A non-standard numerical method for variational data assimilation for a convection-diffusion equation

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Data Assimilation

Given

- Parabolic state equation (with boundary conditions)

$$\mathcal{Y}_t + \mathcal{A}\mathcal{Y} = \mathcal{F}$$

- Distributed measurements $\overline{\mathcal{Y}}$ on (subset) ω

Find

Initial conditions $\mathcal{Y}_0$, s. t. solution $\mathcal{Y}$ of IBVP satisfies $\mathcal{Y}|_\omega = \overline{\mathcal{Y}}$

Ill-posed problem!

Applications

Application

Weather prediction (Navier-Stokes), Geophysics (Boussinesq)

1. Given observations in $[0, T_0]$, compute initial state $\mathcal{Y}_0$ (assimilation)
2. Solve IBVP in $[0, T_1]$, $T_1 > T_0$ (prediction)

Current methods

Tikhonov-regularised optimal control (4DVAR)
Statistical methods (Ensemble Kalman filter)

Forward Data Assimilation

Idea

1. Given observations in $[0, T_0]$, compute *final* state $\mathcal{Y}_{T_0}$ (assimilation)
2. Solve IBVP in $[T_0, T_1]$ (prediction)

$\Rightarrow$ Replace

- ill-posed control problem for state equation

with

- well-posed control problem for adjoint equation

Problem Formulation

$\Omega \subset \mathbb{R}^n$ domain, boundary Γ, $c : \Omega \to \mathbb{R}$, $b : \Omega \times [0, T] \to \mathbb{R}^n$

Convection-Diffusion equation

$$\begin{cases} y_t - c^2 \Delta y + b^T \nabla y = f, & \Omega \times [0, T] \\ y = 0, & \Gamma \times [0, T] \end{cases}$$

$\omega \subset \Omega$ nonempty: **Given $y|_\omega(x, t)$, find $y(T)$!**

Adjoint equation

$$\begin{cases} -\varphi_t - c^2 \Delta \varphi - \operatorname{div}(b\varphi) = v\chi_\omega, & \Omega \times [0, T] \\ \varphi = 0, & \Gamma \times [0, T] \\ \varphi(x, T) = \varphi_T(x), & x \in \Omega \end{cases}$$

χ_ω characteristic function of $\omega \subset \Omega$, control $v : \Omega \times [0, T] \to \mathbb{R}$

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Well-posedness

Theorem (Puel 2002)

Γ class C^2, $f \in L^2(0,T;L^2(\Omega))$, $c \in C^1(\bar{\Omega})$, $b \in L^2(0,T;H^1(\Omega)^n)$

Then: For all $T > 0$, $\omega \subset \Omega$ nonempty, for any $\varphi_T \in L^2(\Omega)$

- there exists $v = v(\varphi_T) \in L^2(0,T;L^2(\Omega))$, s. t. solution of adjoint equation satisfies:

$$\varphi(0) = 0$$

- there exists $C(\Omega, \omega, T) > 0$:

$$\|y(T)\|_{L^2(\Omega)} \leq C \left(\int_0^T \int_\omega |y|^2 dxdt + \int_0^T \int_\Omega |f|^2 dxdt \right)$$

Proof relies on Carleman estimate for 2nd order parabolic equation

Reconstruction Method

Under conditions of last theorem, the following identity holds:

$$\int_\Omega y(T)\varphi_T \, dx = \int_0^T \int_\Omega f\varphi \, dxdt - \int_0^T \int_\omega yv(\varphi_T) \, dxdt$$

for all $\varphi_T \in L^2(\Omega)$ with null controlled (by v) adjoint solution φ

$\Rightarrow$ Reconstruction method for $y(T)$:

Algorithm

Given measurement $y|_\omega$, Hilbert basis $\{\varphi_n\}$ of $L^2(\Omega)$:

1. Calculate null controls $v(\varphi_n)$, adjoint solution φ
2. Calculate coefficients $c_n := \langle y(T), \varphi_n \rangle_{L^2(\Omega)}$
3. Then: $y(T) = \sum_n c_n \varphi_n$

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Exact Distributed Control: Glowinski/Lions

Biadjoint equation

$$\begin{cases} \psi_t - c^2\Delta\psi + b^T\nabla\psi = 0, & \Omega\times[0,T] \\ \psi = 0, & \Gamma\times[0,T] \\ \psi(0) = \psi_0, & \Omega \end{cases}$$

Let $\varphi(0; v)$ solution of adjoint equation controlled by v at $t = 0$

Operator formulation

$$\Lambda : \psi_0 \mapsto \varphi(0; \psi(x, T-t)\chi_\omega)$$

Then: Solution ψ_0^* of $\Lambda\psi_0 = 0$ yields null control $v(\varphi_T) := \psi^*\chi_\omega$

$\Rightarrow$ **Use CG method to compute** ψ_0^*

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Finite Dimensional Approximation

Solve problem in finite dimensional subspace $V_h \subset L^2(\Omega)$:

Algorithm (finite dimensional)

Given discrete measurement $y^h|_\omega$, basis $\{\varphi_n^h\}$ of V_h:

1. Calculate null controls $v^h(\varphi_n^h)$
2. Calculate coefficients

$$c_n := \left\langle f, \varphi_n^h \right\rangle_{V_h \times L^2([0,T])} - \left\langle y^h|_\omega, v^h(\varphi_n^h) \right\rangle_{V_h \times L^2([0,T])}$$

3. Set $y_T^h := \sum_n c_n \varphi_n^h$

Choice of basis

Use Finite Element Space:

- V_h space of piecewise polynomials on mesh
- $\{\varphi_n^h\}$ nodal basis (hat functions)
- Weighted inner product for $x, y \in V_h$

$$\langle x, y \rangle_{V_h} := \xi^T M \eta$$

with $x = \sum \xi_i \varphi_i$, $y = \sum \eta_i \varphi_i$, and

$$M_{ij} := \int_\Omega \varphi_i^h \varphi_j^h$$

Efficient calculation of coefficients, but curse of dimensions!

$\Rightarrow$ **Use model reduction**

Proper Orthogonal Decomposition (POD)

Given set $\{\varphi_i\}_{i=1}^N \subset V_h$, find $l < N$ elements $u_i \in \text{span}\{\varphi_n\}$ solving

$$\max_{u_i \in V_h} \left\{ \sum_{k=1}^{l} \sum_{i=1}^{N} \langle \varphi_i, u_k \rangle_{V_h}^2 \ \text{ s. t. } \ \langle u_i, u_j \rangle_{V_h} = \delta_{ij},\ 1 \le i,j \le l \right\}$$

Matrix representation $\Phi := (\varphi_1 | \cdots | \varphi_N)$, $\varphi_i \in \mathbb{R}^{\dim V_h}$:

Optimality conditions

$$\Phi^T M \Phi v_i = \lambda_i v_i$$

$$u_i := \frac{1}{\sqrt{\lambda_i}} \Phi v_i$$

$\Rightarrow$ Solve symmetric eigenvalue problem

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POD Approximation Error

POD basis $\{u_i\}_{i=1}^{l}$ singular vectors of $M^{\frac{1}{2}}\Phi$

$\Rightarrow \{u_i\}_{i=1}^{l}$ best rank l-approximation of $\{\varphi_i\}_{i=1}^{N}$ (in mean)

Error estimate

V_h Finite Element space, h mesh size, $\mathcal{P}^h$ projector on V_h
$\{u_i\}_{i=1}^{m}$ POD Basis, eigenvalues λ_i, $m = \dim V_h$

$$w_l := \sum_{i=1}^{l} \left\langle \mathcal{P}^h y(T), u_i \right\rangle_{V_h} u_i\,, \qquad 1 \le l \le m$$

Then there exists $C(\Omega, V_h) > 0$, s. t.

$$\|y(T) - w_l\|_{L^2(\Omega)}^2 \le C \left(\sum_{i=l+1}^{m} \lambda_i + h \, \|y(T)\|_{H^1(\Omega)} \right)$$

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Problem specific POD basis

POD basis depends only on Finite Element space:

- Can be precalculated (in parallel)
- But not most efficient: less basis elements sufficient?

$\Rightarrow$ Use optimal problem specific subset

Problem specific POD basis

Iterative POD basis $(\tilde{u}_1, \ldots, \tilde{u}_l)$

1. Calculate FE-POD basis u_i, $i = 1, \ldots, N_0 \leq N$
2. Estimate (e.g. by interpolation) error function

$$e_n := y^h(T) - \sum_{i=1}^{n-1} c_i \tilde{u}_i$$

3. Pick

$$\tilde{u}_n := \underset{u_k \notin \{\tilde{u}_1, \ldots, \tilde{u}_{n-1}\}}{\operatorname{argmax}} \langle e_n, u_k \rangle_{V_h}$$

4. Calculate c_n using exact control
5. Repeat from step 2 while $\|e_n\|_{V_h} >$ tolerance

Test Problem

- Domain $\Omega = [0,1]^2,\ T = 1$
- $c^2 = 0.1,\ b = (1,1)^T$
- Right hand side: $O = \left\{ \sqrt{(x_1 - 0.5)^2 - (x_2 - 0.5)^2} \leq 0.2 \right\}$,

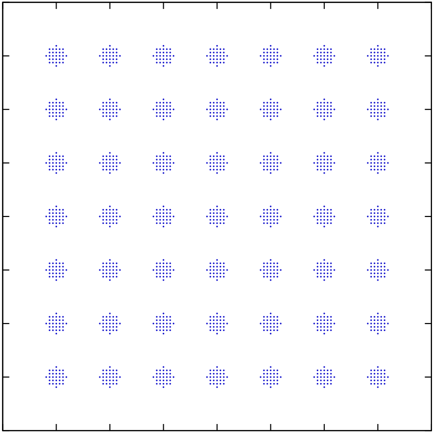
$$f(x,t) = 10\cos(3\pi t)\sqrt{r^2 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2}\chi_O$$

- Initial value

$$y(x,0) = 10\sin(3x_1\pi)\left[\sin(2x_2\pi) + \sin(3x_2\pi) + \sin(4x_2\pi)\right]$$

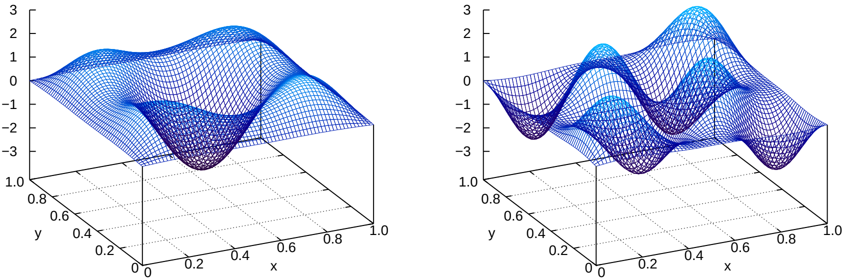
- Discretisation: rectangular grid $h_1 = h_2 = \frac{1}{128},\ h_t = \frac{1}{256}$
- Piecewise bilinear finite elements
- Implementation in `deal.II`

Measurement area



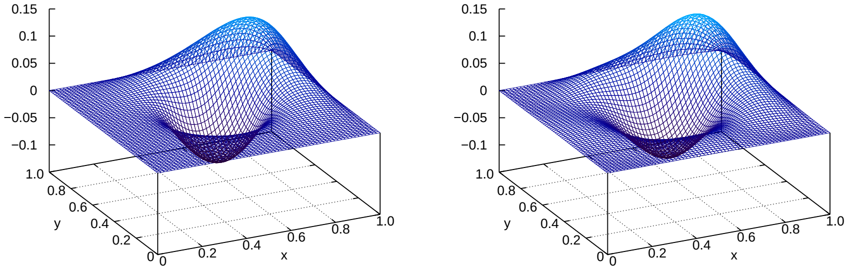
Plot of measurement area ω (blue), $|\omega| \approx 0.087|\Omega|$

POD Basis



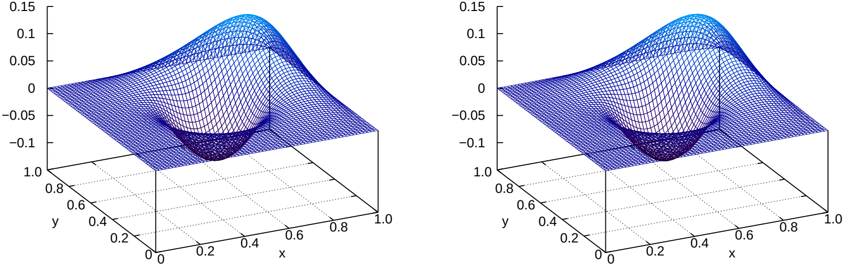
Plot of two POD basis elements φ_5, φ_{12}

Quality of Reconstruction



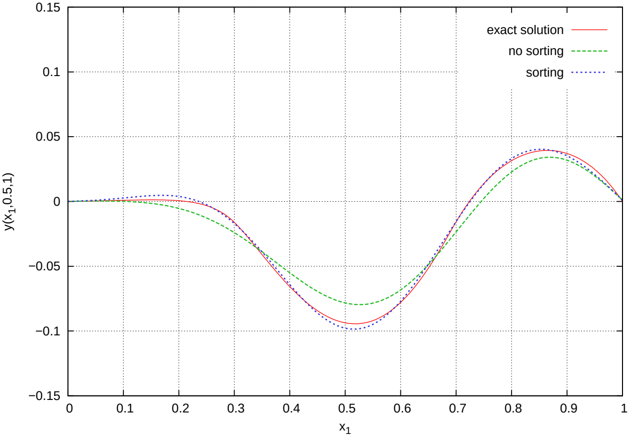
Comparison of exact solution (left) and reconstruction from 10 POD elements (right)

Quality of Reconstruction



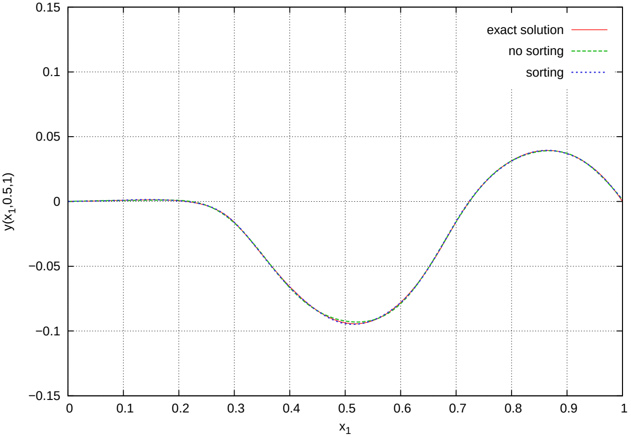
Comparison of exact solution (left) and reconstruction from 100 POD elements (right)

Quality of Reconstruction



Cut of exact solution $y(T)$ and reconstruction from 10 POD elements

Quality of Reconstruction



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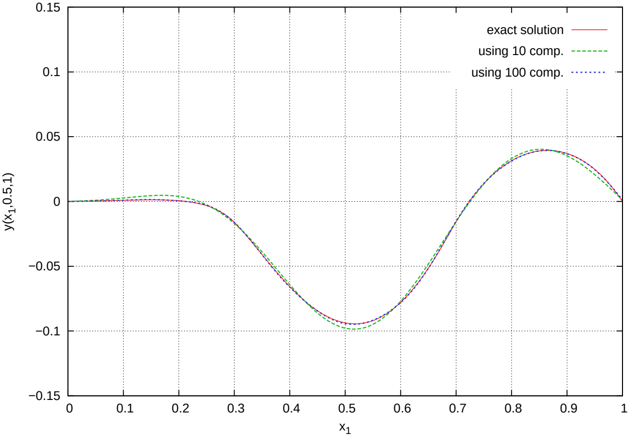
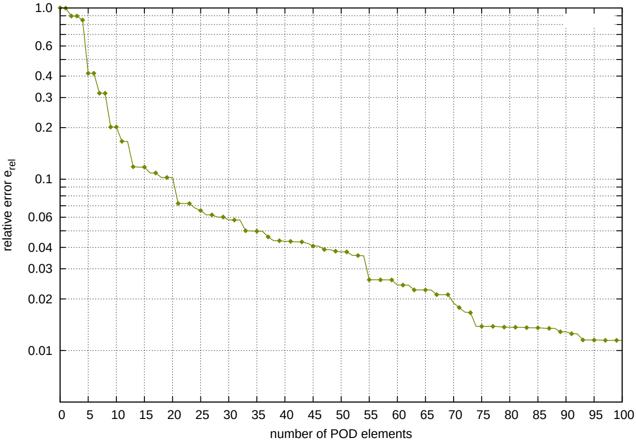


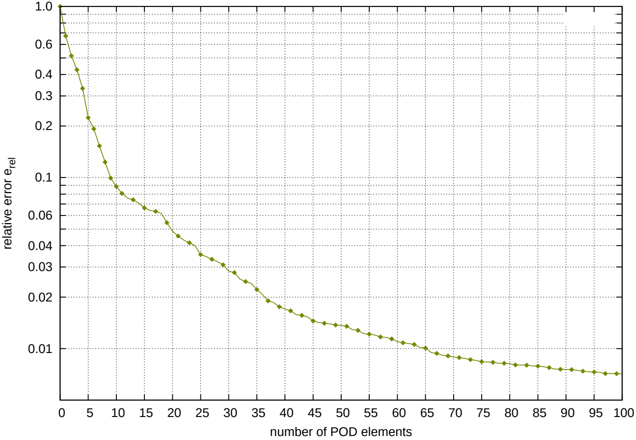
Cut of exact solution $y(T)$ and reconstruction from sorted POD elements

Convergence



Convergence of approximation using unsorted POD elements

Convergence



Convergence of approximation using sorted POD elements

Conclusion

Summary:

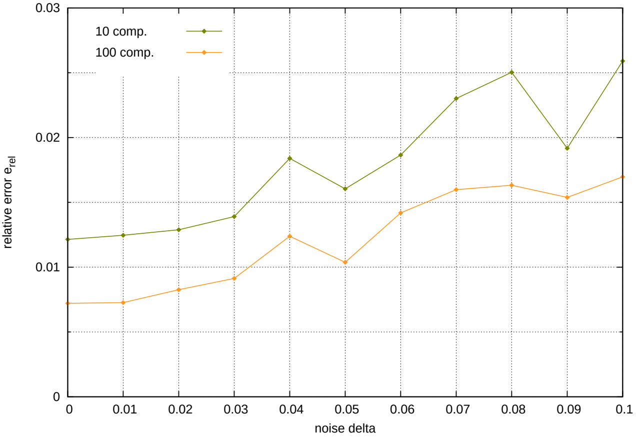
- Reconstructing final conditions is well-posed problem
- Efficiently computable using proper orthogonal decomposition
- Strategies exist for good choice of components
- Fast alternative to 4DVAR

Perspective:

- Adaptive grid refinement: Adaptive POD
- Rates of convergence

Thank you for your attention!

Influence of noise



Reconstruction from noisy measurement

$$y^{\delta}(T) = y^{h}(T) + \delta \| y^{h}(T) \|$$