based a posteriori estimators for rhs

Application to norm penalty

X Banach space (e.g., $X = \mathcal{M}(\Omega) \rightsquigarrow$ sparsity)

$$G_\alpha(u) = \alpha\|u\|_X \qquad \varphi_\alpha(u_1, u_2) = 0$$

$$(G_\alpha)^*(u^*) = \iota_{\alpha B_{X^*}}(u^*) := \begin{cases} 0 & \|u^*\|_{X^*} \leqslant \alpha, \\ \infty & \text{else} \end{cases}$$

Functional estimate with $y = \kappa\bar{y}_h + (1-\kappa)y^\delta$

$$\|S\bar{u} - S\bar{u}_h\|_Y^2 \leqslant 2\alpha\|\bar{u}_h\|_X + 2\langle\bar{u}_h, S^*(y - \kappa(\bar{y}_h - y^\delta))\rangle + \|S\bar{u}_h - \kappa\bar{y}_h + (1-\kappa)y^\delta\|_Y^2$$

- for κ such that $\|S^*y\|_{X^*} \leqslant \alpha$ (can be estimated)

Application to norm penalty

Residual estimate for right-hand side for $\bar{w}_h = S_h(y^\delta - \bar{y}_h)$

$$
\begin{aligned}
\|\bar{y}_h - \bar{y}\|_Y^2 &\leqslant 4(\alpha\|\bar{u}_h\|_X - \langle \bar{u}_h, \bar{w}_h\rangle) + 4\kappa\langle \bar{u}_h, \bar{w}_h - S^*(y^\delta - \bar{y}_h)\rangle \\
&\quad + 4(1-\kappa)\langle \bar{u}_h, \bar{w}_h\rangle + 4\|S\rho_y + (\kappa - 1)(\bar{y}_h - y^\delta)\|_Y^2 \\
J_\alpha(\bar{u}_h) - J_\alpha(\bar{u}) &\leqslant \alpha\|\bar{u}_h\|_X - \langle \bar{u}_h, \bar{w}_h\rangle + \kappa\langle \bar{u}_h, \bar{w}_h - S^*(y^\delta - \bar{y}_h)\rangle \\
&\quad + (1-\kappa)\langle \bar{u}_h, \bar{w}_h\rangle \\
&\quad + \|S\rho_y + (\kappa - 1)(\bar{y}_h - g^\delta)\|_Y\|\bar{y}_h - y^\delta\|_Y
\end{aligned}
$$

- (non-)standard element-based a posteriori estimators for rhs
- $X^* = C_0(\Omega) \rightsquigarrow L^\infty$ a posteriori estimators for κ
- no bound on $\bar{u}_h - \bar{u}$, not needed for convergence!

Problem statement

$$\min_{u \in X} F(K(u)) + G(u)$$

- $F : Y \to \overline{\mathbb{R}}, G : X \to \overline{\mathbb{R}}$ convex, lower semicontinuous
- X, Y Hilbert spaces
- $K \in C^2(X, Y)$ nonlinear (here: $K(u) = S(u) - y^\delta$)
- saddle point formulation:

$$\min_{u \in X} \sup_{v \in Y^*} G(u) + \langle K(u), v \rangle - F^*(v)$$

- $F^* : Y^* \to \overline{\mathbb{R}}$ Fenchel conjugate

Algorithm

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K linear:

$$\begin{cases} u^{k+1} = \mathrm{prox}_{\tau G}(u^k - \tau K^* v^k) \\ \bar{u}^{k+1} = 2u^{k+1} - u^k \\ v^{k+1} = \mathrm{prox}_{\sigma F^*}(v^k + \sigma K \bar{u}^{k+1}) \end{cases}$$

- σ, τ step sizes, $\sigma\tau < \|K\|^{-2}$
- $\mathrm{prox}_{\sigma F}(v) = \arg\min_u \frac{1}{2\sigma}\|u - v\|^2 + F(u)$ proximal mapping

Algorithm

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K nonlinear:

$$\begin{cases} u^{k+1} = \operatorname{prox}_{\tau G}(u^k - \tau K'(u^k)^* v^k) \\ \bar{u}^{k+1} = 2u^{k+1} - u^k \\ v^{k+1} = \operatorname{prox}_{\sigma F^*}(v^k + \sigma K(\bar{u}^{k+1})) \end{cases}$$

- σ, τ step sizes, $\sigma\tau < \sup_{u \in B_R} \|K'(u)\|^{-2}$
- $\operatorname{prox}_{\sigma F}(v) = \arg\min_u \frac{1}{2\sigma}\|u - v\|^2 + F(u)$ proximal mapping
- $K'(u)$ Fréchet derivative, $K'(u)^*$ adjoint

Algorithm

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K nonlinear, accelerated:

$$\begin{cases} u^{k+1} = \operatorname{prox}_{\tau_k G}(u^k - \tau^k K'(u^k)^* v^k) \\ \quad \omega_k = 1/\sqrt{1+2c\tau^k} \quad \tau^{k+1} = \omega_k \tau^k \quad \sigma_{k+1} = \sigma_k/\omega_k \\ \bar{u}^{k+1} = u^{k+1} + \omega_k(u^{k+1} - u^k) \\ v^{k+1} = \operatorname{prox}_{\sigma_{k+1} F^*}(v^k + \sigma_{k+1} K(\bar{u}^{k+1})) \end{cases}$$

- σ, τ step sizes, $\sigma_0 \tau_0 < \sup_{u \in B_R} \|K'(u)\|^{-2}$
- $\operatorname{prox}_{\sigma F}(v) = \arg\min_u \frac{1}{2\sigma}\|u - v\|^2 + F(u)$ proximal mapping
- $K'(u)$ Fréchet derivative, $K'(u)^*$ adjoint
- $c \geqslant 0$ acceleration parameter

Convergence

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Theorem

Iterates converge locally to saddle point $(\bar{u}, \bar{v})$ *if*

1. G *is* c_G*-strongly convex* *(here:* $c_G = 1$*)*
2. $c = c_n \in [0, c_G)$, $c_n = 0$ *for* $n > N \in \mathbb{N}$ *(finite acceleration)*
3. *metric regularity around saddle point*

(cf. [Valkonen 2014])

- allows for inexact evaluation of K ($\rightsquigarrow$ adaptive discretization)

Difficulty:

- metric regularity in function spaces
- $\rightsquigarrow$ requires infinite-dimensional set-valued analysis
- here: only rough outline, no details

Metric regularity

Saddle-point problem

$$\min_{u\in X} \sup_{v\in Y^*} G(u) + \langle K(u), v\rangle - F^*(v)$$

Primal-dual optimality conditions

$$\begin{cases} K(\bar{u}) \in \partial F^*(\bar{v}) \\ -K'(\bar{u})^*\bar{v} \in \partial G(\bar{u}) \end{cases}$$

Metric regularity

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Primal-dual optimality conditions

$$\begin{cases} K(\bar{u}) \in \partial F^*(\bar{v}) \\ -K'(\bar{u})^* \bar{v} \in \partial G(\bar{u}) \end{cases}$$

Set inclusion for $H : L^2(\Omega)^2 \rightrightarrows L^2(\Omega)^2$

$$0 \in H_{\bar{u}}(\bar{u}, \bar{v}) := \begin{pmatrix} \partial G(\bar{u}) + K'(\bar{u})^* \bar{v} \\ \partial F^*(\bar{v}) - K(\bar{u}) \end{pmatrix}$$

Metric regularity

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Set inclusion for $H : L^2(\Omega)^2 \rightrightarrows L^2(\Omega)^2$

$$0 \in H_{\bar u}(\bar u, \bar v) := \begin{pmatrix} \partial G(\bar u) + K'(\bar u)^* \bar v \\ \partial F^*(\bar v) - K(\bar u) \end{pmatrix}$$

Metric regularity at $(\bar u, \bar v)$

$$\inf_{q:\, w \in H_{\bar u}(q)} \|q - (\bar u, \bar v)\| \leqslant L\|w\| \quad \text{for all } \|w\| \leqslant \rho$$

- interpretation: small perturbation w of 0
  $\Rightarrow$ small perturbation q of saddle point $(\bar u, \bar v)$
- Lipschitz property for set-valued $H_{\bar u}^{-1}$ at $((\bar u, \bar v), 0)$

Metric regularity

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Metric regularity at $(\bar{u}, \bar{v})$

$$\inf_{q:\, w \in H_{\bar{u}}(q)} \|q - (\bar{u}, \bar{v})\| \leqslant L\|w\| \quad \text{for all } \|w\| \leqslant \rho$$

Mordukhovich criterion

$$L_R = \inf_{t>0} \sup \left\{ \|\widehat{D}^* R(q'|w')\| \,\middle|\, q' \in B((\bar{u}, \bar{v}), t),\ w' \in R(q') \cap B(w, t) \right\}$$

- Aubin constant $L_{H^{-1}}$ is minimal choice of L
- $\widehat{D}^* R$ regular coderivative of $R (= H^{-1})$ (cf. $L = \|\nabla f\|$ for $f \in C^1$)
- $\rightsquigarrow$ set-valued analysis in function spaces

Set-valued analysis in function space

Difficulties:

- multiple non-equivalent concepts (regular, limiting)
- calculus not tight

Here:

- set-valued mappings from subdifferentials of pointwise functionals
- $\rightsquigarrow$ infinite-dimensional (regular) derivatives pointwise via nice finite-dimensional (regular, graphical) derivatives
- cf. pointwise Fenchel conjugates, subdifferentials [Ekeland]

Application to saddle point inclusion

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$$0 \in H_{\bar u}(\bar u, \bar v) = \begin{pmatrix} \partial G(\bar u) + K'(\bar u)^* \bar v \\ \partial F^*(\bar v) - K(\bar u) \end{pmatrix}$$

$$DH_{\bar u}(q|w)(\Delta q) = \begin{pmatrix} D[\partial G](u|\xi - K'(\bar u)^* v)(\Delta u) + K'(\bar u)^* \Delta v \\ D[\partial F^*](v|\eta + K'(\bar u)u + c_{\bar u})(\Delta v) - K'(\bar u)\Delta u \end{pmatrix}$$

$$\widetilde{DH_{\bar u}}(q|w)(\Delta q) = \begin{cases} T_q \Delta q + V(q|w)^\circ & \Delta q \in V(q|w) \\ \emptyset & \text{otherwise} \end{cases}$$

- $q = (u, v), \quad w = (\xi, \eta), \quad c_{\bar u} = K(\bar u) - K'(\bar u)\bar u$
- T_q linear Operator (independent of w), $\quad V(q|w)$ cone
- $\widehat{D}^* H_{\bar u}(q|w) = [\widetilde{DH_{\bar u}}(q|w)]^{*+}$ upper adjoint of convexification

Mordukhovich criterion

$$\widetilde{DH_{\bar{u}}}(q|w)(\Delta q) = \begin{cases} T_q \Delta q + V(q|w)^\circ & \Delta q \in V(q|w) \\ \emptyset & \text{otherwise} \end{cases}$$

Then: Aubin constant $L_H \leqslant c < \infty$ **iff**

$$\sup_{t>0} \inf_{\substack{(\Delta w, z) \in W^t(q|w), \\ \|\Delta w\| > 0}} \frac{\|T_q^* \Delta w - z\|}{\|\Delta w\|} \geqslant c^{-1} > 0$$

$$W^t(q|w) = \bigcup \left\{ V(q'|w') \times V(q'|w')^\circ \;\middle|\; \begin{array}{l} w' \in H_{\bar{u}}(q'), \\ \|q' - q\| < t,\ \|w' - w\| < t \end{array} \right\}$$

Primal-dual optimality conditions

$$\begin{cases} S(\bar{u}) - y^\delta \in \partial F^*(\bar{v}) \\ -S'(\bar{u})^* \bar{v} = \bar{u} \end{cases}$$

Metric regularity around $(\bar{u}, \bar{v})$ if either

1. $\sup_{t>0} \inf \left\{ \frac{\|S'(\bar{u})S'(\bar{u})^* z - v\|}{\|z\|} \,\middle|\, (z, v) \in V^t_{\partial F^*}(\bar{v}|y^\delta - S(\bar{u})),\ z \neq 0 \right\} > 0$
2. Moreau–Yosida regularization: $F^* \mapsto F^*_\gamma := F^* + \frac{\gamma}{2}\|\cdot\|^2$
3. finite-dimensional data: $Y \rightsquigarrow Y_h$

In case 1: $\|S'(\bar{u})^* z\| \geqslant c\|z\|$ for $z \in V^t_{\partial F^*}(\bar{v}|y^\delta - S(\bar{u}))$ necessary!

L^1 fitting

$$\min_u \frac{1}{\alpha}\|S(u) - y^\delta\|_{L^1} + \frac{1}{2}\|u\|_{L^2}^2$$

- $F(y) = \int_\Omega \alpha^{-1}|y(x)|\,dx \quad \rightsquigarrow \quad f^*(z) = \iota_{[-\alpha^{-1},\alpha^{-1}]}(z)$
- $S : U \subset L^2(\Omega) \to L^2(\Omega), \quad S(u) =: y$ satisfies

$$\begin{cases} -\Delta y + uy = f \\ \partial_\nu y = 0 \end{cases}$$

L^1 fitting: metric regularity

Here: $z \in V^t_{\partial F^*}(v|\eta)$ if

$$z(x) \in \begin{cases} \{0\} & |v'(x)| = \alpha^{-1} \text{ and } \eta'(x) \neq 0 \\ -\operatorname{sign} v'(x)[0,\infty) & |v'(x)| = \alpha^{-1} \text{ and } \eta'(x) = 0 \\ \mathbb{R} & |v'(x)| < \alpha^{-1} \text{ and } \eta'(x) = 0 \end{cases}$$

for some $\|v' - \bar{v}\| \leqslant t, \|\eta' - \bar{\eta}\| \leqslant t$

- S compact operator: $\|S'(\bar{u})^* z\| \geqslant c\|z\|$ only holds for $z = 0$
- $\bar{\eta} = S(\bar{u}) - y^\delta, \quad \alpha\bar{v} \in \operatorname{sign} \bar{\eta}$
- $\rightsquigarrow$ in general not satisfied!

L^1 fitting: algorithm

$$
\begin{cases}
z^{k+1} = S'(u^k)^* v^k \\
u^{k+1} = \dfrac{1}{1+\tau_k}(u^k - \tau_k z^{k+1}) \\
\omega_k = 1/\sqrt{1+2c\tau_k} \quad \tau_{k+1} = \omega_k \tau_k \quad \sigma_{k+1} = \sigma_k/\omega_k \\
\bar{u}^{k+1} = u^{k+1} + \omega_k(u^{k+1} - u^k) \\
v^{k+1} = \mathrm{proj}_{[-\alpha^{-1},\alpha^{-1}]}\left(\frac{1}{1+\sigma_{k+1}\gamma}(v^k + \sigma_{k+1}(S(\bar{u}^{k+1}) - y^\delta))\right)
\end{cases}
$$

- $S'(u^k)^* v^k$ solution of adjoint equation
- proj_C pointwise projection on convex set $C \subset \mathbb{R}$
- Moreau–Yosida parameter $\gamma \geqslant 0$
- local convergence if $\gamma > 0$ or finite-dimensional

L^∞ fitting

$$\min_u \frac{1}{2}\|u\|_{L^2}^2 \quad \text{s.t.} \quad |S(u)(x) - y^\delta(x)| \leqslant \delta \quad \text{a.e. in } \Omega$$

- $F(y) = \iota_{\{|y(x)|\leqslant\delta\}}(y) \quad \rightsquigarrow \quad f^*(z) = \delta|z|$
- $S : U \subset L^2(\Omega) \to L^2(\Omega)$, $S(u) =: y$ satisfies

$$\begin{cases} -\Delta y + uy = f \\ \partial_\nu y = 0 \end{cases}$$

- for $t = 0$: $z(x) = 0$ if $|S(\bar u)(x) - y^\delta(x)| < \delta$ $\rightsquigarrow$ estimate unlikely

L^∞ fitting: algorithm

$$\begin{cases} z^{k+1} = S'(u^k)^* v^k \\ u^{k+1} = \dfrac{1}{1+\tau_k}(u^k - \tau_k z^{k+1}) \\ \omega_k = 1/\sqrt{1+2c\tau_k} \quad \tau_{k+1} = \omega_k \tau_k \quad \sigma_{k+1} = \sigma_k/\omega_k \\ \bar{u}^{k+1} = u^{k+1} + \omega_k(u^{k+1} - u^k) \\ \bar{v}^{k+1} = v^k + \sigma_{k+1}(S(\bar{u}^{k+1}) - y^\delta) \\ v^{k+1} = \frac{1}{1+\sigma_{k+1}\gamma}(|\bar{v}^{k+1}| - \delta\sigma)^+ \operatorname{sign}(\bar{v}^{k+1}) \end{cases}$$

- local convergence if $\gamma > 0$ or finite-dimensional

L^1 fitting

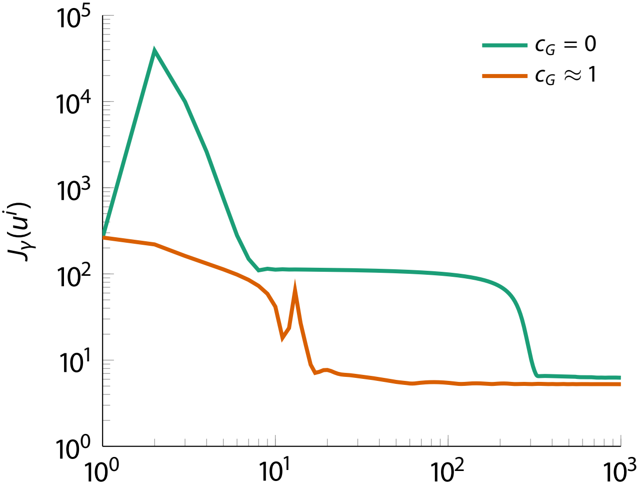
- $\Omega = [-1,1]$, FE (P_1–P_0) discretization of (y,u)
- random impulsive noise:

$$y^\delta(x) = \begin{cases} y^\dagger(x) + \|y^\dagger\|_\infty \xi(x) & \text{with probability } 0.3 \\ y^\dagger(x) & \text{else} \end{cases}$$

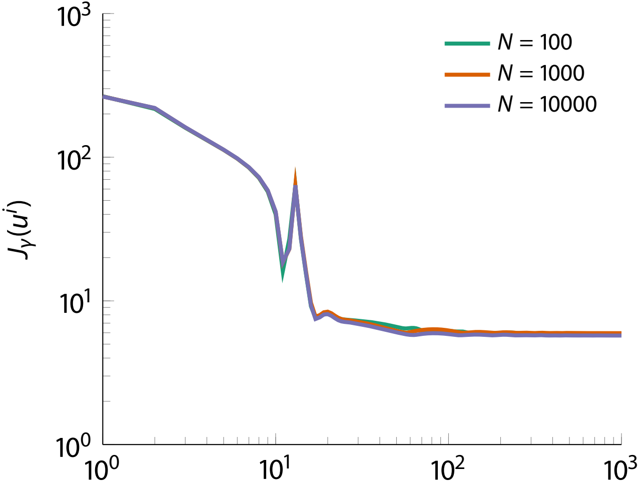
$$y^\dagger = S(u^\dagger), \quad \xi(x) \in \mathcal{N}(0, 0.1), \quad \rightsquigarrow \alpha = 10^{-2}$$

- $\sigma_0 = \tilde{L}^{-1}, \tau_0 = 0.99\tilde{L}^{-1}, \quad \tilde{L} = \|S(u^0)\|/\|u^0\|$
- $\gamma = 10^{-12}, \quad u^0 \equiv 1, v^0 = 0$ (no warmstart!)
- compare $c \in \{0, 1 - 10^{-16}\}$, $N \in \{100, 1000, 10000\}$

L^1 fitting: acceleration (same data)



L^1 fitting: discretization (avg. of 10)



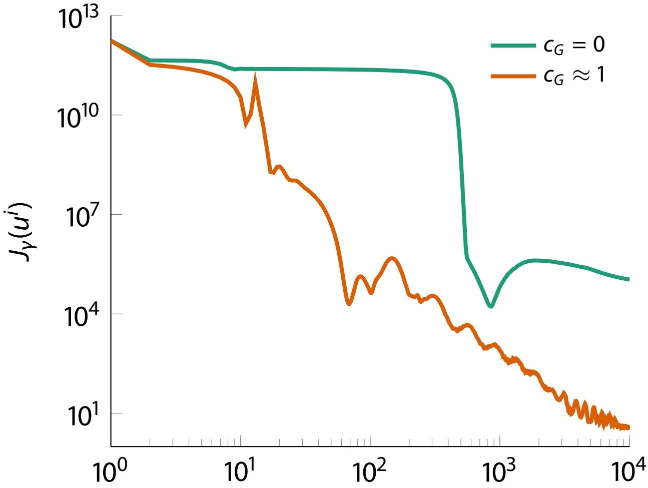
L^∞ fitting

- $\Omega = [-1, 1]$, FE (P_1–P_0) discretization of (y, u)
- quantization noise:

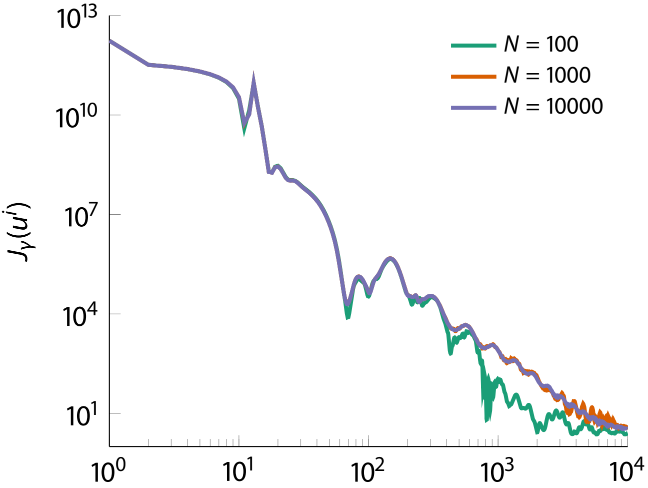
$$\text{round } y^\dagger \text{ to } n_b = 11 \text{ equidistant values}$$

- $\gamma = 10^{-12}$, $\sigma_0 = \tilde{L}^{-1}, \tau_0 = 0.99\tilde{L}^{-1}$, $\tilde{L} = \|S(u^0)\|/\|u^0\|$
- $u^0 \equiv 1, v^0 = 0$ (no warmstart!)
- compare $c \in \{0, 1 - 10^{-16}\}$, $N \in \{100, 1000, 10000\}$

L^∞ fitting: acceleration (same data)



L^∞ fitting: discretization (same data)



Conclusion

Primal-dual extragradient methods in function space:

- can be accelerated
- analyzed using set-valued analysis in function space
- requires Moreau–Yosida regularization
- $\rightsquigarrow$ no norm gap, continuation needed; mesh-independence

Outlook:

- full acceleration
- partial stability (w.r.t. primal variable only)
- adaptive discretization using functional estimators

Preprints/code:

`http://www.uni-due.de/mathematik/agclason/clason_pub.php`