The quasi-reversibility method for thermoacoustic tomography in a heterogeneous medium

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joint work with M. Klibanov

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1. Background
    - Motivation
    - Problem Formulation
2. Method of Quasi-reversibility
    - Derivation
    - Stability
    - Convergence
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    - Galerkin Approximation
    - Implementation
4. Numerical Results
5. Conclusion

Thermoacoustic Tomography

Imaging method, multi-modal:

1. Electromagnetic irradiation (RF, Microwave)
2. Absorption in tissue
3. Heating, expansion
4. Pressure wave in tissue, coupling medium
5. Measurement of acoustic pressure in medium

Absorption dependent on tissue type $\Rightarrow$ recognition of tumours

Mathematical Model for TCT

Model for propagation of pressure waves

No heat conduction, homogeneous excitation pulse:

$$\begin{cases} \frac{1}{c^2(x)}\partial_{tt}u(x,t) - \Delta u(x,t) &= 0 & (x,t) \in \mathbb{R}^3 \times [0,T] \\ u(x,t)|_{t=0} &= \alpha(x) & x \in \mathbb{R}^3 \\ \partial_t u(x,t)|_{t=0} &= 0 & x \in \mathbb{R}^3 \end{cases}$$

($u(x,t)$ acoustic pressure, $c(x)$ wave speed, $\alpha(x)$ absorption parameter)

Inverse problem: Calculate $\alpha(x)$ from measurement of $u(x,t)$!

Goal

Reconstruction method for variable wave speed c

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Lateral Cauchy Problem

Notations: $\Omega \subset \mathbb{R}^n$ domain, $Q_T := \Omega \times [0, T]$, $S_T := \partial\Omega \times [0, T]$.
Initial conditions unknown, boundary measurement $(u, \partial_\nu u)$:

Problem (C)

Given c, f, φ_0, φ_1, find $u(x, t)$ in Q_T so that:

$$\begin{cases} \frac{1}{c(x)^2}\partial_{tt}u(x,t) - \Delta u(x,t) & = f \qquad\qquad (x,t) \in Q_T\,, \\ u(x,t) & = \varphi_0(x,t) \qquad (x,t) \in S_T\,, \\ \partial_\nu u(x,t) & = \varphi_1(x,t) \qquad (x,t) \in S_T\,. \end{cases}$$

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Approximation via Method of Quasi-reversibility

Consider first the case of $\varphi_0 \equiv \varphi_1 \equiv 0$.

Ansatz: Look for best approximation in Hilbert space X
having minimal Y-norm

Tikhonov functional

$$J_\varepsilon(u) := \frac{1}{2}\left\|\frac{1}{c(x)^2}\partial_{tt}u - \Delta u - f\right\|^2_{L^2(Q_T)} + \frac{\varepsilon}{2}\|u\|_Y^2 \to \min_{u\in X}$$

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For which X, Y does the minimisation problem have a unique solution?

Choice of Function Space

Tikhonov functional

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Choose

$$X := H^2_0(Q_T) := \left\{u \in H^2(Q_T) : \; u|_{S_T} = \partial_\nu u|_{S_T} = 0\right\}$$

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Choose

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with inner product

$$\langle u, v\rangle_{QR} := \int_{Q_T} \partial_{tt}u\,\partial_{tt}v\,dq + \int_{Q_T}\sum_{i=1}^{n}\partial_{ii}u\,\partial_{ii}v\,dq + \int_{Q_T} u\,v\,dq$$

and induced norm $\|u\|^2_{QR} := \langle u, u\rangle_{QR}$

Characterisation of Solution

Lemma

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Euler equation

$$J_\varepsilon'(u_\varepsilon)(v) = 0 \text{ for all } v \in H_0^2(Q_T)$$

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$$\int_{Q_T} Lu_\varepsilon\, Lv\, dq + \varepsilon \langle u_\varepsilon, v\rangle_{QR} - \int_{Q_T} Lv\, f\, dq = 0 \text{ for all } v \in H_0^2(Q_T)$$

$\left(Lu := \frac{1}{c(x)^2}\partial_{tt} u - \Delta u\right)$

Non-homogeneous Boundary Conditions

Consider *boundary function* $\Phi \in H^2(Q_T)$ for (φ_0, φ_1):

$$\begin{cases} \Phi(x,t) & = \varphi_0(x,t) \qquad (x,t) \in S_T \\ \partial_\nu \Phi(x,t) & = \varphi_1(x,t) \qquad (x,t) \in S_T \end{cases}$$

$\Rightarrow$ $u^* := u - \Phi$ satisfies

$$\begin{cases} Lu^* & = f - L\Phi \quad (x,t) \in Q_T \\ u^*(x,t) & = 0 \qquad (x,t) \in S_T \\ \partial_\nu u^*(x,t) & = 0 \qquad (x,t) \in S_T \end{cases}$$

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Quasi-reversibility Approximation

Problem (Q)

Define bilinear form

$$M_\varepsilon(u, v) := \int_{Q_T} Lu\, Lv\, dq + \varepsilon \langle u, v\rangle_{QR}$$

Given $\Phi \in H^2(Q_T)$, $c \in C^1(\bar{\Omega})$, $\varepsilon > 0$, find $u_\varepsilon \in H_0^2(Q_T)$ so that

$$M_\varepsilon(u_\varepsilon, v) = -\int_{Q_T} L\Phi\, Lv\, dq$$

for all $v \in H_0^2(Q_T)$

Existence of Unique Solution

Theorem

For $\Phi \in H^2(Q_T)$, $c \in C^1(\bar{\Omega})$, $\varepsilon > 0$:

- Problem (Q) has unique solution u_ε
- There is a $C(Q_T, \|c\|_{L^2(\Omega)}) > 0$ such that

$$\|u_\varepsilon\|_{H^2(Q_T)} \leq \frac{C}{\sqrt{\varepsilon}} \|\Phi\|_{H^2(Q_T)}$$

Proof.

$M_\varepsilon(u, v)$ inner product: Riesz' theorem, estimate via equivalence of norms $\square$

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Regularity

Lemma

$M_\varepsilon(u,v)$ is $H_0^2(Q_T)$ elliptic: There are $c_1(\|c\|_{L^2(\Omega)}), c_2(Q_T) > 0$ such that:

$$\begin{aligned} |M_\varepsilon(u,v)| &\leq (c_1+\varepsilon)\|u\|_{H^2(Q_T)}\|v\|_{H^2(Q_T)} \\ |M_\varepsilon(u,u)| &\geq c_2\varepsilon\|u\|^2_{H^2(Q_T)} \end{aligned}$$

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Problem (Q) has a solution in $H^3(U)$ for all compact $U \subset Q_T$

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Convergence of Approximations

Theorem

- u^* solution of Problem (C) with boundary function Φ
- u_ε^δ solution of Problem (Q) for Φ^δ with $\left\|\Phi - \Phi^\delta\right\|_{H^2(Q_T)} \leq \delta$
- $c(x)$ bounded, satisfies $2c^{-2}(x) + \left\langle \nabla(c^{-2})(x), x - x_0 \right\rangle_n > 0$

If $T > T_0(\Omega, c) > 0$:

$$\left\|u^* - u_\varepsilon^\delta\right\|^2_{H^1(Q_T)} \leq C\left(\delta^2 + \varepsilon \left\|u^*\right\|^2_{H^2(Q_T)}\right)$$

Parameter choice rule $\varepsilon = \tau\delta^2, \tau > 1$

$\Rightarrow$ QR approximation is convergent regularisation method

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Ritz-Galerkin Approximation

Ansatz: Solve Problem (Q) in finite dimensional subspace

Problem (R)

Find $u_h \in S_h \subset H_0^2(Q_T)$ such that for all $v_h \in S_h$

$$M_\varepsilon(u_h, v_h) = -\int_{Q_T} L\Phi \, Lv_h \, dq$$

Here: Cubic splines

- satisfy regularity requirements
- are numerically advantageous (and easy to implement)
- allow construction of boundary function Φ

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Ansatz Space

Cubic splines in one dimension

$$\mathcal{S}_i^4 := \{s : s \text{ piecewise polynomial of order } 4\} \cap C^2$$

Cubic splines in $n+1$ dimensions

$$\mathcal{S}^4 := \operatorname{span}\left\{\prod_{i=1}^{n+1} s_i : \; s_i \in \mathcal{S}_i^4\right\}$$

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Error Estimate

Knots uniformly distributed, distance h:

Theorem

A solution $u_h \in S_h$ of Problem (R) satisfies with $C(\Omega, c, \varepsilon) > 0$:

$$\|u_\varepsilon - u_h\|_{H^2(Q_T)} \leq Ch \|u_\varepsilon\|_{H^3(Q_T)}$$

Proof.

1. M_ε elliptic, hence use Céa's lemma
2. approximation theorems for tensor product splines in Sobolev spaces
3. infimum of interpolation error in $\mathcal{S}^4$ is attained in S_h

□

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Implementation

Choose basis of $\mathcal{S}_i^4$:

Normalised cubic B-splines $B_j^4(x)$

- form partition of unity
- have local support
- can be differentiated analytically with B-splines as derivative
- have inner products which can be evaluated exactly and stably by Gauss quadrature
- allow stable construction of boundary function by *complete cubic spline interpolation*

details

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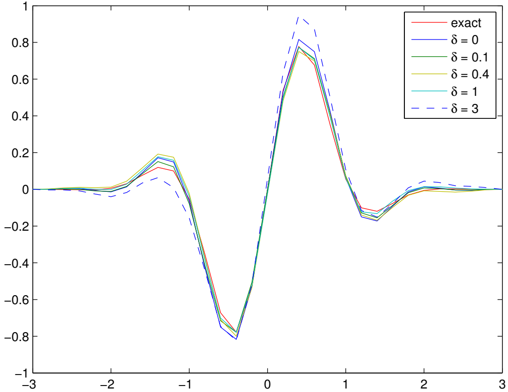
Numerical Results

- Domain $Q_T = [-3, 3]^2 \times [0, 7]$
- Discretisation $h_x = h_y = 0.2$, $h_t = 0.1$
- Given initial conditions $\partial_t u(x, y, 0) \equiv 0$,

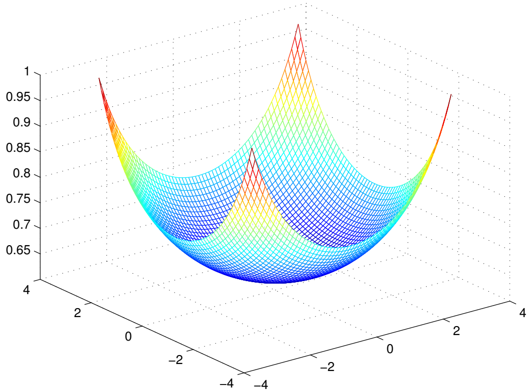
$$u(x, y, 0) = e^{-(x^2+y^2)} \sin(3x) \cos(3y)$$

Constant Coefficients ($c \equiv 1$): Reconstruction



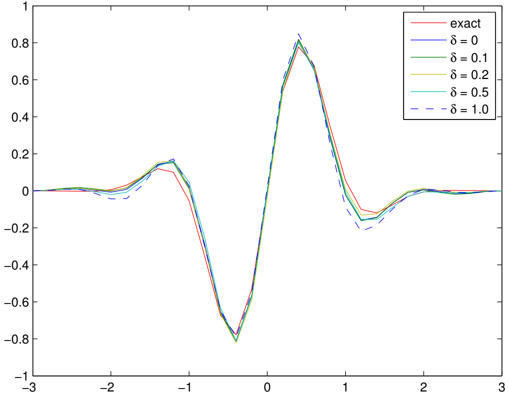
$u_\varepsilon(x,0,0)$, $\varepsilon = 10^{-3}$

Smooth Coefficients



$$\frac{1}{c(x,y)^2} = \frac{5}{2} - \frac{1}{12}(x^2 + y^2)$$

Smooth Coefficients: Reconstruction



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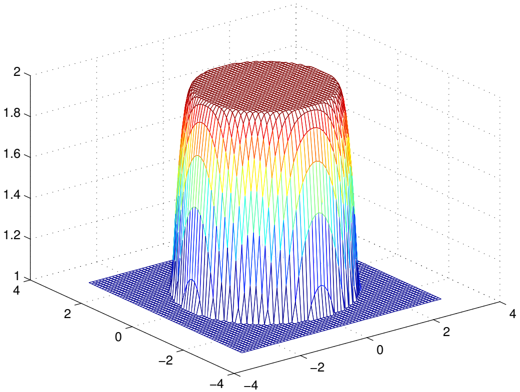
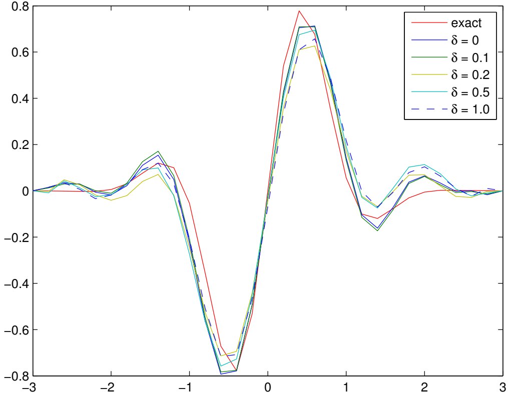
Nondifferentiable Coefficients



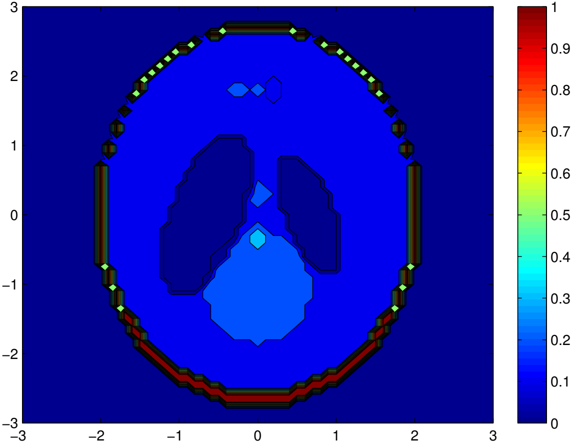
$$c(x, y) = \max\left(2 - \left(\frac{\max\left(2 - 5 + x^2 + y^2, 0\right)}{2}\right)^2, 1\right)$$

Nondifferentiable Coefficients: Reconstruction



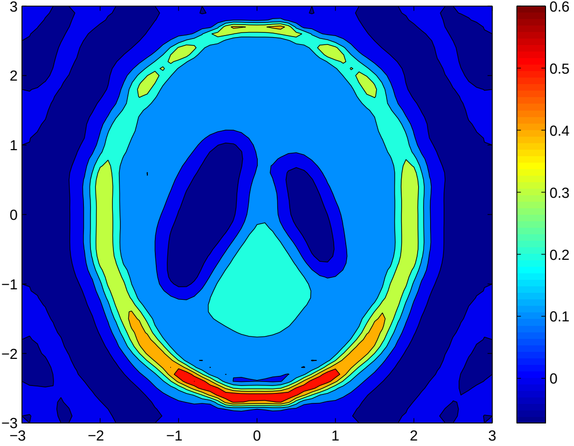
$u_\varepsilon(x,0,0)$, $\varepsilon = 10^{-4}$

Shepp-Logan Phantom



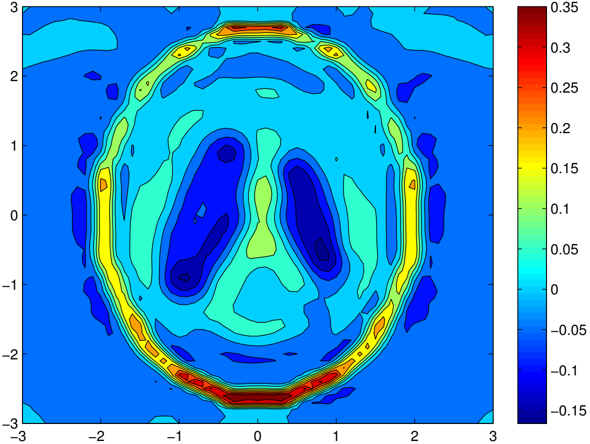
Given $u(x, y, 0)$

Shepp-Logan Phantom



reconstruction (constant coefficients)

Shepp-Logan Phantom



reconstruction (smooth coefficients)

Conclusion

Advantages:

- robust
- not iterative (independent of initial guess, stopping criteria)
- applicable to variable (time dependent) coefficients
- extensible to large class of problems

Disadvantages:

- linear systems only
- high memory requirements

Perspective:

- systems: full elasticity, Maxwell equations
- general domains (weighted B-splines)

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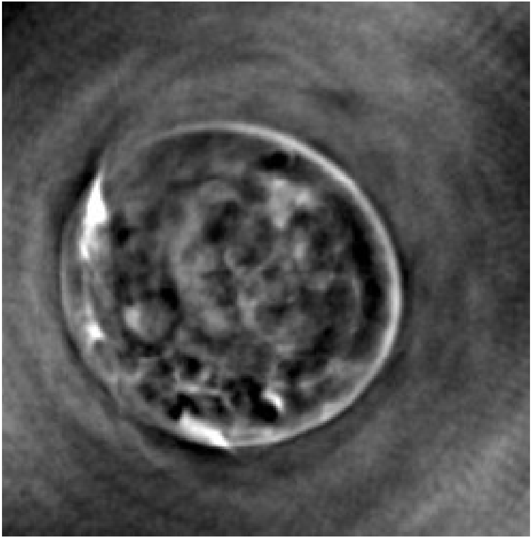
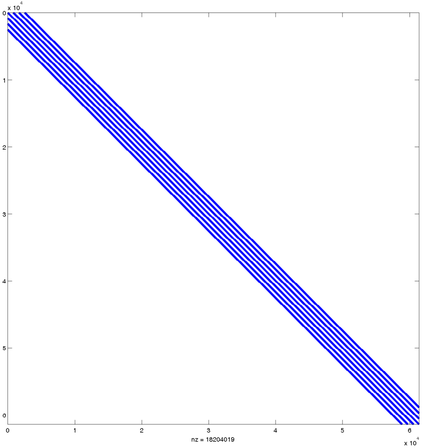
Thank you for your attention!

Mammography with TCT Prototype



Matrix Structure



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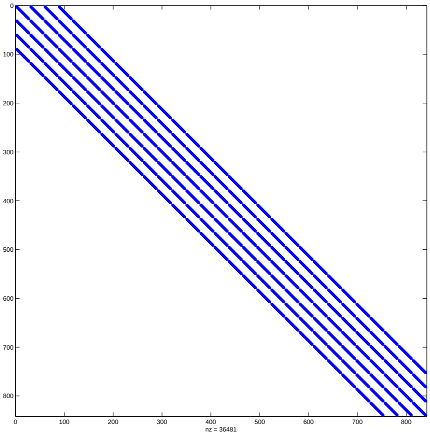

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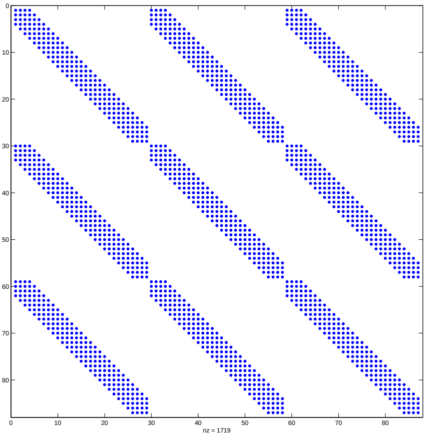

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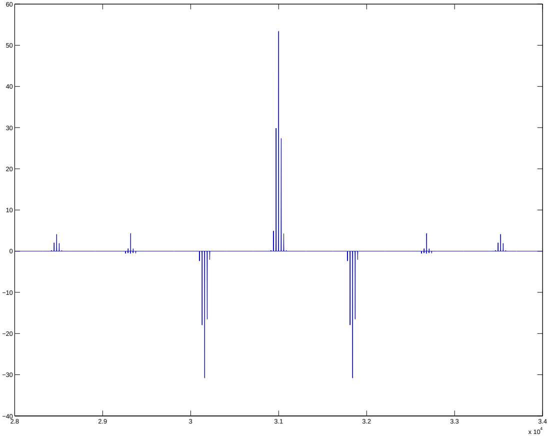


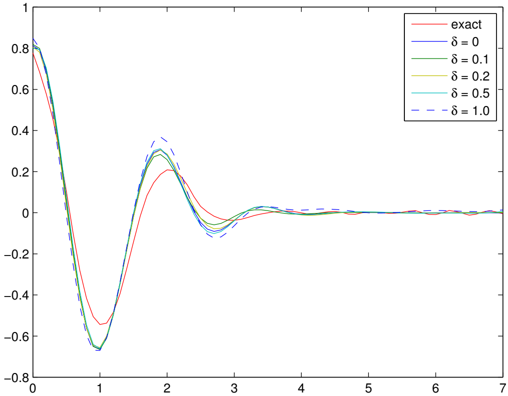
Relative L^2 Errors (Constants Coefficients)

δ, ε	0	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-1}	1.0
0	0.15971	0.15964	0.15901	0.15281	0.09858	0.87138	0.99522
0.05	0.16029	0.16041	0.15768	0.15073	0.09941	0.87178	0.99524
0.1	0.16211	0.15708	0.15837	0.15076	0.09618	0.87057	0.99527
0.2	0.16614	0.15912	0.16017	0.15119	0.09509	0.86999	0.99539
0.4	0.16653	0.18398	0.17926	0.15146	0.11096	0.87364	0.99583
0.5	0.19039	0.14141	0.18613	0.16458	0.13000	0.86619	0.99617
1.0	0.20092	0.23337	0.18634	0.21325	0.12906	0.88003	0.99444
2.0	0.31615	0.38390	0.43442	0.27649	0.34904	0.85463	0.99199
3.0	0.54785	0.43724	0.52161	0.48232	0.55094	0.84709	0.99659
4.0	0.78441	0.69156	0.64085	0.98342	0.64522	0.86108	0.99543
6.0	0.90998	1.08680	0.84162	1.28090	1.26290	0.91392	0.99826

Relative L^2 Errors (Smooth Coefficients)

δ, ε	0	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-1}	1.0
0	0.14984	0.14690	0.14850	0.15036	0.12667	0.76300	0.99530
0.05	0.14253	0.13900	0.15360	0.13643	0.11971	0.76330	0.99540
0.1	0.14457	0.14330	0.14200	0.14502	0.10915	0.76364	0.99513
0.2	0.13169	0.14170	0.15068	0.13298	0.10365	0.76320	0.99554
0.5	0.15338	0.17681	0.16089	0.15436	0.14229	0.75604	0.99508
1.0	0.17350	0.22785	0.22412	0.21235	0.19087	0.75570	0.99376

Time Development (Smooth Coefficients)



Convergence of Approximations

Idea of proof.

1. Use Carleman estimate for wave equation with variable coefficients to derive Lipschitz observability estimate:

$$\|u\|^2_{H^1(Q_T)} \leq C\,\|Lu\|^2_{L^2(Q_T)}$$

2. Difference $w := u^* - u^\delta_\varepsilon$ satisfies

$$\|Lw\|^2_{L^2(Q_T)} + \varepsilon\,\|w\|^2_{QR} = -\left\langle L(\Phi - \Phi^\delta), Lw\right\rangle_{L^2(Q_T)} + \varepsilon\,\langle u^*, w\rangle_{QR}$$

3. Apply observability estimate to w, estimate

$$\|Lw\|^2_{L^2(Q_T)} \leq C\left(\delta^2 + \varepsilon\,\|u^*\|^2_{QR}\right) \qquad \square$$

sketch of proof

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□

sketch of proof

Convergence of Approximations

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$$\|u\|_{H^1(Q_T)}^2 \leq C\,\|Lu\|_{L^2(Q_T)}^2$$

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▸ sketch of proof

Convergence of Approximations

Sketch of proof.

1. Carleman estimate for wave equation with variable coefficients:

$$\lambda^3 \int_{Q_\sigma} |u|^2 e^{2\lambda\varphi}\, dq + \lambda \int_{Q_\sigma} \left(|\nabla u|^2 + |\partial_t u|^2 \right) e^{2\lambda\varphi}\, dq \leq C \int_{Q_T} |Lu|^2 e^{2\lambda\varphi}\, dq$$

   holds for all $u \in H_0^2(Q_\sigma)$, $\lambda > 0$ large enough, φ pseudo-convex function, $Q_\sigma \subset Q_T$ pseudo-convex domain

Convergence of Approximations

Sketch of proof.

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   φ pseudo-convex function, $Q_\sigma \subset Q_T$ pseudo-convex domain

2. Use cutoff function for estimate in Q_σ for $u \in H_0^2(Q_T)$

$$\|u\|^2_{H^1(Q_\sigma)} \leq C \left(e^{-2\lambda c_1} \|u\|^2_{H^1(Q_T)} + e^{2\lambda c_2} \|Lu\|^2_{L^2(Q_T)} \right)$$

Convergence of Approximations

Sketch of proof.

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Sketch of proof.

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4. There is a $\theta \in [t_1, t_2]$ such that

$$\|u\|^2_{H^1(E)} \geq (t_2 - t_1)\left(\|u(\cdot,\theta)\|^2_{H^1(\Omega)} + \|\partial_t u(\cdot,\theta)\|^2_{L^2(\Omega)}\right)$$

Convergence of Approximations

Sketch of proof.

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5. Use standard energy estimate for wave equation:

$$\|u\|^2_{H^1(Q_T)} \leq 2C \left(\|u(\cdot, \theta)\|^2_{H^1(\Omega)} + \|\partial_t u(\cdot, \theta)\|^2_{L^2(\Omega)} + \|Lu\|^2_{L^2(Q_T)} \right)$$

Convergence of Approximations

Sketch of proof.

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6. Insert in step 3 for Lipschitz observability inequality in Q_T

$$\|u\|^2_{H^1(Q_T)} \leq C\,\|Lu\|^2_{L^2(Q_T)}$$

Convergence of Approximations

Sketch of proof.

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Convergence of Approximations

Sketch of proof.

7. Difference $w := u^* - u_\varepsilon^\delta$ satisfies

$$\|Lw\|_{L^2(Q_T)}^2 + \varepsilon \|w\|_{QR}^2 = - \left\langle L(\Phi^\delta - \Phi), Lw \right\rangle_{L^2(Q_T)} + \varepsilon \langle u^*, w \rangle_{QR}$$

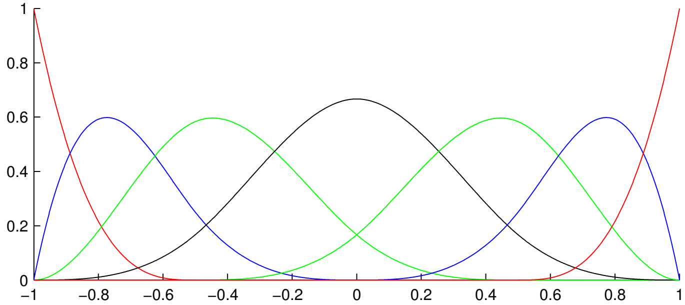
8. Apply Lipschitz observability estimate to w, use step 7 to estimate $\|Lw\|_{L^2(Q_T)}^2$:

$$\|w\|_{H^1(Q_T)}^2 \leq C \left(\delta^2 + \varepsilon \|u^*\|_{QR}^2 \right)$$

$\square$

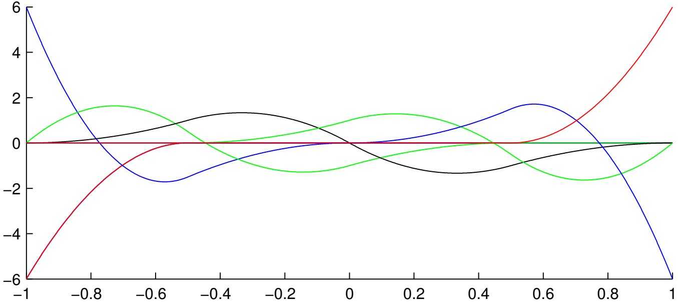
return to numerics

Cubic B-splines



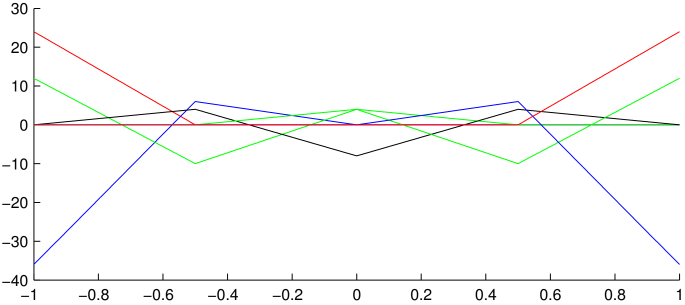
$B_j^4(x)$ for $j = 1, \ldots, 7$

Cubic B-splines (first derivative)



$B_j'^4(x)$ for $j = 1, \ldots, 7$

Cubic B-splines (second derivative)



$B_j''^4(x)$ for $j = 1, \ldots, 7$

Basis of S_h

Simplified problem:

- $\Omega = [-R, R] \times [-R, R] \subset \mathbb{R}^2$
- Uniform discretisation with k_1, k_2, k_3 knots in x, y, t

Basis of $\mathcal{S}^4$

$$\mathcal{B} = \Big\{ B^4_{i,1}(x) B^4_{j,2}(y) B^4_{k,3}(t),$$
$$i \in \{1, \ldots, k_1 + 4\}, j \in \{1, \ldots, k_2 + 4\}, k \in \{1, \ldots, k_3 + 4\} \Big\}$$

Basis of S_h

$$\mathcal{B}^0 = \Big\{ B^4_{i,1}(x) B^4_{j,2}(y) B^4_{k,3}(t),$$
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1. express u_h, v_h as linear combination from $\mathcal{B}^0$
2. express Φ as interpolant from $\mathcal{B}$
3. $\Rightarrow$ system of linear equations for coefficients of u_h

continue with results