Avoiding degeneracy in the Westervelt equation by state constrained optimal control

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Motivation

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Westervelt equation

$$\begin{cases} (1-ky)y_{tt} - c^2\Delta y - b\Delta y_t + dy_t - k(y_t)^2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_\nu y = u & \text{on } (0,T)\times\Gamma \\ y_t + c\partial_\nu y = 0 & \text{on } (0,T)\times\hat{\Gamma} \\ y = y^0,\ y_t = y^1 & \text{in } \{0\}\times\Omega \end{cases}$$

- y: acoustic sound fluctuation, b, c, d, k: material parameters
- describes propagation of high intensity focused ultrasound
- u: boundary control: piezoelectric transducer

Motivation

Westervelt equation

$$
\begin{cases}
(1 - ky)y_{tt} - c^2\Delta y - b\Delta y_t + dy_t - k(y_t)^2 = 0 & \text{in } (0,T)\times\Omega, \\
\partial_\nu y = u & \text{on } (0,T)\times\Gamma \\
y_t + c\partial_\nu y = 0 & \text{on } (0,T)\times\hat{\Gamma} \\
y = y^0,\ y_t = y^1 & \text{in } \{0\}\times\Omega
\end{cases}
$$

- y: acoustic sound fluctuation, b, c, d, k: material parameters
- describes propagation of high intensity focused ultrasound
- u: boundary control: piezoelectric transducer
- nonlinear, degenerate for y large

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Westervelt equation

$$\begin{cases} (1-ky)y_{tt} - c^2\Delta y - b\Delta y_t + dy_t - k(y_t)^2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_\nu y = u & \text{on } (0,T)\times\Gamma \end{cases}$$

Well-posedness

- requires a priori smallness bound $\|y\|_\infty < 1/k$
- usually: $C(0,T;H^2)$ bound, embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$
- $\rightsquigarrow$ strong smoothing, requires smallness of data u
- here: impose pointwise constraints $y^- \le y(x) \le y^+ < 1/k$,
  gradient constraint $\|y_t\|_{L^2} \le M$
- data from solving state constrained optimal control problem

Optimal control problem

$$\begin{cases} \min\limits_{y,u} \dfrac{1}{2}\|y - y^d\|_{L^2}^2 + \dfrac{\alpha}{2}\|u\|_U^2 \\ (1-ky)y_{tt} - c^2\Delta y - b\Delta y_t + dy_t - k(y_t)^2 = 0, \\ \partial_\nu y = u \\ y^- \le y(t,x) \le y^+, \qquad \|y_t\|_{L^2} \le M \end{cases}$$

- $y^d \in L^2(0,T;L^2(\Omega))$, $\quad \alpha > 0$
- some smoothness, no smallness assumption on $u \in U$
- existence of local minimizer $(\bar{u}, \bar{y})$ by direct method
- proof requires compact embeddings, regularity results (technical, not in this talk)
- characterisation of admissible data: optimality conditions

Optimality conditions

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Abstract problem

$$\min_{u,y} J(u,y) \quad \text{s.t.} \quad G(u,y) = 0, \quad H(y) \in K$$

Classical constraint qualification (e.g., [Maurer/Zowe, Tröltzsch '10]):

- introduce control-to-state mapping $S : u \mapsto y$ with $G(u,y) = 0$
- consider reduced problem $j(u) = J(u, S(u))$
- interior point condition: $H(S(u_0)) \in K^o$

Then:

- $j'(\bar{u}) + S'(\bar{u})^* H'(S(\bar{u})) = 0$

But: requires well-posedness of $S \rightsquigarrow$ smallness of u

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Abstract problem

$$\min_{u,y} J(u,y) \quad \text{s.t.} \quad G(u,y)=0, \quad H(y)\in K$$

Alternative [Alibert/Raymond '98]: if there exist u_0, y_0 with

- $G_u(\bar u,\bar y)(u_0-\bar u)+G_y(\bar u,\bar y)y_0=0 \quad \text{with} \quad H(\bar y)+H'(\bar y)y_0\in K^o$
- $G_y(\bar u,\bar y)$ surjective

Then:

- $J_y(\bar u,\bar y)+G_y(\bar u,\bar y)^*p+H'(\bar y)^*\mu=0$
- $J_u(\bar u,\bar y)+G_u(\bar u,\bar y)p=0$
- $\langle\mu, w-H(\bar y)\rangle\leq 0$ for all $w\in K$.

But: u acts only on boundary $\rightsquigarrow$ difficult to satisfy

Optimality conditions

Abstract problem

$$\min_{u,y} J(u,y) \quad \text{s.t.} \quad G(u,y) = 0, \quad H(y) \in K$$

Alternative: Relaxed problem [Bonnans/Casas '89]

1. introduce $z = (1 - ky)$, $w = -k(y_t)^2$ in G
2. penalize $z - (1 - ky)$, $w + k(y_t)^2$ in J
3. obtain optimality conditions (easier for new G)
4. pass to limit in penalization

- but: uses optimality of $(\bar{u}, \bar{y})$, only local minimizers
- $\rightsquigarrow$ localize by penalty on $u - \bar{u}$, $y - \bar{y}$ ([Casas/Tröltzsch '02])

Relaxed problem

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$$\min_{u,y,z,w} J^\varepsilon(u,y,z,w) \quad \text{s.t.} \quad G^\varepsilon(u,y,z,w) = 0, \quad H^\varepsilon(y,z) \in K$$

$$\begin{aligned} J^\varepsilon(u,y,z,w) = {} & \frac{1}{2}\|y - y^d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_U^2 \\ & + \frac{1}{2\varepsilon}\|z + ky - 1\|_Z^2 + \frac{1}{2\varepsilon}\|w + k(y_t)^2\|_W^2 \\ & + \frac{1}{2\delta}\|u - \bar{u}\|_U^2 + \frac{1}{2\delta}\|z + k\bar{y} - 1\|_Z^2 + \frac{1}{2\delta}\|w + k(\bar{y}_t)^2\|_W^2 \end{aligned}$$

- U, Z, W suitable Hilbert spaces (needed for surjectivity)
- $\delta > 0$ sufficiently small, fixed

Relaxed problem

$$\min_{u,y,z,w} J^{\varepsilon}(u,y,z,w) \quad \text{s.t.} \quad G^{\varepsilon}(u,y,z,w) = 0, \quad H^{\varepsilon}(y,z) \in K$$

$$G^{\varepsilon}(u,y,z,w) = \begin{pmatrix} zy_{tt} - c^2\Delta y - b\nabla y_t + w \\ \partial_{\nu} y - u \end{pmatrix}$$

$$H_{\varepsilon}(y,z) = \begin{pmatrix} 1 - z - ky^{+} \\ z - ky^{-} - 1 \\ \|y_t\|_{L^2} - M \end{pmatrix}$$

- K negative cone in $C \times C \times \mathbb{R}$

Relaxed problem

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$$\min_{u,y,z,w} J^\varepsilon(u,y,z,w) \quad \text{s.t.} \quad G^\varepsilon(u,y,z,w) = 0, \quad H^\varepsilon(y,z) \in K$$

- Existence of global solution $x_\varepsilon := (u_\varepsilon, y_\varepsilon, z_\varepsilon, w_\varepsilon)$
- Localization: x_ε close to $(\bar{u}, \bar{y})$
- (Regularity of solution)

Regularity condition 1

There exists (u_0, y_0, z_0, w_0) with

$$G_u^\varepsilon(x_\varepsilon)(u_0 - u_\varepsilon) + G_y^\varepsilon(x_\varepsilon)y_0 + G_z^\varepsilon(x_\varepsilon)z_0 + G_w^\varepsilon(x_\varepsilon)(w_0 - w_\varepsilon) = 0$$

- $z_\varepsilon(y_0)_{tt} - c^2\Delta y_0 - b\nabla(y_0)_t + z_0(y_\varepsilon)_{tt} + (w_0 - w_\varepsilon) = 0$
- $\partial_\nu y_0 = (u_0 - u_\varepsilon) = 0$
- $1 + ky^- < z_0 + z^\varepsilon < 1 + ky^+, \qquad \|(y_0 + y_\varepsilon)_t\|_{L^2} < M$
- (necessary regularity)

Satisfied for, e.g.,

$$u_0 = u_\varepsilon, \quad y_0 = -y_\varepsilon, \quad z_0 = 1 - z_\varepsilon, \quad w_0 = -(1 - z_\varepsilon)(y_\varepsilon)_{tt} + w_\varepsilon$$

Regularity condition 2

$G_{(y,z)}(x_\varepsilon)$ is surjective: there is solution (y,z) for given f of

$$\begin{cases} z(y_\varepsilon)_{tt} + z_\varepsilon y_{tt} - c^2\Delta y - b\Delta y_t + dy_t = f \\ \partial_\nu y = 0 \text{ on } (0,T)\times\Gamma, \\ y_t + c\partial_\nu y = 0 \text{ on } (0,T)\times\hat{\Gamma}, \\ y = 0,\ y_t = 0 \text{ in } \{0\}\times\Omega \end{cases}$$

- linearized Westervelt equation
- well-posed (proof similar to [C./Kaltenbacher/Veljović '09])
- (necessary regularity)

Relaxed optimality system

There exist p_ε, measures $\mu_\varepsilon^+, \mu_\varepsilon^- \geq 0$, $\lambda_\varepsilon \geq 0$ satisfying

Relaxed state equation (weak form)

$$\langle z_\varepsilon (y_\varepsilon)_{tt}, v\rangle + \langle c^2 \nabla y_\varepsilon + b\nabla (y_\varepsilon)_t, \nabla v\rangle + \langle d(y_\varepsilon)_t + w, v\rangle + \langle c y_t + \frac{b}{c} y_{tt}, v\rangle_{\hat{\Gamma}} = \langle c^2 u_\varepsilon + b(u_\varepsilon)_t, v\rangle_\Gamma$$

Relaxed optimality system

There exist p_ε, measures $\mu_\varepsilon^+, \mu_\varepsilon^- \geq 0$, $\lambda_\varepsilon \geq 0$ satisfying

Relaxed adjoint equation

$$\begin{aligned}\langle z_\varepsilon p_\varepsilon, v_{tt}\rangle + \langle c^2 \nabla p_\varepsilon, \nabla v\rangle + \langle b \nabla p_\varepsilon, \nabla v\rangle + \langle d p_\varepsilon, v_t\rangle \\ + \langle c p_\varepsilon, v_t\rangle_{\hat{\Gamma}} + \langle \frac{b}{c} p_\varepsilon, v_{tt}\rangle_{\hat{\Gamma}} = \langle y^d - y_\varepsilon, v\rangle \\ - \frac{1}{\varepsilon}\langle z_\varepsilon + k y_\varepsilon - 1, kv\rangle_Z \\ - \frac{1}{\varepsilon}\langle w_\varepsilon + k(y_\varepsilon)_t^2), 2k y_\varepsilon v_t\rangle_W \\ - 2\lambda_\varepsilon \langle \|(y_\varepsilon)_t(t)\|_{L^2}^2 (y_\varepsilon)_t, v_t\rangle\end{aligned}$$

- $\langle \cdot, \cdot \rangle_X$ inner product in Hilbert space X

Relaxed optimality system

There exist p_ε, measures $\mu_\varepsilon^+, \mu_\varepsilon^- \geq 0$, $\lambda_\varepsilon \geq 0$ satisfying

Stationarity in z

$$\frac{1}{\varepsilon}\langle z_\varepsilon + k y_\varepsilon - 1, v\rangle_Z + \frac{1}{\delta}\langle z_\varepsilon + k\bar{y} - 1, v\rangle_Z + \langle (y_\varepsilon)_{tt} p_\varepsilon, v\rangle = \mu_\varepsilon^+(v) - \mu_\varepsilon^-(v)$$

Stationarity in w

$$\frac{1}{\varepsilon}\langle w_\varepsilon + k(y_\varepsilon)_t^2, v\rangle_W + \frac{1}{\delta}\langle w_\varepsilon + k(\bar{y}_t)^2, v\rangle_W + \langle p_\varepsilon, v\rangle = 0$$

Relaxed optimality system

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There exist p_ε, measures $\mu_\varepsilon^+, \mu_\varepsilon^- \geq 0$, $\lambda_\varepsilon \geq 0$ satisfying

Optimality

$$\langle \alpha u_\varepsilon \rangle_U + \frac{1}{\delta}\langle u_\varepsilon - \bar{u}, v\rangle_U = \langle c^2 p_\varepsilon, v\rangle_\Gamma + \langle b p_\varepsilon, v_t\rangle_\Gamma$$

Complementarity

$$\mu_\varepsilon^+(z_\varepsilon + k y^+ - 1) = 0, \qquad \mu_\varepsilon^-(z_\varepsilon - k y^- - 1) = 0,$$

$$\lambda_\varepsilon(\|(y_\varepsilon)_t\|_{L^2}^2 - M) = 0$$

Passing to the limit: minimizers

Consider global minimizers $x_\varepsilon = (u_\varepsilon, y_\varepsilon, z_\varepsilon, w_\varepsilon)$ as $\varepsilon \to 0$

1. Optimality of x_ε:

$$J^\varepsilon(u_\varepsilon, y_\varepsilon, z_\varepsilon, w_\varepsilon) \leq J^\varepsilon(\bar{u}, \bar{y}, 1 - k\bar{y}, k(\bar{y}_t)^2) = J(\bar{u}, \bar{y})$$

$\rightsquigarrow \{x_\varepsilon\}_{\varepsilon>0}$ bounded, subsequence $x_{\varepsilon_n} \rightharpoonup \hat{x}$

2. Boundedness of penalty terms:

$$\|z_\varepsilon + ky_\varepsilon - 1\|_Z \to 0, \qquad \|w_\varepsilon + k(y_\varepsilon)_t^2\|_W \to 0$$

$\rightsquigarrow \hat{w} + k(\hat{y}_t)^2 = 0, \qquad \hat{z} + k\hat{y} - 1 = 0$

3. Compact embeddings: $y_\varepsilon \to \hat{y}, \quad z_\varepsilon \to \hat{z}$ pointwise (along further subsequence)

Passing to the limit: minimizers

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3. Compact embeddings: $y_\varepsilon \to \hat{y}$, $z_\varepsilon \to \hat{z}$ pointwise (along further subsequence)

4. Hence: $z_\varepsilon (y_\varepsilon)_{tt} \to \hat{z}\hat{y}_{tt}$,

$$G(\hat{u}, \hat{y}) = G^\varepsilon(\hat{u}, \hat{y}, \hat{z}, \hat{w}) = \lim_{n\to\infty} G^\varepsilon(u_{\varepsilon_n}, y_{\varepsilon_n}, z_{\varepsilon_n}, w_{\varepsilon_n}) = 0$$

5. Also: $H(\hat{y}) = H^\varepsilon(\hat{y}, \hat{z}) \in K$

6. $\rightsquigarrow (\hat{u}, \hat{y})$ feasible for original problem, close to $(\bar{u}, \bar{y})$

Passing to the limit: minimizers

6. $\rightsquigarrow (\hat{u}, \hat{y})$ feasible for original problem, close to $(\bar{u}, \bar{y})$

7. Local optimality of $(\bar{u}, \bar{y})$, bound on $J^\varepsilon(x_\varepsilon)$:

$$
\begin{aligned}
J(\hat{u}, \hat{y}) \geq J(\bar{u}, \bar{y}) &\geq \liminf_{\varepsilon \to 0} J^\varepsilon(u_\varepsilon, y_\varepsilon, z_\varepsilon, w_\varepsilon) \\
&\geq \liminf_{\varepsilon \to 0} J(u_\varepsilon, y_\varepsilon) + \frac{1}{2\delta} \|u_\varepsilon - \bar{u}\|_U^2 \\
&\quad + \frac{1}{2\delta} \|z_\varepsilon + k\bar{y} - 1\|_Z^2 + \frac{1}{2\delta} \|w_\varepsilon + k(\bar{y}_t)^2\|_W^2 \\
&\geq J(\hat{u}, \hat{y}) + \frac{1}{2\delta} \|\hat{u} - \bar{u}\|_U^2 \\
&\quad + \frac{1}{2\delta} \|\hat{z} + k\bar{y} - 1\|_Z^2 + \frac{1}{2\delta} \|\hat{w} + k(\bar{y}_t)^2\|_W^2
\end{aligned}
$$

8. $\rightsquigarrow \hat{u} = \bar{u}, \quad k(\hat{y}_t)^2 = \hat{w} = -k(\bar{y}_t)^2, \quad 1 - k\hat{y} = \hat{z} = 1 - k\bar{y}$

Passing to the limit: minimizers

8. $\rightsquigarrow \hat{u} = \bar{u}, \quad k(\hat{y}_t)^2 = \hat{w} = -k(\bar{y}_t)^2, \quad 1 - k\hat{y} = \hat{z} = 1 - k\bar{y}$

9. $\rightsquigarrow$ strong convergence of $u_\varepsilon, w_\varepsilon, z_\varepsilon$

- $u_\varepsilon \to \bar{u}$ in U
- $w_\varepsilon \to k(\bar{y}_t)^2$ in W
- $z_\varepsilon \to 1 - k\bar{y}$ in Z
- $y_\varepsilon \to \bar{y}$ pointwise

Passing to the limit: optimality system

Consider optimality system for $x_\varepsilon = (u_\varepsilon, y_\varepsilon, z_\varepsilon, w_\varepsilon)$ as $\varepsilon \to 0$

1. Stationarity in $z_\varepsilon, w_\varepsilon$:

$$\frac{1}{\varepsilon}\langle z_\varepsilon + ky_\varepsilon - 1, kv\rangle_Z = -\frac{1}{\delta}\langle z_\varepsilon + k\bar{y} - 1, kv\rangle_Z - \langle (y_\varepsilon)_{tt} p_\varepsilon, kv\rangle + \mu_\varepsilon^+(kv) - \mu_\varepsilon^-(kv)$$

$$\frac{1}{\varepsilon}\langle w_\varepsilon + k(y_\varepsilon)_t^2, 2k(y_\varepsilon)_t v_t\rangle_W = -\frac{1}{\delta}\langle w_\varepsilon + k(\bar{y}_t)^2, 2k(y_\varepsilon)_t v_t\rangle_W - \langle p_\varepsilon, 2k(y_\varepsilon)_t v_t\rangle$$

2. Insert in relaxed adjoint equation:

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2. Insert in relaxed adjoint equation:

$$
\begin{aligned}
&\langle z_\varepsilon p_\varepsilon, v_{tt}\rangle + \langle c^2 \nabla p_\varepsilon, \nabla v\rangle + \langle b\nabla p_\varepsilon, \nabla v\rangle + \langle d p_\varepsilon, v_t\rangle \\
&\qquad + \langle c p_\varepsilon, v_t\rangle_{\hat\Gamma} + \langle \frac{b}{c} p_\varepsilon, v_{tt}\rangle_{\hat\Gamma} = \langle y^d - y_\varepsilon, v\rangle \\
&\qquad\quad + \frac{1}{\delta}\langle z_\varepsilon + k\bar{y} - 1, kv\rangle_Z + \langle (y_\varepsilon)_{tt} p_\varepsilon, kv\rangle \\
&\qquad\quad + \mu_\varepsilon^+(kv) - \mu_\varepsilon^-(kv) \\
&\qquad\quad + \frac{1}{\delta}\langle w_\varepsilon + k(\bar{y}_t)^2, 2k(y_\varepsilon)_t v_t\rangle_W + \langle p_\varepsilon, 2k(y_\varepsilon)_t v_t\rangle \\
&\qquad\quad - 2\lambda_\varepsilon \langle \|(y_\varepsilon)_t(t)\|_{L^2}^2 (y_\varepsilon)_t, v_t\rangle
\end{aligned}
$$

Passing to the limit: optimality system

3. First: assume $\{p_\varepsilon, \mu_\varepsilon^+, \mu^-, \lambda_\varepsilon\}_{\varepsilon>0}$ bounded $\rightsquigarrow$ subsequence

$$p_{\varepsilon_n} \rightharpoonup \bar{p}, \qquad \mu_{\varepsilon_n}^+ \rightharpoonup^\star \bar{\mu}^+, \qquad \mu_{\varepsilon_n}^- \rightharpoonup^\star \bar{\mu}^-, \qquad \lambda_\varepsilon \to \bar{\lambda}$$

4. Strong convergence of $y_\varepsilon, z_\varepsilon, w_\varepsilon$ (and embeddings)

Adjoint equation

$$\begin{aligned}\langle (1-k\bar{y})\bar{p}, v_{tt}\rangle &+ \langle c^2\nabla\bar{p}, \nabla v\rangle + \langle b\nabla\bar{p}, \nabla v\rangle + \langle d\bar{p}, v_t\rangle \\ &+ \langle c\bar{p}, v_t\rangle_{\hat{\Gamma}} + \langle \frac{b}{c}\bar{p}, v_{tt}\rangle_{\hat{\Gamma}} + \langle 2k\bar{y}_t\bar{p}, v_t\rangle + \langle k\bar{y}_{tt}\bar{p}, v_t\rangle \\ &= \langle y^d - \bar{y}, v\rangle + \bar{\mu}^+(kv) - \bar{\mu}^-(kv) \\ &\quad - 2\lambda_\varepsilon \langle \|(\bar{y})_t(t)\|_{L^2}^2 (\bar{y})_t, v_t\rangle\end{aligned}$$

Passing to the limit: optimality system

3. First: assume $\{p_\varepsilon, \mu_\varepsilon^+, \mu^-, \lambda_\varepsilon\}_{\varepsilon>0}$ bounded $\rightsquigarrow$ subsequence

$$p_{\varepsilon_n} \rightharpoonup \bar{p}, \qquad \mu_{\varepsilon_n}^+ \rightharpoonup^\star \bar{\mu}^+, \qquad \mu_{\varepsilon_n}^- \rightharpoonup^\star \bar{\mu}^-, \qquad \lambda_\varepsilon \to \bar{\lambda}$$

4. Strong convergence of $y_\varepsilon, z_\varepsilon, w_\varepsilon$ (and embeddings)

Optimality

$$\langle \alpha \bar{u}, v\rangle_U = \langle c^2 \bar{p}, v\rangle_\Gamma + \langle b\bar{p}, v_t\rangle_\Gamma$$

Complementarity

$$\bar{\mu}^+(\bar{y} - \bar{y}^+) = 0, \qquad \bar{\mu}^-(\bar{y} + \bar{y}^-) = 0,$$
$$\bar{\lambda}(\|\bar{y}_t\|_{L^2}^2 - M) = 0$$

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5. If $\{p_\varepsilon, \mu_\varepsilon^+, \mu_\varepsilon^-, \lambda_\varepsilon\}_{\varepsilon>0}$ not bounded: define

$$r_\varepsilon := \|p_\varepsilon\|_P + \|\mu_\varepsilon^+\|_{\mathcal{M}} + \|\mu_\varepsilon^-\|_{\mathcal{M}} + |\lambda_\varepsilon|$$

$$\{\underline{p}_\varepsilon, \underline{\mu}_\varepsilon^+, \underline{\mu}_\varepsilon^-, \underline{\lambda}_\varepsilon\}_{\varepsilon>0} := \{p_\varepsilon/r_\varepsilon, \mu_\varepsilon^+/r_\varepsilon, \mu_\varepsilon^-/r_\varepsilon, \lambda_\varepsilon/r_\varepsilon\}_{\varepsilon>0}$$

6. $\rightsquigarrow$ bounded; subsequence converges to $(\bar{p}, \bar{\mu}^+, \bar{\mu}^-, \bar{\lambda})$

7. Insert in relaxed adjoint equation

Passing to the limit: optimality system

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6. $\rightsquigarrow$ bounded; subsequence converges to $(\bar{p}, \bar{\mu}^+, \bar{\mu}^-, \bar{\lambda})$

7. Insert in relaxed adjoint equation

$$
\begin{aligned}
&\langle z_\varepsilon \underline{p}_\varepsilon, v_{tt} \rangle + \langle c^2 \nabla \underline{p}_\varepsilon, \nabla v \rangle + \langle b \nabla \underline{p}_\varepsilon, \nabla v \rangle + \langle d \underline{p}_\varepsilon, v_t \rangle \\
&\quad + \langle c \underline{p}_\varepsilon, v_t \rangle_{\hat{\Gamma}} + \langle \frac{b}{c} \underline{p}_\varepsilon, v_{tt} \rangle_{\hat{\Gamma}} = \frac{1}{r_\varepsilon} \langle y^d - y_\varepsilon, v \rangle \\
&\quad + \frac{1}{r_\varepsilon \delta} \langle z_\varepsilon + k\bar{y} - 1, kv \rangle_Z + \langle (y_\varepsilon)_{tt} \underline{p}_\varepsilon k, v \rangle \\
&\quad + \underline{\mu}_\varepsilon^+ (kv) - \underline{\mu}_\varepsilon^- (kv) \\
&\quad + \frac{1}{r_\varepsilon \delta} \langle w_\varepsilon + k(\bar{y}_t)^2, 2k(y_\varepsilon)_t v_t \rangle_W + \langle \underline{p}_\varepsilon, 2k(y_\varepsilon)_t v_t \rangle \\
&\quad - 2 \underline{\lambda}_\varepsilon \langle \| (y_\varepsilon)_t(t) \|_{L^2}^2 (y_\varepsilon)_t, v_t \rangle
\end{aligned}
$$

Passing to the limit: optimality system

6. $\rightsquigarrow$ bounded; subsequence converges to $(\bar{p}, \bar{\mu}^+, \bar{\mu}^-, \bar{\lambda})$

7. Insert in relaxed adjoint equation

Adjoint equation

$$
\begin{aligned}
\langle (1 - k\bar{y})\bar{p}, v_{tt}\rangle + \langle c^2 \nabla \bar{p}, \nabla v\rangle + \langle b \nabla \bar{p}, \nabla v\rangle + \langle d\bar{p}, v_t\rangle \\
+ \langle c\bar{p}, v_t\rangle_{\hat{\Gamma}} + \langle \frac{b}{c}\bar{p}, v_{tt}\rangle_{\hat{\Gamma}} + \langle 2k\bar{y}_t\bar{p}, v_t\rangle + \langle k\bar{y}_{tt}\bar{p}, v_t\rangle \\
= \bar{\mu}^+(kv) - \bar{\mu}^-(kv) \\
- 2\lambda_\varepsilon \langle \|(\bar{y})_t(t)\|^2_{L^2} (\bar{y})_t, v_t\rangle
\end{aligned}
$$

8. Insert into relaxed stationarity condition

$$\frac{1}{r_\varepsilon}\langle \alpha u_\varepsilon \rangle_U + \frac{1}{r_\varepsilon \delta}\langle u_\varepsilon - \bar{u}, v\rangle_U = \langle c^2 \underline{p}_\varepsilon, v\rangle_\Gamma + \langle b\underline{p}_\varepsilon, v_t\rangle_\Gamma$$

Passing to the limit: optimality system

8. Insert into relaxed stationarity condition

Optimality

$$\langle c^2\bar{p}, v\rangle_\Gamma + \langle b\bar{p}, v_t\rangle_\Gamma = 0$$

Complementarity

$$\bar{\mu}^+(\bar{y} - \bar{y}^+) = 0, \qquad \bar{\mu}^-(\bar{y} + \bar{y}^-) = 0,$$
$$\bar{\lambda}(\|\bar{y}_t\|_{L^2}^2 - M) = 0$$

Conclusion

Optimal control of singular equation with state constraints

- yields large solution to Westervelt equation
- requires no smallness assumption on data
- is of practical interest (lithotripsy)
- is applicable to other problems in nonlinear acoustics

Preprints:
http://www.uni-due.de/agclason/clason_pubs.php