Bouligand–Landweber iteration for a non-smooth ill-posed problem

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Open-Minded

Inverse problem: find

- unknown parameter u
  e.g., heat source, diffusion constant, thermal conductivity, heat capacity, latent heat density, …

given

- measurement y
- model $S : u \mapsto y$, e.g., solution of PDE

$\rightsquigarrow$ solve

$$S(u) = y$$

Problem: measurement $y = y^\delta$ noisy, range of S not closed

$\rightsquigarrow$ ill-posed, needs regularization

Regularization

Solve approximate, stable problem:

1. Tikhonov regularization $\rightsquigarrow$ optimal control
2. iterative regularization, e.g., Landweber iteration

$$u_{n+1} = u_n + w_n S'(u_n)^*(y^\delta - S(u_n)) \qquad n = 1, \dots, N$$

- $S'(u)$ Fréchet derivative, $S'(u)^*$ adjoint
- stopping index $N = N(\delta) < \infty$ regularization parameter
- regularization: $N(\delta) \to \infty$, $u_{N(\delta)} \to u^\dagger$ for $\delta \to 0$

Here: S solution operator for non-smooth PDE

$\rightsquigarrow$ not Fréchet differentiable

Non-smooth equation

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$$-\Delta y + \max(0, y) = u$$

solution operator $S : u \mapsto y \ \ (:= S(u))$

- well-posed (in suitable – standard – spaces)
- Lipschitz continuous
- completely continuous ($\rightsquigarrow$ ill-posed)
- not Fréchet differentiable unless $|\{x : y(x) = 0\}| = 0$

- model for membrane under water
- can be extended to arbitrary $f(y)$ piecewise differentiable
- simplified model for sharp phase transition (Stefan problem)

Non-smooth equation

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$$-\Delta y + \max(0, y) = u$$

solution operator $S : u \mapsto y \ \ (:= S(u))$

- not Fréchet differentiable unless $|\{x : y(x) = 0\}| = 0$
- but: directionally differentiable

Directional derivative $S'(u; h) =: \eta$ solves

$$-\Delta \eta + \mathbb{1}_{\{y=0\}} \max(0, \eta) + \mathbb{1}_{\{y>0\}} \eta = h$$

not linear in $h \rightsquigarrow$ not useful for algorithm

Non-smooth equation

Bouligand subdifferential

$$\partial_B S(u) := \left\{ G \text{ linear} \;\middle|\; \begin{array}{l} \text{there is } \{u_n\} \text{ Gâteaux differentiable with} \\ u_n \to u \text{ and } S'(u_n; h) \to G\,h \text{ for all } h \end{array} \right\}$$

$$-\Delta\eta + \mathbb{1}_{\{y>0\}}\eta = h$$

$G_u : h \mapsto \eta$

- $G_u \in \partial_B S(u)$ Bouligand derivative [Christof/Meyer/Walter/C.]
- $u \mapsto G_u$ uniformly bounded (in right spaces)
- linear $\rightsquigarrow$ use for Landweber in place of $S'(u)$

1 Motivation

2 Non-smooth equation

3 Bouliganda–Landweber iteration

4 Numerical examples

Bouligand–Landweber iteration

$$u^\delta_{n+1} = u^\delta_n + w_n G^*_{u^\delta_n}\left(y^\delta - S(u^\delta_n)\right), \quad n = 0, 1, 2, \dots, N(\delta)$$

- $S : u \mapsto y$ non-smooth
- y^δ with $\|y^\delta - y^\dagger\| \leqslant \delta, \quad y^\dagger = S(u^\dagger)$ (assume unique)
- $u^\delta_0 = u_0$ starting point
- w_n step sizes
- stopping index $N(\delta)$ by discrepancy principle:

$$\|y^\delta - S(u^\delta_{N(\delta)})\|_Y \leqslant \tau\delta < \|y^\delta - S(u^\delta_n)\|_Y, \quad 0 \leqslant n < N(\delta)$$

(modified Landweber iteration [Scherzer '95])

Well-posedness

Assume:

1. $\{G_u\}$ uniformly bounded
2. generalized tangential cone condition (GTCC)

$$\|S(u') - S(u) - G_u(u' - u)\| \leqslant \mu\|S(u') - S(u)\| \quad \text{for all } u, u' \in B_\rho(u^\dagger)$$

non-smooth PDE: satisfied for $1 > \mu > C(\{x : y^\dagger(x) = 0|\})$

3. $u_0 \in B_\rho(u^\dagger)$

Then (under conditions on μ, τ, w_n):

- $u_n^\delta \in B_\rho(u^\dagger)$ for all $n \leqslant N(\delta)$
- $\delta > 0$: $N(\delta) < \infty$ and $\|u_n^\delta - u^\dagger\| < \|u_{n-1}^\delta - u^\dagger\|$ for $n \leqslant N(\delta)$
- $\delta = 0$: $N(\delta) = \infty$ and $u_n^0 \to u^\dagger$ for $n \to \infty$

Regularization

Goal: show that $u^\delta_{N(\delta)} \to u^\dagger$ for $\delta \to 0$

Standard proof: combine

1. monotonicity: $\|u^\delta_n - u^\dagger\| < \|u^\delta_{n-1} - u^\dagger\|$ for $n \leqslant N(\delta)$
2. stability: $u^\delta_n \to u^0_n$ for all $n = 1, \ldots$

Problem:

- stability requires continuity of $u \mapsto G_u$
- $u \mapsto G_u$ not continuous for S non-smooth
- $\rightsquigarrow$ use asymptotic stability

Asymptotic stability

Definition

Iterative method generating $\{u_n^\delta\}_{n\leqslant N(\delta)}$ asymptotically stable for $\delta \to 0$ if exists subsequence $\{\delta_k\}$ with:

- For all $0 \leqslant n \leqslant \overline{N} := \lim_{k\to\infty} N(\delta_k) \in \mathbb{N} \cup \{\infty\}$

  $$u_n^{\delta_k} \to \tilde{u}_n \quad \text{as } k \to \infty$$

  for some $\tilde{u}_n \in \overline{B}_U(u^\dagger, \rho)$
- If $\overline{N} = \infty$,

  $$\tilde{u}_n \to u^\dagger \quad \text{as } n \to \infty$$

- $\tilde{u}_n$ generated by perturbed noise-free iteration
- perturbation needs to vanish for $n \to \infty$

Asymptotic stability

Bouligand–Landweber iteration

$$u_{n+1}^\delta = u_n^\delta + w_n G_{u_n^\delta}^* \left(y^\delta - S(u_n^\delta)\right), \quad n = 0, 1, 2, \ldots, N(\delta)$$

- asymptotically stable for non-smooth PDE

  (proof uses GTCC and compact embedding for $\mathcal{R}(G_u^*)$)
- $\rightsquigarrow$ regularization (under conditions on μ, τ, w_n):

$$u_{N(\delta)}^\delta \to u^\dagger \quad \text{for} \quad \delta \to 0$$

Numerical example

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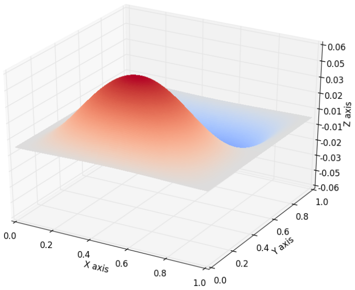
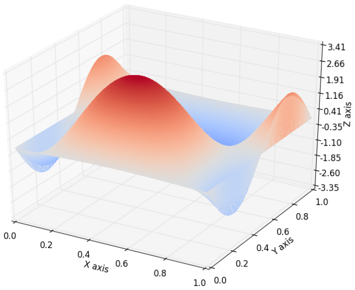
$$-\Delta y + \max(0, y) = u$$

- finite element discretization
- semismooth Newton method for solution (evaluation of S)
- constructed exact solution $u^\dagger$ with $|\{x : y^\dagger(x) = 0\}| > 0$
- random Gaussian noise: $\|y^\delta - y^\dagger\| = \delta$
- $\mu = 0.1, \quad \tau = 1.4, \quad \rho = 5, \quad w_n = \frac{2-2\mu}{\bar{L}^2}, \quad \bar{L} = 5 \times 10^{-2}$
- compare two starting points:
  1. $u_0 \equiv 0$
  2. $\bar{u}_0$ satisfying $u^\dagger - u_0 \in \mathcal{R}(G^*_{u^\dagger})$ (generalized source condition)

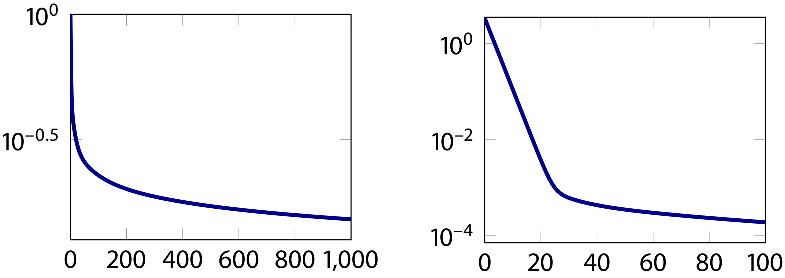


(a) exact data $y^\dagger$



(b) exact solution $u^\dagger$

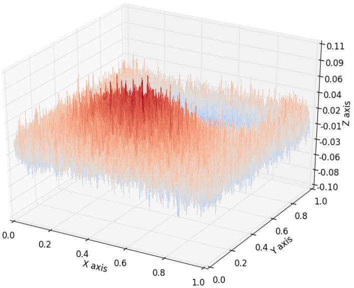
Numerical example: noise-free convergence



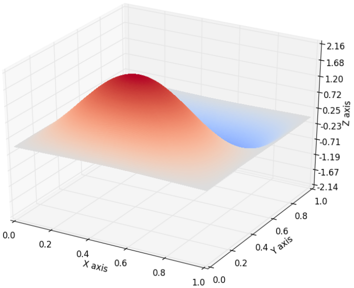
(a) starting point $u_0 \equiv 0$ (b) starting point $\bar{u}_0$

Figure: relative error per iteration

Numerical example: results with u_0

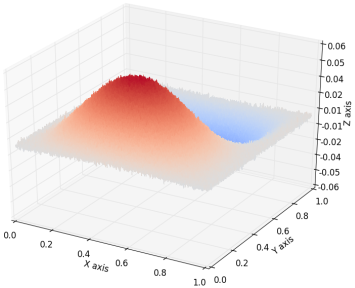


(a) y^δ, $\delta \approx 10^{-2}$

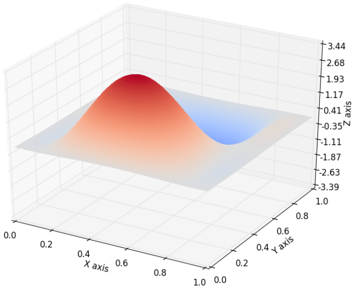


(b) $u^\delta_{N(\delta)}$, $N(\delta) = 4$

Numerical example: results with u_0

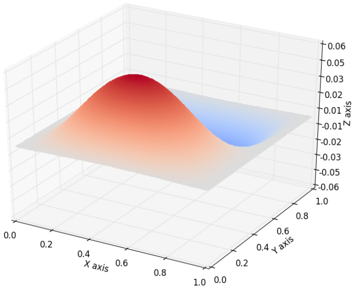


(a) y^δ, $\delta \approx 10^{-3}$

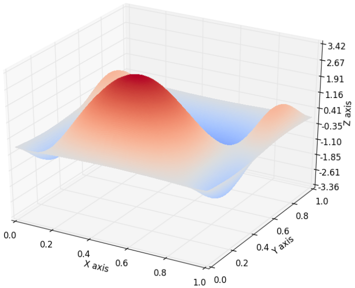


(b) $u^\delta_{N(\delta)}$, $N(\delta) = 42$

Numerical example: results with u_0

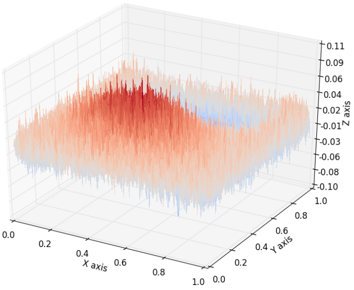


(a) y^δ, $\delta \approx 10^{-4}$

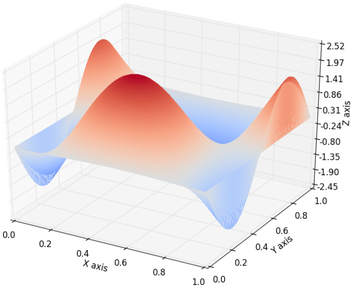


(b) $u^\delta_{N(\delta)}$, $N(\delta) = 1119$

Numerical example: results with $\bar{u}_0$

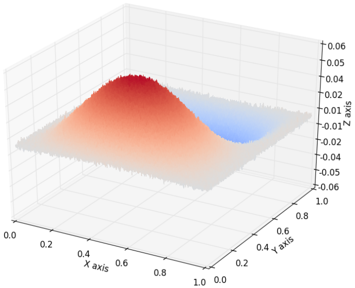


(a) y^δ, $\delta \approx 10^{-2}$

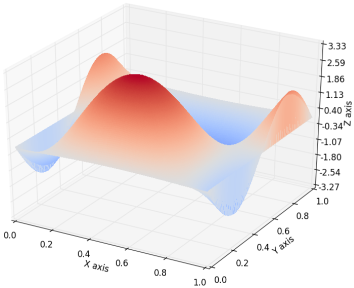


(b) $u^\delta_{N(\delta)}$, $N(\delta) = 7$

Numerical example: results with $\bar{u}_0$

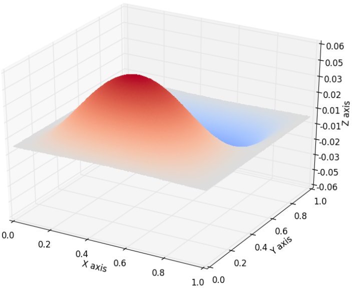


(a) y^δ, $\delta \approx 10^{-3}$

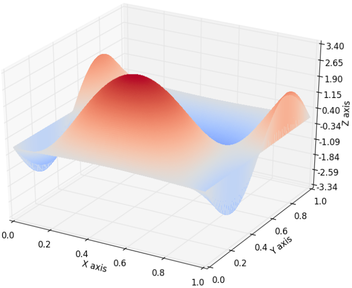


(b) $u^\delta_{N(\delta)}$, $N(\delta) = 14$

Numerical example: results with $\bar{u}_0$



(a) y^δ, $\delta \approx 10^{-4}$



(b) $u^\delta_{N(\delta)}$, $N(\delta) = 21$

Numerical example: quantitative results

	$u_0 \equiv 0$			$u_0 = \bar{u}_0$			
δ	N_δ	E_N^δ	# SSN	N_δ	E_N^δ	R_N^δ	# SSN
$1.06 \cdot 10^{-2}$	3	$5.35 \cdot 10^{-1}$	7	7	$3.04 \cdot 10^{-1}$	4.43	23
$5.29 \cdot 10^{-3}$	6	$4.18 \cdot 10^{-1}$	16	9	$1.54 \cdot 10^{-1}$	3.17	31
$1.06 \cdot 10^{-3}$	42	$2.66 \cdot 10^{-1}$	125	14	$2.76 \cdot 10^{-2}$	1.27	50
$5.30 \cdot 10^{-4}$	106	$2.25 \cdot 10^{-1}$	342	16	$1.41 \cdot 10^{-2}$	0.92	55
$1.06 \cdot 10^{-4}$	1120	$1.49 \cdot 10^{-1}$	4409	21	$2.68 \cdot 10^{-3}$	0.39	75
$5.29 \cdot 10^{-5}$	3224	$1.23 \cdot 10^{-1}$	12 562	23	$1.49 \cdot 10^{-3}$	0.31	84
$1.06 \cdot 10^{-5}$	—	—	—	28	$6.80 \cdot 10^{-4}$	0.31	102
$5.29 \cdot 10^{-6}$	—	—	—	35	$4.93 \cdot 10^{-4}$	0.32	133

E_N^δ: relative L^2 error; R_N^δ: empirical rate $\mathcal{O}(\sqrt{\delta})$

Conclusion

Summary

- iterative regularization using Bouligand derivatives
- inverse source problems for non-smooth PDEs
- convergence under asymptotic stability

Outlook

- convergence rates under source condition
- acceleration, regularized (semismooth) Newton
- other non-smooth equations, variational inequalities
- coefficient inverse problems

Preprint, Python/Julia codes:

`http://www.uni-due.de/mathematik/agclason/clason_pubs.php`