

# THE QUASI-REVERSIBILITY METHOD FOR THERMOACOUSTIC TOMOGRAPHY IN A HETEROGENEOUS MEDIUM

CHRISTIAN CLASON\* AND MICHAEL V. KLIBANOV†

**Abstract.** In this paper we consider thermoacoustic tomography as the inverse problem of determining from lateral Cauchy data the unknown initial conditions in a wave equation with spatially varying coefficients. This problem also occurs in several applications in the area of medical imaging and non-destructive testing. Using the method of quasi-reversibility, the original ill-posed problem is replaced with a boundary value problem for a fourth order partial differential equation. We find a weak  $H^2$  solution of this problem and show that it is a well-posed elliptic problem. Error estimates and convergence of the approximation follow from observability estimates for the wave equation, which are proven using a Carleman estimate. We derive a numerical scheme for the solution of the quasi-reversibility problem by a  $B$ -spline Galerkin method, for which we give error estimates. Finally, we present numerical results supporting the robustness of this method for the reconstruction of initial conditions from full and limited boundary data.

**Key words.** inverse problem, ill-posed cauchy problem, quasi-reversibility, wave equation

**AMS subject classifications.** 65M32, 65M60

**1. Introduction.** Many practical problems are concerned with determining the strength and location of sources of disturbances in a medium, when only boundary measurements are available. Examples include medical imaging, seismic observations, geodynamics, or tracing electromagnetic pulses. If the sources can be temporally localized, this problem is equivalent to the determination of the initial conditions in a wave equation. One such application is the medical imaging method of thermoacoustic computed tomography (TCT), where electromagnetic radiation induces a pressure wave in a sample, which is proportional to the varying energy absorption by different types of tissue. From time dependent measurements, one wishes to calculate the absorption coefficient and from this, the tissue distribution (e.g., healthy and cancerous tissue). If it would be possible to completely characterize a ‘final’ state, then the time reversibility of the wave equation can be employed to calculate the wave field backwards in time to the moment of interest. However, in practical applications, it is usually either not possible to measure a wave field in a complete region, or dissipative terms break the time invariance of the equation. In both cases, the problem is then the reconstruction of initial conditions from boundary measurements only. The problem is known to be ill-posed in general, but under certain conditions observability estimates can be proved, which warrant a unique and stable solution. An additional difficulty appears when the wave propagates in a medium with spatially varying wave speed, as for instance in bone and soft tissue or water. In this case, the wave propagation will no longer happen along straight lines, a fact which increases the difficulties in proving the stability estimates as well as in computing the numerical solution. Previous works on the numerical determination of the initial condition in a hyperbolic equation from lateral Cauchy data were [11] and [16], which only applied to constant coefficients in the principal part of the operator. The variable coefficient case was considered in [18], but no numerical studies were done. For numerical approaches to TCT in a homogeneous medium, see e.g. [6], [7] and [11].

---

\*Zentrum Mathematik, TU München, 85748 Garching bei München, Germany, (clason@ma.tum.de).

†Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA, (mklibanv@unc.edu).

Our main contribution in this paper is therefore the presentation and justification of a robust numerical method for the solution of the inverse problem of reconstruction of initial conditions for the wave equation in a heterogeneous medium. Specifically, we give a complete proof for the Lipschitz stability for the solution of the inverse problem and derive a new numerical method which is especially suitable for the solution of the inverse problem. We also consider the reconstruction from Cauchy data given only on a part of the boundary.

This paper is organized as follows: In section 2, after introducing some notations, we present the problem of TCT as an inverse problem for recovering the initial conditions in the wave equation, and give its mathematical formulation as a lateral Cauchy problem (section 2.2). Then we introduce the method of quasi-reversibility for the solution of such a problem in section 3. We show that this problem is well-posed and briefly discuss the regularity of its solution. Error estimates and convergence results for the method of quasi-reversibility can be derived by Carleman estimates, which is done in section 4. The computational method used for the numerical solution of the inverse problem is discussed in section 5. In this paper, we consider a direct approach based on a  $B$ -spline Galerkin method, which we derive for the problem of quasi-reversibility and for which we give an error estimate. We illustrate the robustness of the method with several numerical examples, which are presented in section 6. Finally we discuss the results and give concluding remarks in section 7.

## 2. Statement of the problem.

**2.1. Notations.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with boundary  $\partial\Omega$ . For  $T > 0$ , consider the space-time cylinder  $Q_T := \Omega \times [0, T]$  with lateral boundary  $S_T := \partial\Omega \times [0, T]$ . To simplify notations, we write  $q := (x, t) \in Q_T$  for the integration variable over  $Q_T$ . We let  $\partial_i$  stand for the partial derivative with respect to  $x_i$ , with  $\partial_{n+1} := \partial_t$ , the derivative with respect to time. Higher order derivatives are represented by repeated indices, e.g.,  $\partial_{tt}$ . The spatial gradient is written as  $\nabla := (\partial_1, \dots, \partial_n)^T$ , while  $\Delta := \sum_{i=1}^n \partial_{ii}$  stands for the Laplacian in  $\mathbb{R}^n$ . For convenience, we also use the notation  $\nabla^2 := (\partial_{11}, \dots, \partial_{nn})^T$  for the vector of second spatial derivatives. Similarly,  $\partial_\nu$  denotes the normal derivative at  $\partial\Omega$ .

Where not stated otherwise, we denote the inner product in the Hilbert space  $X$  by  $\langle \cdot, \cdot \rangle_X$ . For brevity, the inner product in  $\mathbb{R}^n$  is simply written as  $\langle \cdot, \cdot \rangle_n$ . Finally,  $B_r(x)$  is the open ball around  $x \in \mathbb{R}^n$  with radius  $r > 0$ .

**2.2. Thermoacoustic tomography.** Thermoacoustic computed tomography is a new imaging method that uses different modalities for illumination of the target and measurement of its response (see, e.g., [20, 21, 22, 23, 34, 35] and references therein). Specifically, the target is subjected to a short electromagnetic impulse, which is absorbed, leading to a temperature increase and hence to expansion. This induces a pressure wave in the target, which can be measured as a change in the acoustic field outside the sample. If the absorption of the electromagnetic energy is spatially varying, the resulting wave field will carry the signature of this inhomogeneity. Applications in medical imaging utilize the fact that cancerous tissue absorbs more energy per volume than healthy tissue. The problem is hence to calculate this absorption density of the target from time dependent acoustic measurements outside it.

In order to make this precise, we introduce the radiation intensity of the illuminating pulse,  $I(x, t)$ , and the spatially varying absorption coefficient of the target and the surrounding medium,  $\alpha(x)$ . The generated acoustic pressure wave  $v(x, t)$  propagating in  $\mathbb{R}^3$  can then be described by the inviscid liquid model, ignoring thermal

diffusion [20, 27]:

$$(2.1) \quad \frac{1}{c^2(x)} \partial_{tt} v(x, t) - \Delta v(x, t) = \alpha(x) \frac{\beta}{c_p} \partial_t I(x, t), \quad (x, t) \in \mathbb{R}^3 \times [0, T],$$

where  $c(x)$  is the sound speed,  $\beta$  the thermal expansion coefficient, and  $c_p$  the specific heat capacity of the medium. We take the medium to be at rest prior to the irradiation, i.e.

$$(2.2) \quad v(x, t) = \partial_t v(x, t) \equiv 0, \quad x \in \mathbb{R}^3, \quad t < 0.$$

If we assume the pulse to be of very short duration (cf. [20]) and spatially homogeneous, we can approximate its temporal shape by the delta distribution:

$$(2.3) \quad I(x, t) \approx I_0 \delta(t).$$

In this case, equation (2.1) is assumed to hold in the weak sense. We denote the generalized derivative by  $D_t$  and let  $\delta'(t) := D_t \delta(t)$ .

Since the right hand side  $f(x, t) := \alpha(x) \beta c_p^{-1} I_0 \delta'(t)$  is now a generalized function (which is defined for all  $t \in \mathbb{R}$ ), equation (2.1) is then a generalized Cauchy problem for the (generalized) function  $u$  (cf. [32, §12.2]). On  $\mathbb{R}^3 \times [0, T]$ , we can find a solution  $v$  of (2.1) by solving the following classical Cauchy problem:

$$(2.4) \quad \begin{cases} \frac{1}{c^2(x)} \partial_{tt} u(x, t) - \Delta u(x, t) & = 0 & (x, t) \in \mathbb{R}^3 \times [0, T], \\ u(x, t)|_{t=0} & = \alpha(x) I_0 \frac{\beta}{c_p} & x \in \mathbb{R}^3, \\ \partial_t u(x, t)|_{t=0} & = 0 & x \in \mathbb{R}^3. \end{cases}$$

To see this, we take a solution  $u$  of (2.4) (continued as zero for  $t < 0$ ) and any  $\varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$  and integrate:

$$(2.5) \quad \begin{aligned} & \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (c^{-2}(x) \partial_{tt} u(x, t) - \Delta u(x, t)) \varphi(x, t) \, dx dt \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (c^{-2}(x) \partial_{tt} \varphi(x, t) - \Delta \varphi(x, t)) u(x, t) \, dx dt \\ &= \int_0^{\infty} \int_{\mathbb{R}^3} (c^{-2}(x) \partial_{tt} \varphi(x, t) - \Delta \varphi(x, t)) u(x, t) \, dx dt \\ &= \int_0^{\infty} \int_{\mathbb{R}^3} (c^{-2}(x) \partial_{tt} u(x, t) - \Delta u(x, t)) \varphi(x, t) \, dx dt \\ &\quad + \int_{\mathbb{R}^3} \partial_t u(x, 0) \varphi(x, 0) \, dx - \int_{\mathbb{R}^3} u(x, 0) \partial_t \varphi(x, 0) \, dx \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left( \alpha(x) I_0 \frac{\beta}{c_p} (x) \delta'(t) \right) \varphi(x, t) \, dx dt, \end{aligned}$$

using partial integration and the definition of the generalized derivative and of the delta distribution. As a standard result (e.g. [32, §12.3]), we know that a solution  $u$  of (2.4) exists, is unique, and depends continuously on  $\alpha(x)$ . Furthermore, this solution is contained among the solutions of (2.1).

The problem of TCT is the inverse problem of determining the absorption coefficient  $\alpha(x)$  from time dependent measurement of the pressure field  $u$  on a surface  $\partial\Omega$

outside the target, when  $c$ ,  $I_0$ ,  $\beta$  and  $c_p$  are known. For specific geometries of  $\partial\Omega$  such as planes, cylinders and spheres, analytic expressions for the absorption density can be directly calculated (cf. [20, 21, 34, 35]). This problem was also solved in [6] and [7] by inversion of the spherical mean operator, in [8] by a filtered back projection algorithm derived by the method of approximate inverse, and in [11] using Fourier expansion. These methods, however, assume a constant sound speed, while we consider thermoacoustic tomography in a heterogeneous medium. For this reason, we pose the problem of thermoacoustic tomography as a lateral Cauchy problem for a wave equation with spatially varying coefficients. Another advantage of this approach is the applicability to thermoacoustic tomography with arbitrary scanning geometries.

**PROBLEM 2.1** (Lateral Cauchy problem in bounded domains). *Given  $c$ ,  $\varphi_0$ , and  $\varphi_1$ , find  $u(x, t)$  in  $Q_T$ , which solves the lateral Cauchy problem:*

$$(2.6) \quad \begin{cases} \frac{1}{c^2(x)} \partial_{tt} u(x, t) - \Delta u(x, t) & = 0 & (x, t) \in Q_T, \\ u(x, t) & = \varphi_0(x, t) & (x, t) \in S_T, \\ \partial_\nu u(x, t) & = \varphi_1(x, t) & (x, t) \in S_T. \end{cases}$$

Of course, for TCT (and similar applications), we are primarily interested in recovering  $u(x, 0)$  in  $\Omega$ , and from this,  $\alpha(x)$ .

Since the lateral Cauchy problem is ill-posed in general (see, e.g. the example constructed in [16]), the problem posed above does not necessarily have a solution. For numerical calculations, we therefore have to consider approximations of this problem, whose solutions are unique and depend continuously on the data, but are still close to that of the original problem. The theory of this problem was considered previously in [12], [15] for the case of constant coefficients. Numerical solutions for this case were presented in [16] and [11]. We discuss one possible approach allowing variable coefficients, which is based on the minimization of a functional in a suitable Hilbert space, in the next section.

**3. The method of quasi-reversibility.** The method of quasi-reversibility was introduced in the book of Lattès and Lions [24] as an approach for the numerical solution of ill-posed boundary value problems for partial differential equations. This method consists in replacing the ill-posed second order problem with a well-posed fourth order problem, and was previously applied to ill-posed Cauchy problems for elliptic [1, 2, 17, 24], parabolic [14, 24] and hyperbolic [16] equations as well as coefficient inverse problems [13, 15]. We employ an hyperbolic variant of the method described in [24, Ch. 4.8]. In contrast with the indirect approach considered there (which entailed the derivation of a partial differential equation equivalent to a variational problem), we use a direct method for calculating the solution of this weak formulation. This allows us to make weaker requirements on the regularity of the solution ( $H^2$  instead of  $H^4$ ). Furthermore, our derivation (which was first described in [13] and [18]) has the advantage of avoiding the introduction of additional boundary conditions in the Cauchy problem.

Let  $L$  be a linear second order hyperbolic differential operator in a bounded domain  $\Omega \subset \mathbb{R}^n$  with Lipschitz boundary  $\partial\Omega$ . Specifically, for  $c \in C^1(\bar{\Omega})$  and  $f \in L^2(Q_T)$ , we consider the wave equation:

$$(3.1) \quad Lu := \frac{1}{c(x)^2} \partial_{tt} u - \Delta u = f.$$

The method is applicable for general operators, and our results carry over to the case of presence of lower order terms.

We introduce the function space

$$(3.2) \quad H_0^2(Q_T) := \{u \in H^2(Q_T) : u|_{S_T} = \partial_\nu u|_{S_T} = 0\} .$$

Furthermore, on  $H^2(Q_T)$ , consider for our quasi-reversibility formulation the following inner product:

$$(3.3) \quad \langle u, v \rangle_{Q_R} := \int_{Q_T} \partial_{tt} u \partial_{tt} v \, dq + \int_{Q_T} \langle \nabla^2 u, \nabla^2 v \rangle_n \, dq + \int_{Q_T} u v \, dq ,$$

and with it the norm  $\|u\|_{Q_R}^2 := \langle u, u \rangle_{Q_R}$ . It can be shown by the Poincaré inequality that this norm is equivalent on  $H_0^2(Q_T)$  to the standard norm  $\|\cdot\|_{H^2(Q_T)}$ , which is the crucial feature of the method of quasi-reversibility.

First, assume that  $\varphi_0 \equiv \varphi_1 \equiv 0$  in (2.6). Now consider for a small  $\varepsilon > 0$  the Tikhonov functional

$$(3.4) \quad J_\varepsilon(u) := \frac{1}{2} \|Lu - f\|_{L^2(Q_T)}^2 + \frac{\varepsilon}{2} \|u\|_{Q_R}^2 .$$

We seek a minimizer  $u_\varepsilon$  of  $J_\varepsilon$  in  $H_0^2(Q_T)$  of which we will show in section 4 that—for a suitable choice of  $f$  given below—it is close to the solution of Problem 2.1. Since the functional  $J_\varepsilon$  is convex, coercive, and Fréchet differentiable with derivative  $J'_\varepsilon$ , this problem is equivalent with the abstract Euler equation:

$$(3.5) \quad J'_\varepsilon(u_\varepsilon)(v) = 0 \text{ for all } v \in H_0^2(Q_T) .$$

Straightforward calculation yields the explicit Euler equation for a minimizer  $u_\varepsilon \in H_0^2(Q_T)$ :

$$(3.6) \quad \int_{Q_T} Lu_\varepsilon Lv \, dq + \varepsilon \langle u_\varepsilon, v \rangle_{Q_R} - \int_{Q_T} Lv f \, dq = 0 \text{ for all } v \in H_0^2(Q_T) .$$

In order to apply this to the problem described in section 2.2, we first have to arrange for homogeneous boundary conditions on  $u$ . The standard method is to introduce a function  $\Phi \in H^2(Q_T)$  satisfying:

$$(3.7) \quad \begin{cases} \Phi(x, t) &= \varphi_0(x, t) & (x, t) \in S_T , \\ \partial_\nu \Phi(x, t) &= \varphi_1(x, t) & (x, t) \in S_T . \end{cases}$$

There exist infinitely many of these functions for a given pair  $\varphi_0, \varphi_1$ . The main difficulty in the practical application of the method of quasi-reversibility is of course the construction of such a function. We discuss one method in section 5.2.

Now, if  $u$  is a solution of Problem 2.1, the function  $u^* := u - \Phi$  satisfies

$$(3.8) \quad \begin{cases} Lu^* &= -L\Phi & (x, t) \in Q_T , \\ u^*(x, t) &= 0 & (x, t) \in S_T , \\ \partial_\nu u^*(x, t) &= 0 & (x, t) \in S_T . \end{cases}$$

Therefore, setting  $f := -L\Phi$  in (3.6), we arrive at the direct quasi-reversibility approximation of Problem 2.1:

PROBLEM 3.1 (Problem of quasi-reversibility). *Set*

$$(3.9) \quad M_\varepsilon(u, v) := \int_{Q_T} Lu Lv dq + \varepsilon \langle u, v \rangle_{Q_R}.$$

Given  $\Phi \in H^2(Q_T)$ ,  $c \in C^1(\bar{\Omega})$ ,  $\varepsilon > 0$ , find  $u_\varepsilon \in H_0^2(Q_T)$  so that for all  $v \in H_0^2(Q_T)$

$$(3.10) \quad M_\varepsilon(u_\varepsilon, v) = - \int_{Q_T} L\Phi Lv dq.$$

Because  $\|\cdot\|_{Q_R}$  and  $\|\cdot\|_{H^2(Q_T)}$  are equivalent on  $H_0^2(Q_T)$ , this problem is now elliptic:

LEMMA 3.2. *The bilinear form  $M_\varepsilon(u, v)$  is  $H_0^2(Q_T)$ -elliptic, i.e. there exist constants  $c_1, c_2 > 0$ , depending only on the  $L^2(Q_T)$  norm of  $c$  and on  $Q_T$ , respectively, such that*

$$(3.11) \quad |M_\varepsilon(u, v)| \leq (c_1 + \varepsilon) \|u\|_{H^2(Q_T)} \|v\|_{H^2(Q_T)} \quad \text{for all } u, v \in H_0^2(Q_T),$$

$$(3.12) \quad |M_\varepsilon(u, u)| \geq c_2 \varepsilon \|u\|_{H^2(Q_T)}^2 \quad \text{for all } u \in H_0^2(Q_T).$$

The well-posedness of Problem 3.1 can therefore be established from the Lax-Milgram lemma. The proof is standard and we omit it here.

THEOREM 3.3. *Given  $\Phi \in H^2(Q_T)$ ,  $c \in C^1(\bar{\Omega})$ ,  $\varepsilon > 0$ , there exists a unique solution  $u_\varepsilon$  of Problem 3.1 in  $H_0^2(Q_T)$ . Furthermore, there exists a constant  $C > 0$ , depending only on  $Q_T$  and the  $L^2(Q_T)$  norm of  $c$ , such that*

$$(3.13) \quad \|u_\varepsilon\|_{H^2(Q_T)} \leq \frac{C}{\sqrt{\varepsilon}} \|\Phi\|_{H^2(Q_T)}.$$

The theorem guarantees the existence of a unique solution which depends continuously on the data, hence Problem 3.1 is well-posed in the sense of Hadamard. As  $M_\varepsilon(u, v)$  is  $H_0^2(Q_T)$ -elliptic and *a fortiori*  $H_0^2(Q_T)$ -coercive, by Gårding's theorem, Problem 3.1 is strongly elliptic. We can therefore apply the theory of elliptic partial differential operators to determine the regularity of solutions to Problem 3.1, which will be of use in deriving error estimates for the numerical solution. The following is a corollary of [33, Ths. 20.1, 20.4]:

THEOREM 3.4. *A solution  $u_\varepsilon$  of Problem 3.1 satisfies  $u_\varepsilon \in H^3(\tilde{Q}_T)$  for every compact subset  $\tilde{Q}_T \subset Q_T$ . If  $\partial Q_T$  is of class  $C^4$ , this also holds for  $Q_T$  itself.*

The question of convergence of  $u_\varepsilon$  to the solution  $u$  of the lateral Cauchy problem as  $\varepsilon \rightarrow 0$  will be the subject of section 4.

**4. Rates of convergence and error estimates.** We now discuss error estimates of Lipschitz type for the quasi-reversibility method, which will also yield convergence rates for the approximation of the lateral Cauchy problem 2.1 by the quasi-reversibility problem 3.1. The proof makes use of a one parameter family of weighted  $L^2$  estimates known as Carleman estimates. This method has been employed previously (cf. [12, 15, 18]), and generalizes results for constant coefficients in the principal part of the operator obtainable by the method of multipliers (for an excellent introduction to which we refer to [19]). The main new result is the proof of Lipschitz stability for the quasi-reversibility approximation for the case of variable coefficients in the principal part of the operator.

In this section, we consider  $Q'_T := \Omega \times [-T, T]$  instead of  $Q_T$ , which facilitates the proof of the Lipschitz stability estimate. Similarly, we write  $S'_T := \partial\Omega \times [-T, T]$ . Since by (2.4) we know that  $\partial_t u|_{t=0} \equiv 0$ , we can take the even extension of  $u^*$ ,  $u_\varepsilon$  and  $\Phi$  in  $\Omega \times [-T, 0]$  (which we will also denote by  $u^*$ ,  $u_\varepsilon$  and  $\Phi$ , respectively).

First we will state the main tool, a Carleman estimate for the wave equation, which can be derived from pseudodifferential calculus [10] or directly by partial integration [25, 18]. The latter admits less regular boundaries and thus will be used here. For this, we make use of a pointwise derived estimate given in [18, Th. 2.2.4].

We take  $x_0 \in \mathbb{R}^n$  and set  $r := \max_{x \in \overline{\Omega}} |x - x_0|$ . Assume

$$(4.1) \quad 0 < c_{\min} \leq c(x) \leq c_{\max}, \quad \frac{1}{2} + \langle \nabla(c^{-2})(x), x - x_0 \rangle_n \geq 0, \quad \text{for all } x \in \overline{\Omega},$$

and choose  $\beta > 0$  so that

$$(4.2a) \quad \sqrt{\beta} < \frac{1}{4(c_{\min}^{-4} + r|\nabla(c^{-2})(x)|)},$$

$$(4.2b) \quad \beta < \frac{3}{n+3} c_{\min}^2.$$

If  $c(x) \equiv c$  is constant, these conditions are reduced to  $\beta < c$ .

We define the function

$$(4.3) \quad \varphi(x, t) := |x - x_0|^2 - \beta t^2,$$

and, for  $\sigma \geq 0$ , the domain bounded by their level sets:

$$(4.4) \quad Q_\sigma := \{(x, t) \in Q'_T : \varphi(x, t) > \sigma\}.$$

Then we have the following

LEMMA 4.1 (Carleman estimate). *Choose  $\sigma > 0$  so that  $Q_\sigma \cap \{|t| = T\} = \emptyset$ . Then there exist constants  $C = C(\sigma) > 0$  and  $\lambda_0 > 0$ , such that*

$$(4.5) \quad \lambda^3 \int_{Q_\sigma} |u|^2 e^{2\lambda\varphi} dq + \lambda \int_{Q_\sigma} (|\nabla u|^2 + |\partial_t u|^2) e^{2\lambda\varphi} dq \leq C \int_{Q_\sigma} |Lu|^2 e^{2\lambda\varphi} dq$$

for all  $u \in H_0^2(Q_\sigma)$  and all  $\lambda > \lambda_0$ .

REMARK 4.2. *Since the Carleman estimate depends only on the principal part of the operator  $L$ , the following results also hold for hyperbolic operators of second order including terms of lower order spatial derivatives. The method of quasi-reversibility consequently is applicable for TCT in absorbing or dissipative media, as well.*

The proof of the Lipschitz stability of the solution of Problem 3.1 with regard to the boundary data  $\Phi$  consists of several steps. First, we will show weaker estimates inside domains bounded by pseudo-convex level sets. The key step is then their combination in two carefully chosen domains to obtain a Lipschitz observability estimate for the wave equation, independent of initial conditions. From this we can derive the desired convergence and stability estimates for the quasi-reversibility approximation.

LEMMA 4.3. *Assume that there exists an  $x_0 \in \mathbb{R}^n$ , such that (4.1) is satisfied. Choose  $a \geq 0$  and  $\sigma \geq 0$  so that  $Q_{a+3\sigma} \neq \emptyset$  and  $\overline{Q_a} \cap \{|t| = T\} = \emptyset$ . Then there exist*

constants  $C > 0, \lambda_0 > 0$ , such that for all  $w \in H_0^2(Q'_T)$  and all  $\lambda > \lambda_0$  the following inequality holds:

$$(4.6) \quad \|w\|_{H^1(Q_{a+3\sigma})}^2 \leq C \left( e^{-2\lambda\sigma} \|w\|_{H^1(Q'_T)}^2 + e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2 \right).$$

*Proof.* We set  $m := \max_{\overline{Q_a}} \varphi(x, t)$ . Then we have:

$$(4.7) \quad \int_{Q'_T} (Lw)^2 dq = \int_{Q'_T} (Lw)^2 e^{2\lambda\varphi} e^{-2\lambda\varphi} dq \geq e^{-2\lambda m} \int_{Q'_T} (Lw)^2 e^{2\lambda\varphi} dq.$$

Consequently:

$$(4.8) \quad \int_{Q'_T} (Lw)^2 e^{2\lambda\varphi} dq \leq C_1 e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2.$$

To apply the Carleman estimate (4.5), we have to arrange for zero boundary conditions for  $w$  on  $Q_{a+\sigma}$ . Therefore we introduce a cut-off function  $\chi_\sigma \in C^\infty(Q'_T)$  with:

$$(4.9) \quad \chi_\sigma(x, t) \begin{cases} = 1 & \text{for } (x, t) \in Q_{a+2\sigma}, \\ = 0 & \text{for } (x, t) \in Q'_T \setminus Q_{a+\sigma}, \\ \in (0, 1) & \text{for } (x, t) \in Q_{a+\sigma} \setminus Q_{a+2\sigma}, \end{cases}$$

and set  $w_\sigma := \chi_\sigma w$ . Since  $\|\chi_\sigma\|_{H^2(Q_a \setminus Q_{a+\sigma})} = 0$ , it follows that

$$(4.10) \quad (Lw_\sigma)^2 \leq 2(Lw)^2 + C_2(1 - \chi_\sigma)(w + \partial_t w + |\nabla w|)^2,$$

where the constant  $C_2$  depends only on the  $H^2(Q_{a+\sigma})$  norm of  $\chi_\sigma$ . Hence, multiplication of (4.10) with  $e^{2\lambda\varphi}$ , integration over  $Q_{a+\sigma}$ , and use of (4.8) yields

$$(4.11) \quad \int_{Q_{a+\sigma}} (Lw_\sigma)^2 e^{2\lambda\varphi} dq \leq C_2 \int_{Q_{a+\sigma} \setminus Q_{a+2\sigma}} (w + \partial_t w + |\nabla w|)^2 e^{2\lambda\varphi} dq + C_1 e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2.$$

We can now apply the Carleman estimate (4.5) to obtain:

$$(4.12) \quad \lambda^3 \int_{Q_{a+\sigma}} w_\sigma^2 e^{2\lambda\varphi} dq + \lambda \int_{Q_{a+\sigma}} (|\nabla w_\sigma|^2 + |\partial_t w_\sigma|^2) e^{2\lambda\varphi} dq \leq C_2 e^{4\lambda\sigma} \|w\|_{H^1(Q_{a+\sigma})}^2 + C_1 e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2.$$

Here we have used that  $\varphi(x, t) < a + 2\sigma$  on  $Q_\sigma \setminus Q_{a+2\sigma}$ . Replacing  $Q_{a+\sigma}$  with  $Q_{a+3\sigma}$  on the left hand side and estimating  $\varphi(x, t)$  by  $a + 3\sigma$  from below there, we have for  $\lambda > 1$ :

$$(4.13) \quad \|w\|_{H^1(Q_{a+3\sigma})}^2 \leq C e^{-2\lambda\sigma} \|w\|_{H^1(Q'_T)}^2 + C e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2.$$

□

We now eliminate the  $H^1$ -norm of the function on the right hand side and extend this estimate to  $Q'_T$  by applying standard energy estimates for hyperbolic operators:



LEMMA 4.4. *Assume that  $\Omega \subset \{x \in \mathbb{R}^n : |x| < R\}$ , that condition (4.1) holds for an  $x_0 \in \Omega$ , and that  $T > \frac{2R}{\sqrt{\beta}}$ . Then there exists a constant  $C > 0$ , such that for all  $w \in H_0^2(Q'_T)$ , the following estimate holds:*

$$(4.14) \quad \|w\|_{H^1(Q'_T)}^2 \leq C \|Lw\|_{L^2(Q'_T)}^2 .$$

*Proof.* We start by combining two estimates from Lemma 4.3, chosen so that the corresponding domains overlap  $\Omega \times [-\delta, \delta]$  for a  $\delta > 0$ , while their boundaries intersect  $\partial\Omega$  only for  $|t| < T$ .

First we pick an  $x_0 \in \Omega$  so that (4.1) is satisfied. By continuity of the inner product, there exists an  $\varepsilon > 0$ , such that  $B_\varepsilon(x_0) \subset \Omega$  and for all  $x_1 \in B_\varepsilon(x_0)$ , condition (4.1) is also satisfied. Now choose  $a > 0$  and  $\sigma > 0$  so that the following hold:

$$(4.15) \quad 6\sqrt{a+4\sigma} < \varepsilon, \quad \frac{(2R+3\sqrt{a+4\sigma})^2+a}{\beta} < T^2 .$$

This choice is possible from the assumption that  $T > \frac{2R}{\sqrt{\beta}}$ . Then we fix  $x_1 \in B_\varepsilon(x_0)$  with  $|x_0 - x_1| = 3\sqrt{a+4\sigma}$ , and set

$$(4.16a) \quad Q_\sigma(x_0) := \{(x, t) \in Q'_T : |x - x_0|^2 - \beta t^2 > \sigma\} ,$$

$$(4.16b) \quad Q_\sigma(x_1) := \{(x, t) \in Q'_T : |x - x_1|^2 - \beta t^2 > \sigma\} .$$

Since  $B_\varepsilon(x_0) \setminus B_{\sqrt{a+4\sigma}}(x_0) \neq \emptyset$  by (4.15), the same holds for the sets  $Q_{a+4\sigma}(x_0) \subset Q_{a+3\sigma}(x_0) \subset Q_{a+2\sigma}(x_0) \subset Q_{a+\sigma}(x_0) \subset Q_a(x_0)$ , in which the former is contained. Similarly,  $Q_{a+4\sigma}(x_1) \neq \emptyset$ , since for all  $x \in B_{\sqrt{a+4\sigma}}(x_0)$ , we can show that  $(x, 0) \in Q_{a+4\sigma}(x_1)$ :

$$(4.17) \quad |x - x_1| = |(x_1 - x_0) - (x - x_0)| \geq 3\sqrt{a+4\sigma} - \sqrt{a+4\sigma} > \sqrt{a+4\sigma} .$$

On the other hand, as  $T > \frac{2R}{\sqrt{\beta}}$ , we have that  $|x - x_0|^2 - \beta T^2 < 4R^2 - 4R^2 = 0$ , and so for  $a > 0$ , it holds that  $\overline{Q}_a(x_0) \cap \{|t| = T\} = \emptyset$ . Because of (4.15), for all  $x \in \Omega$ ,

$$(4.18) \quad |x - x_1|^2 - \beta T^2 \leq (2R + 3\sqrt{a+4\sigma})^2 - (2R + 3\sqrt{a+4\sigma})^2 - a < 0 ,$$

hence for  $a > 0$ , it holds that  $\overline{Q}_a(x_1) \cap \{|t| = T\} = \emptyset$  as well.

Therefore, we can apply the estimate (4.6) in  $Q_{a+3\sigma}(x_0)$  and  $Q_{a+3\sigma}(x_1)$  to obtain for all  $w \in H_0^2(Q'_T)$ :

$$(4.19a) \quad \|w\|_{H^1(Q_{a+3\sigma}(x_0))}^2 \leq C \left( e^{-2\lambda\sigma} \|w\|_{H^1(Q'_T)}^2 + e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2 \right) ,$$

$$(4.19b) \quad \|w\|_{H^1(Q_{a+3\sigma}(x_1))}^2 \leq C \left( e^{-2\lambda\sigma} \|w\|_{H^1(Q'_T)}^2 + e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2 \right) .$$

We next show that we can combine these estimates so that the left hand sides becomes an integral over a domain containing  $\Omega \times [-\delta, \delta]$  for some  $\delta > 0$ . Set

$$(4.20a) \quad G_0 := \{(x, t) \in Q'_T : |x - x_0|^2 > a + 4\sigma\} \cap Q_{a+3\sigma}(x_0) ,$$

$$(4.20b) \quad G_1 := \{(x, t) \in Q'_T : |x - x_1|^2 > a + 4\sigma\} \cap Q_{a+3\sigma}(x_1) .$$

By the definition of  $Q_{a+3\sigma}(x_0)$  and  $Q_{a+3\sigma}(x_1)$ , it is clear that

$$(4.21a) \quad \{(x, t) \in Q'_T : \beta t^2 < \sigma, |x - x_0|^2 > a + 4\sigma\} \subset G_0,$$

$$(4.21b) \quad \{(x, t) \in Q'_T : \beta t^2 < \sigma, |x - x_1|^2 > a + 4\sigma\} \subset G_1.$$

But since for  $x \in Q'_T$  with  $|x - x_0|^2 < a + 4\sigma$ , we have by (4.17) that  $|x - x_1|^2 > a + 4\sigma$ , it follows that  $(Q'_T \cap \{\beta t^2 < \sigma\}) \setminus G_0 \subset G_1$ . Hence, if we set  $\delta := \sqrt{\frac{\sigma}{\beta}}$ , we have that  $E_\delta := \{(x, t) \in Q'_T : t^2 < \delta^2, x \in \Omega\} \subset G_0 \cup G_1$ , and so by combining the inequalities (4.19a) and (4.19b):

$$(4.22) \quad \|w\|_{H^1(E_\delta)}^2 \leq 2C \left( e^{-2\lambda\sigma} \|w\|_{H^1(Q'_T)}^2 + e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2 \right).$$

By elementary properties of the Lebesgue integral, there exists a  $t_1 \in (-\delta, \delta)$  such that

$$(4.23) \quad \|w\|_{H^1(E_\delta)}^2 \geq 2\delta \left( \|w(\cdot, t_1)\|_{H^1(\Omega)}^2 + \|\partial_t w(\cdot, t_1)\|_{L^2(\Omega)}^2 \right),$$

and thus, by way of (4.22),

$$(4.24) \quad \|w(\cdot, t_1)\|_{H^1(\Omega)}^2 + \|\partial_t w(\cdot, t_1)\|_{L^2(\Omega)}^2 \leq \frac{C}{\delta} \left( e^{-2\lambda\sigma} \|w\|_{H^1(Q'_T)}^2 + e^{2\lambda m} \|Lw\|_{L^2(Q'_T)}^2 \right).$$

We complete the proof by using the above to derive the desired estimate from a standard energy inequality. Obviously, any  $w \in H_0^2(Q_T)$  satisfies the following hyperbolic equation:

$$(4.25) \quad \begin{cases} L(w(x, t)) & = (Lw)(x, t) & (x, t) \in \Omega \times [t_1, T], \\ w(x, t)|_{t=t_1} & = w(x, t_1) & x \in \Omega, \\ \partial_t w(x, t)|_{t=t_1} & = \partial_t w(x, t_1) & x \in \Omega, \\ w(x, t) = \partial_\nu w(x, t) & \equiv 0 & (x, t) \in \partial\Omega \times [t_1, T]. \end{cases}$$

Now we can apply the standard energy estimate to this equation for  $t \in (t_1, T)$ :

$$(4.26) \quad \int_{t_1}^T \|w(\cdot, t)\|_{H^1(\Omega)}^2 + \|\partial_t w(\cdot, t)\|_{L^2(\Omega)}^2 dt \\ \leq C_3 \left( \|w(\cdot, t_1)\|_{H^1(\Omega)}^2 + \|\partial_t w(\cdot, t_1)\|_{L^2(\Omega)}^2 + \int_{t_1}^T \|Lw\|_{L^2(\Omega)}^2 dt \right),$$

where  $C_3 > 0$  is a constant independent of  $w$ ,  $t_1$  and  $t$ . Since  $L$  is a time reversible operator, we can repeat this procedure for  $t \in (-T, t_1)$  after the transformation  $\tau = -t$  and get a similar estimate on  $\Omega \times [-T, t_1]$ . Summing up both estimates gives:

$$(4.27) \quad \|w\|_{H^1(Q'_T)}^2 \leq 2C_3 \left( \|w(\cdot, t_1)\|_{H^1(\Omega)}^2 + \|\partial_t w(\cdot, t_1)\|_{L^2(\Omega)}^2 + \|Lw\|_{L^2(Q'_T)}^2 \right).$$

Finally, we insert the estimate (4.24) on the right hand side, yielding

$$(4.28) \quad \|w\|_{H^1(Q'_T)}^2 \leq C_4 \left( e^{-2\lambda\sigma} \|w\|_{H^1(Q'_T)}^2 + \|Lw\|_{L^2(Q'_T)}^2 \right).$$

Taking now  $\lambda > \lambda_0$  large enough so that  $C_4 e^{-2\lambda\sigma} < 1$  holds, we can absorb the  $H^1(Q'_T)$  norm into the left hand side and arrive at the desired estimate.  $\square$

REMARK 4.5. If (4.1) holds for  $|x_0| < c$ , then  $|x - x_0| < R + c$  for all  $x \in \Omega$ , and so we can replace  $2R$  by  $R$  in the proof above and weaken the hypothesis on  $T$  here and below to  $T > \frac{R}{\sqrt{\beta}}$ .

In reality, we have to assume that the Cauchy data  $\varphi_0, \varphi_1$  are contaminated by noise, and we have, instead,  $\varphi_0^\delta$  and  $\varphi_1^\delta$ . The following theorem consequently addresses the convergence and also the stability of the method of quasi-reversibility.

THEOREM 4.6 (Lipschitz stability and convergence). *Let  $u^* \in H_0^2(Q'_T)$  be a solution of (3.8). Assume there exists a function  $\Phi^\delta \in H^2(Q'_T)$  which satisfies*

$$(4.29) \quad \begin{cases} \Phi^\delta(x, t) &= \varphi_0^\delta(x, t) & (x, t) \in S'_T, \\ \partial_\nu \Phi^\delta(x, t) &= \varphi_1^\delta(x, t) & (x, t) \in S'_T, \end{cases}$$

and that

$$(4.30) \quad \|\Phi - \Phi^\delta\|_{H^2(Q'_T)} \leq \delta.$$

Denote by  $u_\varepsilon^\delta$  the solution of Problem 3.1 with  $\Phi^\delta$  replacing  $\Phi$ .

Under the assumptions of Theorem 4.4, there exists a  $C > 0$ , such that

$$(4.31) \quad \|u^* - u_\varepsilon^\delta\|_{H^1(Q'_T)}^2 \leq C \left( \delta^2 + \varepsilon \|u^*\|_{H^2(Q'_T)}^2 \right).$$

*Proof.* Set  $w := u^* - u_\varepsilon^\delta$ . Then  $w \in H_0^2(Q'_T)$ , so we can apply the observability estimate (4.14) to the difference:

$$(4.32) \quad \|w\|_{H^1(Q'_T)}^2 \leq C_1 \|Lw\|_{L^2(Q'_T)}^2.$$

Now we estimate the right hand side. From (3.8), we know  $u^*$  satisfies for all  $v \in H_0^2(Q'_T)$ :

$$(4.33) \quad M_\varepsilon(u^*, v) = - \int_{Q'_T} L\Phi Lv \, dq + \varepsilon \langle u^*, v \rangle_{QR}.$$

Since  $(u^* - u_\varepsilon^\delta) \in H_0^2(Q'_T)$  as well, we can subtract (3.10) with  $\Phi^\delta$  replacing  $\Phi$ , and take  $v = w$ :

$$(4.34) \quad \|Lw\|_{L^2(Q_T)}^2 + \varepsilon \left( \|\partial_{tt}w\|_{L^2(Q_T)}^2 + \|\nabla^2 w\|_{L^2(Q_T)}^2 + \|w\|_{L^2(Q_T)}^2 \right) = \\ \int_{Q'_T} L(\Phi^\delta - \Phi) Lw \, dq \\ + \varepsilon \left( \int_{Q'_T} \partial_{tt}u^* \partial_{tt}w \, dq + \int_{Q'_T} \langle \nabla^2 u^*, \nabla^2 w \rangle_n \, dq + \int_{Q'_T} u^* w \, dq \right).$$

By the Cauchy-Schwarz inequality and Young's inequality, we can subtract the terms containing  $w$  and use (4.30), which shows:

$$(4.35) \quad \|Lw\|_{L^2(Q_T)}^2 + \varepsilon \left( \|\partial_{tt}w\|_{L^2(Q_T)}^2 + \|\nabla^2 w\|_{L^2(Q_T)}^2 + \|w\|_{L^2(Q_T)}^2 \right) \\ \leq \delta^2 + \varepsilon \left( \|\partial_{tt}u^*\|_{L^2(Q'_T)}^2 + \|\nabla^2 u^*\|_{L^2(Q'_T)}^2 + \|u^*\|_{L^2(Q'_T)}^2 \right).$$

Including the missing derivatives on the right hand side of the inequality, and dropping the term  $\|w\|_{Q_R}$  on the left hand side, we finally obtain:

$$(4.36) \quad \int_{Q'_T} (Lw)^2 dq \leq C_2 \left( \delta^2 + \varepsilon \|u^*\|_{H^2(Q'_T)}^2 \right).$$

Combining the two estimates (4.32) and (4.36) completes the proof.  $\square$

REMARK 4.7. *It follows from this result that  $\varepsilon = \beta\delta^2$  for  $\beta \geq 1$  constitutes a regularization parameter choice rule, hence the method of quasi-reversibility is a convergent regularization method for the lateral Cauchy problem (cf. [5]).*

The rest of the paper is concerned with the numerical solution of the problem of quasi-reversibility.

**5. Numerical solution.** To solve Problem 3.1 numerically, we employ a Ritz-Galerkin approximation; that is, we look for the solution in a finite dimensional subspace  $S_h$  of  $H_0^2(Q_T)$ . The classical choice are the spaces of piecewise polynomials which are at least once continuously differentiable everywhere. In section 5.1, we construct a suitable space  $S_h$  of this type and give an error estimate for the corresponding Ritz-Galerkin approximation. As a local basis of this space, we choose  $B$ -splines, which provide the necessary regularity and allow us to deal with the boundary conditions via (3.7) in a natural way. This is discussed in section 5.2, together with the reduction of the approximating problem to a system of linear equations.

**5.1. Ritz-Galerkin approximation.** The finite dimensional subspace taken for our Ritz-Galerkin approximation is the classical spline space of Schoenberg [28]; to approximate our multivariate functions, we make use of tensor products of univariate splines. For a thorough discussion of splines, we refer to [30].

Assume  $Q_T$  is contained in a cube  $R := \prod_{i=1}^{n+1} [a_i, b_i] \subset (\mathbb{R}^n \times [0, T])$ . We start by introducing a partition  $\Delta$  of  $R$ :

$$(5.1) \quad \Delta_i := \{a_i = x_{i,0} < x_{i,1} < \dots < x_{i,k_i} < x_{i,k_i+1} = b_i\},$$

$$(5.2) \quad \Delta := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in \Delta_i\}.$$

The  $\Delta_i$  generate the set of intervals  $I_{i,j} := [x_{i,j}, x_{i,j+1})$  for  $j = 0, \dots, k_i - 1$ , and  $I_{i,k_i} := [x_{i,k_i}, x_{i,k_i+1}]$ . With these partitions, we associate mesh widths

$$(5.3) \quad h_i := \max_{j \in \{0, \dots, k_i\}} (x_{i,j+1} - x_{i,j}), \quad \bar{h} := \max_{i \in \{1, \dots, n+1\}} h_i, \quad \underline{h} := \min_{i \in \{1, \dots, n+1\}} h_i.$$

A set of partitions  $\mathfrak{S}$  is called quasi-uniform if there exists a constant  $K > 0$ , such that for each partition  $\Delta \in \mathfrak{S}$ , the mesh widths satisfy  $\bar{h}/\underline{h} \leq K$ . A partition belonging to such a set will also be called quasi-uniform. The univariate spline spaces  $\mathcal{S}_i^d$  on a partition  $\Delta_i$  are then defined as:

$$(5.4) \quad \mathcal{S}_i^d := \{v \in C^{d-2}([a_i, b_i]) : v|_{I_{i,j}} \in \mathcal{P}^d \text{ for all } j = 0, \dots, k_i\},$$

where  $\mathcal{P}^d := \left\{ p(x) : p(x) = \sum_{i=1}^d c_i x^{i-1}, c_1, \dots, c_d \in \mathbb{R}, x \in \mathbb{R} \right\}$  is the space of polynomials of order  $d$ . The multivariate spline space  $\mathcal{S}^d$  is then the tensor product of these spaces:

$$(5.5) \quad \mathcal{S}^d := \text{span} \left\{ \prod_{i=1}^{n+1} s_i : s_i \in \mathcal{S}_i^d \right\}.$$

This is a linear space of dimension  $\prod_{i=1}^{n+1} (d + k_i)$ . For arbitrary domains  $\Omega \subset R$ , we define the spline space

$$(5.6) \quad \mathcal{S}^d(Q_T) := \{s|_{Q_T} : s \in \mathcal{S}^d\}.$$

By construction, each  $s \in \mathcal{S}^d(Q_T)$  is in  $C^{d-2}(Q_T)$ , and, if all but one variable are fixed, its restriction on a cube  $\prod_{i=1}^{n+1} I_{i,j_i}$  is a polynomial of order  $d$ . Since piecewise smooth functions which are  $d-2$  times continuously differentiable are also in  $H^{d-1}$ , we have that  $\mathcal{S}^d(Q_T) \subset H^{d-1}(Q_T)$ .

It remains to give bounds for the approximation error of functions in the Sobolev spaces  $H^r(Q_T)$  by functions in the spline spaces  $\mathcal{S}^d(Q_T)$ . The following theorem is a corollary of Theorem 4.5 in [29]:

**THEOREM 5.1.** *Assume that  $Q_T = R$  or that  $Q_T \subset R$  is a Lipschitz domain. If  $\mathcal{S}^d(Q_T)$  is defined on a quasi-uniform partition of mesh width  $\bar{h}$ , then there exists a  $C > 0$ , such that for all  $d \geq 2$  and  $u \in H^{d-1}(Q_T)$ :*

$$(5.7) \quad \inf_{s \in \mathcal{S}^d} \|u - s\|_{H^{d-2}(Q_T)} \leq C\bar{h} \|u\|_{H^{d-1}(Q_T)},$$

*Furthermore, no other approximation spaces of the same dimension give asymptotically smaller errors.*

Thus, if we wish to construct an approximation of  $u \in H^2(Q_T)$ , we have to look at functions in  $\mathcal{S}^d(Q_T)$  with  $d = 4$ . An approximate solution  $u_h$  to (3.10) in a finite dimensional subspace of  $H_0^2(Q_T)$  then has to be a function in  $\mathcal{S}^4(Q_T)$  which vanishes together with its normal derivative on  $\partial\Omega$ . For this subspace, we can give an error estimate for the Ritz-Galerkin approximation of Problem 3.1. For a given partition  $\Delta$  of  $Q_T$  with mesh width  $\bar{h}$ , we define

$$(5.8) \quad S_h := \mathcal{S}^4(Q_T) \cap H_0^2(Q_T).$$

**PROBLEM 5.2** (Ritz-Galerkin approximation). *Given  $\Phi \in H^2(Q_T)$ ,  $c \in C^1(\bar{\Omega})$ ,  $\varepsilon > 0$ , find  $u_h \in S_h$  which satisfies for all  $v_h \in S_h$ :*

$$(5.9) \quad M_\varepsilon(u_h, v_h) = - \int_{Q_T} L\Phi Lv_h \, dq.$$

The existence and uniqueness of the solution to this problem follows again immediately from the Lax-Milgram lemma.

**THEOREM 5.3.** *Let  $\mathfrak{S}$  be a quasi-uniform set of partitions of  $Q_T$ , and  $u_\varepsilon$  be a solution of Problem 3.1. For each partition  $\Delta \in \mathfrak{S}$  with mesh width  $\bar{h}$ , let  $u_h \in S_h$  denote the solution of Problem 5.2. Then there exists a constant  $C > 0$ , such that the following estimate holds:*

$$(5.10) \quad \|u_\varepsilon - u_h\|_{H^2(Q_T)} \leq C\bar{h} \|u_\varepsilon\|_{H^3(Q_T)}.$$

*Proof.* Since by Lemma 3.2 we know that  $M_\varepsilon$  is a continuous  $H_0^2(Q_T)$ -elliptic bilinear form and the right hand side of (5.9) is a continuous linear functional, by Céa's lemma the approximation error is bounded by the interpolation error. Because we know from Theorem 3.3 that  $u_\varepsilon \in H^3(Q_T) \cap H_0^2(Q_T)$ , the interpolation error in turn can be estimated in  $H_0^2(Q_T)$  by Theorem 5.1 and the orthogonal projection theorem:

$$(5.11) \quad \|u_\varepsilon - u_h\|_{H^2(Q_T)} \leq \frac{c_1 + \varepsilon}{c_2 \varepsilon} \inf_{s_h \in S_h} \|u_\varepsilon - s_h\|_{H^2(Q_T)} \leq \frac{c_3}{c_2} K\bar{h} \|u_\varepsilon\|_{H^3(Q_T)},$$

where  $c_1, c_2$  are the constants from Lemma 3.2 and  $c_3$  is chosen so that  $c_1 + \varepsilon \leq c_3 \varepsilon$ .  $\square$

For general domains, the boundary conditions can be enforced by multiplication by a suitable weight function. This leads to weighted spline spaces, for which similar error bounds can be shown (cf. [9] for an overview). Since the required functions can be more easily characterized on rectangular domains  $\Omega = R$  by taking an appropriate subset of  $\mathcal{S}^4(Q_T)$ , we restrict ourselves from now on to such domains.

**5.2. Implementation.** For the numerical solution of the Ritz-Galerkin approximation, we transform Problem 5.2 into a system of linear equations by expressing  $u_h$  and  $v_h$  by the elements of a basis of  $S_h$ . A convenient choice are the cubic  $B$ -splines. We start by recalling the definition of univariate  $B$ -splines of order  $d$  and their relevant properties, following [4], to which we refer for proofs.

To a partition<sup>1</sup>  $\Delta = \{x_0, \dots, x_{k+1}\}$  of the interval  $[a, b] \subset \mathbb{R}$ , we associate a knot vector  $[t_0, \dots, t_{k+2d}]$  with  $t_0 = t_1 = \dots = t_{d-1} = x_0$ ,  $t_{j+d} = x_j$  for  $j \in \{0, \dots, k\}$ , and  $t_{k+d+1} = \dots = t_{k+2d} = x_{k+1}$ . For  $j \in \{0, \dots, k+d+1\}$ , the  $j$ -th normalized  $B$ -splines of order  $d$  can then be defined by the following recurrence relation:

$$(5.12a) \quad B_j^d(x) := w_j^d(x) B_j^{d-1}(x) + (1 - w_{j+1}^d(x)) B_{j+1}^{d-1}(x),$$

$$(5.12b) \quad w_j^d(x) := \begin{cases} \frac{x-t_j}{t_{j+d-1}-t_j}, & \text{if } t_j \neq t_{j+d-1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$(5.12c) \quad B_j^1(x) := \begin{cases} 1, & \text{if } t_j \leq x < t_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the  $B_j^d$  are identically zero for  $j = 0$  and  $j = k+d+1$ .

These  $B_j^d$  are linear independent and form a partition of unity. Each  $B_j^d$  is non-negative and a polynomial of order  $d$  on each interval  $[x_j, x_{j+1})$ . Furthermore, the derivative of  $B_j^d$  is again a  $B$ -spline of order  $d-1$ , which can be calculated analytically:

$$(5.13) \quad \frac{d}{dx} B_j^d(x) = \frac{d-1}{t_{j+d-1}-t_j} B_j^{d-1}(x) - \frac{d-1}{t_{j+d}-t_{j+1}} B_{j+1}^{d-1}(x).$$

From these properties, it follows that the set  $\mathcal{B}^d := \{B_1^d, \dots, B_{k+d}^d\}$  is a basis of  $\mathcal{S}^d$ . Also, from the definition, we see that  $B_j^d(x_0) \neq 0$  only for  $j = 1$  and  $B_j^d(x_{k+1}) \neq 0$  only for  $j = k+d$ . Using (5.13), a similar consideration for the derivatives of  $B_j^d$  at the boundary knots  $x_0$  and  $x_{k+1}$  shows that apart from the derivatives of  $B_1^d$  and  $B_{k+d}^d$ , only  $\frac{d}{dx} B_2^d$  and  $\frac{d}{dx} B_{d+k-1}^d$  are not zero there. If we remove these, any linear combination of the remaining elements will vanish at  $x_0$  and  $x_{k+1}$  together with its first derivative. The values at points not lying on the boundary of  $R$ , and those of higher derivatives of linear combinations, are not affected.

Hence, if we define a knot vector  $[t_{0,i}, \dots, t_{k_i+2d,i}]$  on each partition  $\Delta_i$ , the tensor product of the  $\mathcal{B}_i^d$  constitutes a basis  $\mathcal{B}^d$  of the multivariate splines  $\mathcal{S}^d(Q_T)$ . Similarly, if we set  $\mathcal{B}_i^0 := \{B_3^d, \dots, B_{k+d-2}^d\}$ , then the tensor product  $\mathcal{B}^0$  of the

<sup>1</sup>For convenience, we drop the index  $i$  while discussing the univariate case.

$\mathcal{B}_1^{d,0}, \dots, \mathcal{B}_n^{d,0}, \mathcal{B}_{n+1}^d$  is a basis of  $S_h$ . (We recall that  $u, \partial_\nu u$  vanish only on the time-like boundary.) Specifically, for cubic  $B$ -splines in three dimensions, we have with  $(x, y, t) \in \mathbb{R}^3$  the following representation:<sup>2</sup>

$$(5.14) \quad \mathcal{B} = \left\{ B_{i,1}^4(x) B_{j,2}^4(y) B_{k,3}^4(t), \right. \\ \left. i \in \{1, \dots, k_1 + 4\}, j \in \{1, \dots, k_2 + 4\}, k \in \{1, \dots, k_3 + 4\} \right\},$$

$$(5.15) \quad \mathcal{B}^0 = \left\{ B_{i,1}^4(x) B_{j,2}^4(y) B_{k,3}^4(t), \right. \\ \left. i \in \{3, \dots, k_1 + 2\}, j \in \{3, \dots, k_2 + 2\}, k \in \{1, \dots, k_3 + 4\} \right\}.$$

Using this basis  $\mathcal{B}$ , we can construct a function  $\Phi \in H^2(Q_T)$  satisfying (3.7) by complete cubic spline interpolation (i.e., prescribing the function value and the normal derivative at the boundary  $\partial\Omega$ , and setting the function value at partition points in the interior to zero). This interpolation problem has a unique solution, which gives the coefficients  $\varphi_{ijk}$  of the cubic  $B$ -spline interpolant  $\Phi_h$  of  $\Phi$ :

$$(5.16) \quad \Phi_h(x, y, t) = \sum_{i=1}^{k_1+4} \sum_{j=1}^{k_2+4} \sum_{k=1}^{k_3+4} \varphi_{ijk} B_{i,1}^4(x) B_{j,2}^4(y) B_{k,3}^4(t).$$

The expression of  $\Phi$  as a cubic  $B$ -spline will guarantee the necessary regularity of  $u_\varepsilon$  for Theorem 5.3 via Theorem 3.4, as well as facilitate the assembly of the system of linear equations.

Since  $S_h$  is a linear space, we can demand that equation (5.9) holds only for all  $v_h \in \mathcal{B}^0$ . We express  $u_h$  as a linear combination of elements of  $\mathcal{B}^0$  as well:

$$(5.17) \quad u_h(x, y, t) = \sum_{i=3}^{k_1+2} \sum_{j=3}^{k_2+2} \sum_{k=1}^{k_3+4} a_{ijk} B_{i,1}^4(x) B_{j,2}^4(y) B_{k,3}^4(t).$$

Problem 5.2 can now be reduced to a system of linear equations for the coefficients  $a_{ijk}$ . In order to write this system in matrix-vector form, we set  $N := (k_1 + 2)(k_2 + 2)(k_3 + 4)$  and use linear indices  $\iota = \iota(i, j, k)$  of the coefficient tensor  $a_{ijk}$  and of the elements of  $\mathcal{B}^0$ :

$$(5.18) \quad \iota(i, j, k) := (k_1 + 2)(k_2 + 2)(k - 1) + (k_1 + 2)(j - 3) + i - 2.$$

In the same way we define  $j = j(m, n, o)$ . Then the system matrix  $M$  can be written as:

$$(5.19) \quad M := (M_\varepsilon(u_\iota, u_j))_{\iota, j}, \quad u_\iota, u_j \in \mathcal{B}^0, \quad \iota, j = 1, \dots, N.$$

As the  $B$ -splines have compact support contained in  $Q_T$ , we can extend the integrals to  $\mathbb{R}^3$ , and the matrix entries decompose into sums and products of inner products (for clarity, we denote the inner product of  $u$  and  $v$  on  $L^2(\mathbb{R})$  by  $\langle u, v \rangle$ , the inner product

<sup>2</sup>From here on,  $i, j, k$  will always denote indices of univariate spline basis elements, while  $k_i + 2$  still denotes the number of breakpoints for the univariate splines of  $\mathcal{B}$ .

on  $L^2(\mathbb{R}^2)$  by  $\langle u, v \rangle_2$ , drop the arguments, and write  $B''_{1,i}$  for the second derivative with respect to the argument):

$$\begin{aligned}
(5.20) \quad M_\varepsilon(u_i, u_j) = & \langle c^{-2} B_{i,1} B_{j,2}, c^{-2} B_{m,1} B_{n,2} \rangle_2 \langle B''_{k,3}, B''_{o,3} \rangle \\
& - \langle c^{-2} B_{i,1} B_{j,2}, B''_{m,1} B_{n,2} \rangle_2 \langle B_{k,3}, B''_{o,3} \rangle \\
& - \langle c^{-2} B_{i,1} B_{j,2}, B_{m,1} B''_{n,2} \rangle_2 \langle B''_{k,3}, B_{o,3} \rangle \\
& - \langle c^{-2} B''_{i,1} B_{j,2}, B_{m,1} B_{n,2} \rangle_2 \langle B''_{k,3}, B_{o,3} \rangle \\
& - \langle c^{-2} B_{i,1} B''_{j,2}, B_{m,1} B_{n,2} \rangle_2 \langle B_{k,3}, B''_{o,3} \rangle \\
& + \langle B''_{i,1}, B_{m,1} \rangle \langle B_{j,2}, B''_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle \\
& + \langle B_{i,1}, B''_{m,1} \rangle \langle B''_{j,2}, B_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle \\
& + (1 + \varepsilon) \langle B''_{i,1}, B''_{m,1} \rangle \langle B_{j,2}, B_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle \\
& + (1 + \varepsilon) \langle B_{i,1}, B_{m,1} \rangle \langle B''_{j,2}, B''_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle \\
& + \varepsilon \langle B_{i,1}, B_{m,1} \rangle \langle B_{j,2}, B_{n,2} \rangle \langle B''_{k,3}, B''_{o,3} \rangle \\
& + \varepsilon \langle B_{i,1}, B_{m,1} \rangle \langle B_{j,2}, B_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle .
\end{aligned}$$

Similarly, we calculate the right hand side

$$(5.21) \quad F := (F_i)_i, \quad F_i := - \langle L\Phi, Lu_i \rangle_{L^2(Q_T)}, \quad u_i \in \mathcal{B}^0, \quad i = 1, \dots, N,$$

by inserting (5.16), the cubic  $B$ -spline interpolant of  $\Phi$  (which does not vanish on the spatial boundary):

$$\begin{aligned}
(5.22) \quad F_i = & \sum_{m=1}^{k_1+4} \sum_{n=1}^{k_2+4} \sum_{o=1}^{k_3+4} \varphi_{mno} \cdot \left( - \langle c^{-2} B_{i,1} B_{j,2}, c^{-2} B_{m,1} B_{n,2} \rangle_2 \langle B''_{k,3}, B''_{o,3} \rangle \right. \\
& + \langle c^{-2} B_{i,1} B_{j,2}, B''_{m,1} B_{n,2} \rangle_2 \langle B_{k,3}, B''_{o,3} \rangle \\
& + \langle c^{-2} B_{i,1} B_{j,2}, B_{m,1} B''_{n,2} \rangle_2 \langle B''_{k,3}, B_{o,3} \rangle \\
& + \langle c^{-2} B''_{i,1} B_{j,2}, B_{m,1} B_{n,2} \rangle_2 \langle B''_{k,3}, B_{o,3} \rangle \\
& + \langle c^{-2} B_{i,1} B''_{j,2}, B_{m,1} B_{n,2} \rangle_2 \langle B_{k,3}, B''_{o,3} \rangle \\
& - \langle B''_{i,1}, B_{m,1} \rangle \langle B_{j,2}, B''_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle \\
& - \langle B_{i,1}, B''_{m,1} \rangle \langle B''_{j,2}, B_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle \\
& - \langle B''_{i,1}, B''_{m,1} \rangle \langle B_{j,2}, B_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle \\
& \left. - \langle B_{i,1}, B_{m,1} \rangle \langle B''_{j,2}, B''_{n,2} \rangle \langle B_{k,3}, B_{o,3} \rangle \right).
\end{aligned}$$

These inner products can be precalculated efficiently in several ways. Since  $B$ -splines are piecewise polynomials, Gaussian quadrature is the most stable of these which is exact for  $B$ -splines and generalizes easily to products of arbitrary and multivariate functions. For low order splines, the performance is comparable to other methods based on recurrence relations or partial integration (cf. [31]). Between their break points  $x_i$ , cubic  $B$ -splines are polynomials of order 4, so their product is a polynomial of order 7. These can be exactly integrated by a quadrature rule of order 4. The products involving the second derivatives of  $B$ -splines are of order 5, so a quadrature rule of order 3 suffices. Hence we split the integrals (e.g., over  $x$ ) into sums of integrals over the  $k_1 + 1$  intervals defined by the break points  $x_\nu \in \Delta_1$ , and



after scaling to the interval  $[-1, 1]$ , apply the appropriate quadrature rule of order  $d \in \{3, 4\}$ :

$$(5.23) \quad \langle B_{i,1}, B_{m,1} \rangle = \sum_{\nu=0}^{k_1} \left( \frac{(x_{\nu+1} - x_\nu)}{2} \sum_{\mu=1}^d \gamma_\mu B_{i,1}(x_\mu) B_{m,1}(x_\mu) \right),$$

with  $x_\mu := \frac{1}{2}((x_{\nu+1} - x_\nu)t_\mu + (x_{\nu+1} + x_\nu))$ , where  $t_\mu$  are the Gauss-Legendre points with corresponding weights  $\gamma_\mu$ .

Similarly, for the inner product on  $L^2(\mathbb{R}^2)$ , we use the tensor product Gauss rule

$$(5.24) \quad \langle c^{-2} B_{i,1} B_{j,2}, c^{-2} B_{m,1} B_{n,2} \rangle_2 \\ \approx \sum_{\nu_1=0}^{k_1} \sum_{\nu_2=0}^{k_2} \left( h_{\nu_1, \nu_2} \sum_{\mu_1, \mu_2=1}^d \gamma_{\mu_1} \gamma_{\mu_2} \frac{B_{i,1}(x_{\mu_1}) B_{j,2}(y_{\mu_2}) B_{m,1}(x_{\mu_1}) B_{n,2}(y_{\mu_2})}{c^4(x_{\mu_1}, y_{\mu_2})} \right),$$

where  $h_{\nu_1, \nu_2} := \frac{1}{4}(x_{\nu_1+1} - x_{\nu_1})(y_{\nu_2+1} - y_{\nu_2})$ , and  $x_{\mu_1}, y_{\mu_2}$  are again the one dimensional transformed Gauss-Legendre points with the weights  $\gamma_{\mu_1}, \gamma_{\mu_2}$ .

By the above, we arrive at the system of linear equations  $Ma = F$  for the unknown  $a = (a_i)_i$  in (5.17). As the support of  $B_i$  and  $B_j$  (and *a fortiori* of their derivatives) intersect only for  $i-3 \leq j \leq i+3$ , at most  $343N$  entries of  $M$  are nonzero, so the matrix is sparse, banded, and diagonally dominant. The bilinear form  $M_\varepsilon$  is symmetric and elliptic, therefore  $M$  is also symmetric and positive definite, and a preconditioned conjugate gradient method can be used as a fast solver for this system.

**6. Numerical results.** We have implemented the Ritz-Galerkin scheme presented above using Matlab. Specifically, on the square  $\Omega = [-3, 3] \times [-3, 3]$ , we prescribe a sound speed  $c(x, y)$  and an initial condition  $u_0(x, y)$ . The data is then generated by solving the standard well-posed Cauchy problem with the initial conditions  $u(x, y, t)|_{t=0} = u_0(x, y)$ ,  $\partial_t u(x, y, t)|_{t=0} \equiv 0$ . This is done on a larger domain, where we impose homogeneous Dirichlet boundary conditions due to the finite speed of propagation. Both sound speed and initial condition are continued on this extended domain by linear extrapolation. To avoid committing an inverse crime, we solve the forward problem using a semidiscretized finite element method on an unstructured discretization. As our initial condition to be reconstructed, we use

$$(6.1) \quad u_0(x, y) = e^{-(x^2+y^2)} \sin(3x) \cos(3y),$$

which is negligible outside  $\Omega$ . The solution and its gradient is then extracted at a given partition  $\Delta$  of  $\Omega \times [0, T]$  (if necessary, by interpolation) and  $\Phi_h$  is calculated. To simulate errors in the measurement data, we introduce noise to the calculated  $B$ -spline coefficients  $\varphi_{ijk}$ :

$$(6.2) \quad \varphi_{ijk}^\delta = (1 + \delta\xi)\varphi_{ijk},$$

where  $\delta \geq 0$  is a given noise level and  $\xi$  is a random number uniformly distributed between  $-1$  and  $1$ . With these coefficients, we compute the matrix  $M$  and right hand side vector  $F$  as described above. The system of linear equations  $Ma = F$  for the spline coefficients of  $u_h$  is solved to a tolerance<sup>3</sup> of  $10^{-6}$  by a stabilized biconjugate

<sup>3</sup>Since the  $B$ -splines are non-negative and sum to 1 at any point, this tolerance also holds for the function  $u_h$  by way of the triangle inequality.

$\delta, \varepsilon$	0	$10^{-6}$	$10^{-5}$	$10^{-4}$	$10^{-3}$	$10^{-1}$	1.0
0	0.15971	0.15964	0.15901	0.15281	0.09858	0.87138	0.99522
0.05	0.16029	0.16041	0.15768	0.15073	0.09941	0.87178	0.99524
0.1	0.16211	0.15708	0.15837	0.15076	0.09618	0.87057	0.99527
0.2	0.16614	0.15912	0.16017	0.15119	0.09509	0.86999	0.99539
0.4	0.16653	0.18398	0.17926	0.15146	0.11096	0.87364	0.99583
0.5	0.19039	0.14141	0.18613	0.16458	0.13000	0.86619	0.99617
1.0	0.20092	0.23337	0.18634	0.21325	0.12906	0.88003	0.99444
2.0	0.31615	0.38390	0.43442	0.27649	0.34904	0.85463	0.99199
3.0	0.54785	0.43724	0.52161	0.48232	0.55094	0.84709	0.99659
4.0	0.78441	0.69156	0.64085	0.98342	0.64522	0.86108	0.99543

TABLE 6.1

Relative  $L^2(\Omega)$ -errors of reconstructions of (6.1) in a homogeneous medium for various noise levels  $\delta \in [0, 4]$  and regularization parameters  $\varepsilon \in [0, 1]$ .

gradient method (BICGSTAB, provided by Matlab). Due to the matrix being diagonally dominant, we employ a Jacobi prescaling (ie. solving  $DMDa = DF$  with the diagonal matrix  $D_{ii} = \frac{1}{\sqrt{M_{ii}}}$ ,  $i = 1, \dots, N$ ) in order to improve performance and stability. This has proved more efficient than specialized preconditioners, since the matrix is diagonally dominant. We illustrate the effectiveness and robustness of our approach in different situations with several tests. For all these tests, we choose a uniform discretization of  $\Omega$  with mesh width  $h_{1,2} = 0.2$  and of  $[0, T]$  with mesh width  $h_3 = 0.1$ . With this discretization, the calculations took around 25 minutes on a 2.2 GHz Opteron workstation, using 2 gigabytes of memory.

**6.1. Test 1: Constant coefficients.** To evaluate the influence of a heterogeneous medium, we first test our method with a constant coefficient, which we take to be  $c(x) \equiv 1$ . Consequently, we set  $T := 7$ . The reconstructions  $u_\varepsilon(x, y, 0)$  for various noise levels  $\delta \in [0, 3]$  are compared in Figure 6.1, while the relative  $L^2(\Omega)$ -errors for several values of  $\delta \in [0, 4]$  and  $\varepsilon \in [0, 1]$  are given in Table 6.1.

As can be seen, the reconstructions of the initial conditions are good, even with very high noise levels of up to 300%. The Lipschitz stability estimate of Theorem 4.6 provides an indication of this. Of note is that the optimal parameter  $\varepsilon = 10^{-3}$  is the same for all noise levels up to  $\delta = 1$ , and gives comparable results for the higher levels. This further emphasizes the robustness of the method.

**6.2. Test 2: Smooth coefficients.** Next, we repeat these experiments for a heterogeneous medium. Taking (cf. Figure 6.2, left)

$$(6.3) \quad \frac{1}{c^2(x, y)} = \frac{5}{2} - \frac{1}{12}(x^2 + y^2),$$

we satisfy the conditions (4.1) with  $x_0 = 0$ . According to Theorem 4.6, we should take  $\sqrt{\beta} < \frac{1}{197}$ , hence  $T := 1672$ . However, for numerical calculations,  $T := 7$  (a lower bound for the time needed for a wave to propagate from any point in  $\Omega$  to a point on  $\partial\Omega$ ) turned out to be sufficient. Since the bounds from Carleman estimates for hyperbolic equations with variable coefficients are known not to be sharp, this was to be expected. Again, the relative  $L^2(\Omega)$ -errors are given in Table 6.2, while the solutions are compared in Figure 6.3.

Here as well the reconstructions are close to the prescribed initial conditions, although the method is now a little less tolerant to noise.

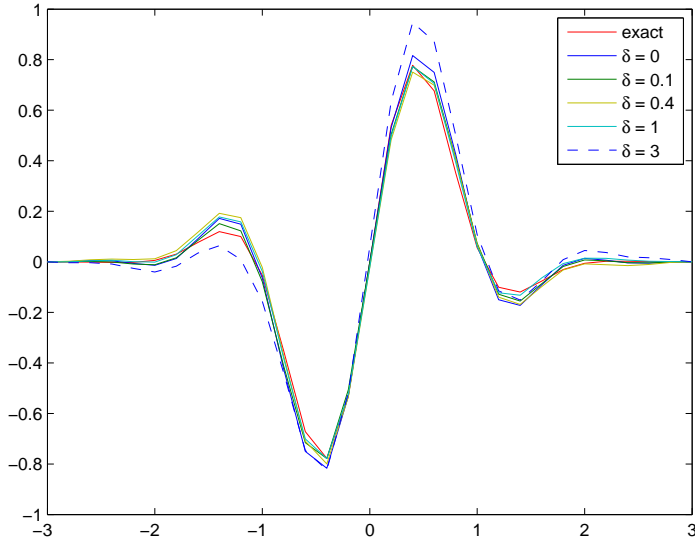


FIG. 6.1. Comparisons of reconstructed initial condition  $u_\varepsilon(x, y, 0)$  in a homogeneous medium for various noise levels  $\delta$  and regularization parameter  $\varepsilon = 10^{-3}$ . Shown are slices  $u_\varepsilon(x, 0, 0)$ ,  $x \in [-3, 3]$ .

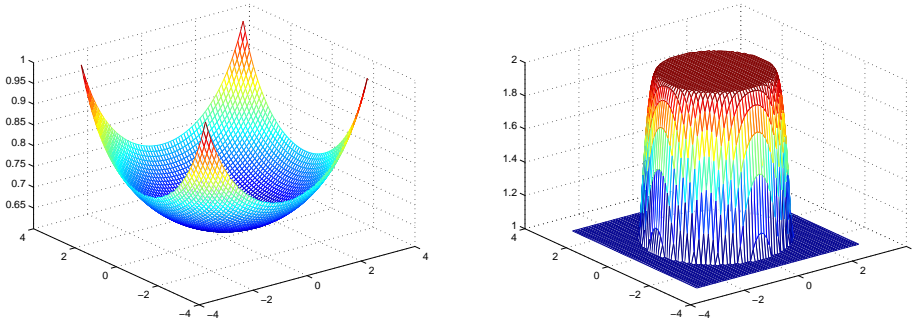


FIG. 6.2. Heterogeneous medium: sound speed  $c$  from (6.3) (left), from (6.4) (right).

$\delta, \varepsilon$	0	$10^{-6}$	$10^{-5}$	$10^{-4}$	$10^{-3}$	$10^{-1}$	1.0
0	0.14984	0.14690	0.14850	0.15036	0.12667	0.76300	0.99530
0.05	0.14253	0.13900	0.15360	0.13643	0.11971	0.76330	0.99540
0.1	0.14457	0.14330	0.14200	0.14502	0.10915	0.76364	0.99513
0.2	0.13169	0.14170	0.15068	0.13298	0.10365	0.76320	0.99554
0.5	0.15338	0.17681	0.16089	0.15436	0.14229	0.75604	0.99508
1.0	0.17350	0.22785	0.22412	0.21235	0.19087	0.75570	0.99376

TABLE 6.2

Relative  $L^2(\Omega)$ -errors of reconstructions of (6.1) in a heterogeneous medium for various noise levels  $\delta$  and regularization parameters  $\varepsilon$ .

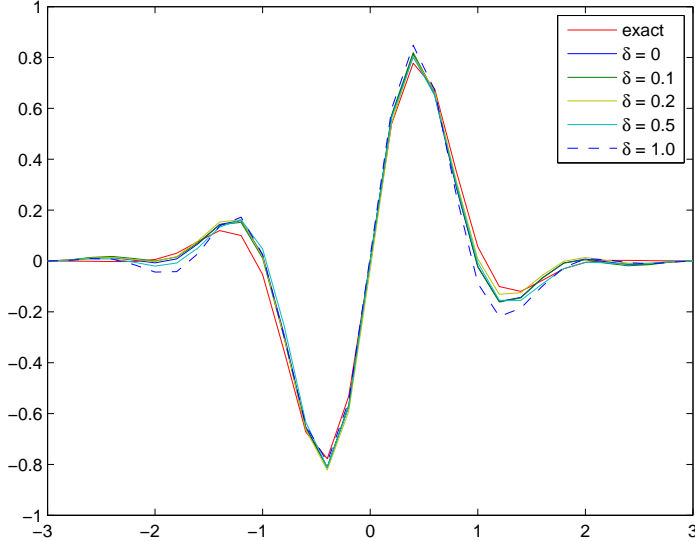


FIG. 6.3. Comparisons of reconstructed initial condition  $u_\varepsilon(x, y, 0)$  in a heterogeneous medium (6.3) for various noise levels  $\delta$  and regularization parameter  $\varepsilon = 10^{-3}$ . Shown are slices  $u_\varepsilon(x, 0, 0)$ ,  $x \in [-3, 3]$ .

**6.3. Test 3: Nondifferentiable coefficients.** Since we are interested in applications in TCT, we also test our method with a more realistic sound speed, which is not covered by our theoretical results, such as a bone surrounded by soft tissue or water. Since the speed of sound in bone is roughly twice of that in water, we model this situation by the following speed distribution (cf. Figure 6.2, right):

$$(6.4) \quad c(x, y) = \max \left( 2 - \left( \frac{\max(2 - 5 + x^2 + y^2, 0)}{2} \right)^2, 1 \right).$$

The results are shown in Figure 6.4.

Since the constant in the error estimate (4.31) depends (by the Carleman estimate (4.5)) on the gradient of  $c^{-2}$ , we expect in this case the inevitable numerical errors to be larger. Still, the reconstruction is only slightly worse than for smooth coefficients.

**6.4. Test 4: Smooth coefficients, limited boundary data.** If the boundary data is given only on a part  $\Gamma$  of the boundary, we expect the reconstruction to deteriorate. As a heterogeneous medium, we again take the smooth coefficient (6.3). Although the results of section 3 can be extended<sup>4</sup> to show that the quasi-reversibility approximation has a unique solution in this case as well, the strong Lipschitz convergence no longer holds. We still attempt a numerical reconstruction, considering in section 5, instead of  $S_h$ , the space  $\mathcal{S}^4 \cap \{u \in H^2(Q_T) : u|_\Gamma \equiv \partial_\nu u|_\Gamma = 0\}$ , where we take  $\Gamma := \{(x, y) \in \partial\Omega : x = -3 \text{ or } y = -3\}$ . Existence and uniqueness of the

<sup>4</sup>The critical step is the equivalence of the norms, which holds (due to the Poincaré inequality) as long as  $\Gamma \subset \partial\Omega$  has a positive  $n$ -dimensional Lebesgue measure.

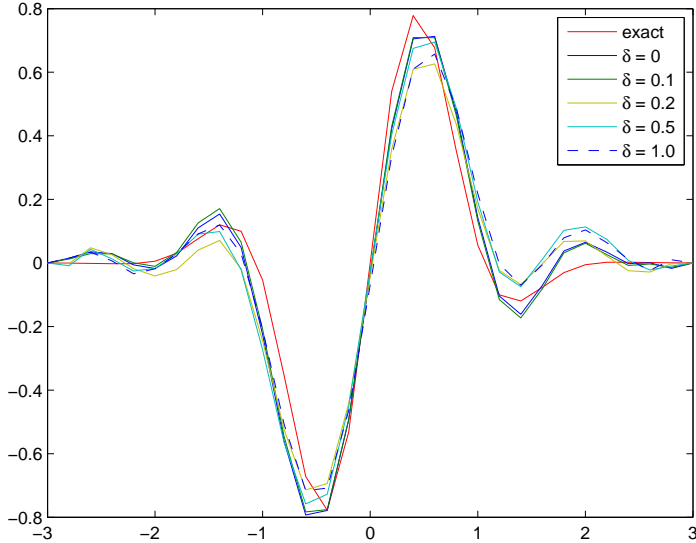


FIG. 6.4. Comparisons of reconstructed initial condition  $u_\varepsilon(x, y, 0)$  in a heterogeneous medium (6.4) for various noise levels  $\delta$  and regularization parameter  $\varepsilon = 10^{-4}$ . Shown are slices  $u_\varepsilon(x, 0, 0)$ ,  $x \in [-3, 3]$ .

Ritz-Galerkin solution, as well as the error estimate, still hold in this case. For our calculations, it was again sufficient to take the minimal time for a wave starting in any point of  $\Omega$  to reach  $\Gamma$ , which is  $T := 14$ . The results are shown in Figure 6.5.

Due to energy loss at the boundary  $\partial\Omega \setminus \Gamma$ , the quasi-reversibility solution has smaller amplitude. The shape of the initial condition, however, is still recovered well for all noise levels.

**7. Conclusion.** We have presented a direct and robust method for the numerical solution of the problem of thermoacoustic tomography in a heterogeneous medium. The main advantages are the high resistance to noise in the lateral Cauchy data and the relative independence from parameter choices (since taking, e.g.,  $\varepsilon = 10^{-3}$  yielded the best results in most of our calculations, and comparable results in the other cases). Since the error estimates and convergence results in this paper also hold for more general hyperbolic operators of second order, our method is also applicable when lower order terms are present. Our approach theoretically is also feasible for time-dependent coefficients, which would allow medical imaging of moving targets (e.g., beating hearts, breathing lungs, or kidney stones).

The computational method presented can be applied to arbitrary (Lipschitz) domains by using weighted  $B$ -splines which vanish to second order on the boundary (cf. [9]). The only difficulty is then the construction of the function  $\Phi$ , which could be handled by a transformation to a rectangular domain. The restriction to uniform knot vectors can also be lifted, which together with the efficient knot insertion algorithms for  $B$ -splines opens the way to adaptive refinement strategies based on *a posteriori* error estimates.

Finally, it would be interesting to derive quasi-reversibility approximations for

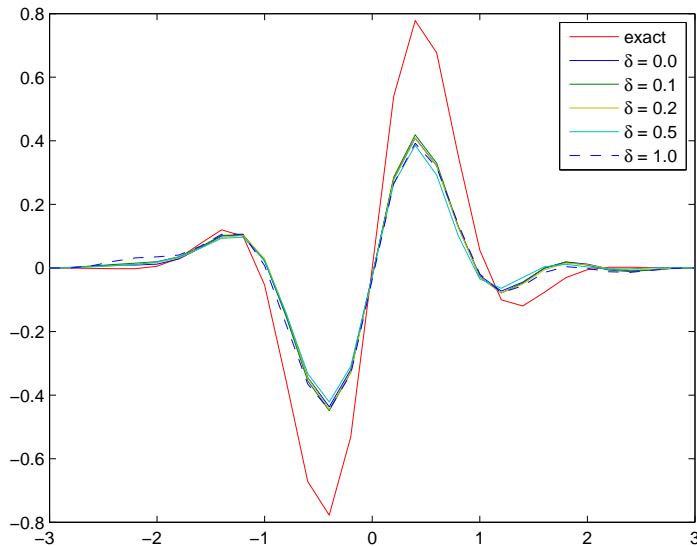


FIG. 6.5. Comparisons of reconstructed solution  $u(x, y, t)$  from limited boundary data in a heterogeneous medium (6.3) for various noise levels  $\delta$  and regularization parameter  $\varepsilon = 10^{-3}$ . Shown are slices  $u_\varepsilon(x, 0, 0)$ ,  $x \in [-3, 3]$ .

systems of linear equations and investigate their numerical solutions. Since observability estimates have been proven for the systems of elasticity and electromagnetism (cf. [3] and [26], respectively), it is to be expected that stability and convergence estimates similar to the ones derived here can be obtained.

**Acknowledgments.** The authors wish to thank Alexander Kallischko for valuable advice on different preconditioners. They would also like to express thanks to the referees for their comments which helped improve the presentation, and for bringing the book of Komornik to their attention. The work of M.V. Klibanov was supported by, or in part by the U.S. Army Research Laboratory and U.S. Army Research Office under contract/grant number W911NF-05-1-0378.

#### REFERENCES

- [1] L. BOURGEOIS, *A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace's equation*, *Inverse Problems*, 21 (2005), pp. 1087–1104.
- [2] ———, *Convergence rates for the quasi-reversibility method to solve the Cauchy problem for Laplace's equation*, *Inverse Problems*, 22 (2006), pp. 413–430.
- [3] J. CHENG, V. ISAKOV, M. YAMAMOTO, AND Q. ZHOU, *Lipschitz stability in the lateral Cauchy problem for elasticity system*, *J. Math. Kyoto Univ.*, 43 (2003), pp. 475–501.
- [4] C. DE BOOR, *A practical guide to splines*, Springer-Verlag, New York, revised ed., 2001.
- [5] H. W. ENGL, M. HANKE, AND A. NEUBAUER, *Regularization of inverse problems*, Kluwer Academic Publishers Group, Dordrecht, 1996.
- [6] D. FINCH, S. K. PATCH, AND RAKESH, *Determining a function from its mean values over a family of spheres*, *SIAM J. Math. Anal.*, 35 (2004), pp. 1213–1240.
- [7] M. HALTMEIER, O. SCHERZER, P. BURGHOLZER, AND G. PALTAUF, *Thermoacoustic computed tomography with large planar receivers*, *Inverse Problems*, 20 (2004), pp. 1663–1673.
- [8] M. HALTMEIER, T. SCHUSTER, AND O. SCHERZER, *Filtered backprojection for thermoacous-*

- tic computed tomography in spherical geometry*, Math. Methods Appl. Sci., 28 (2005), pp. 1919–1937.
- [9] K. HÖLLIG, *Finite element methods with B-splines*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.
- [10] L. HÖRMANDER, *The analysis of linear partial differential operators. IV*, Springer-Verlag, Berlin, 1994.
- [11] S. I. KABANIKHIN, M. A. BEKTEMESOV, AND D. V. NECHAEV, *Numerical solution of 2d thermoacoustic problem*, J. Inv. Ill-Posed Problems, 13 (2005), pp. 1–12.
- [12] M. A. KAZEMI AND M. V. KLIBANOV, *Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities*, Appl. Anal., 50 (1993), pp. 93–102.
- [13] M. V. KLIBANOV AND P. G. DANILAEV, *On the solution of coefficient inverse problems by the method of quasi-inversion.*, Sov. Math., Dokl., 41 (1990), pp. 83–87.
- [14] ———, *Estimates of initial conditions of parabolic equations and inequalities via lateral Cauchy data*, Inverse Problems, 22 (2006), pp. 495–514.
- [15] M. V. KLIBANOV AND J. MALINSKY, *Newton-Kantorovich method for three-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time-dependent data*, Inverse Problems, 7 (1991), pp. 577–596.
- [16] M. V. KLIBANOV AND RAKESH, *Numerical solution of a time-like Cauchy problem for the wave equation*, Math. Methods Appl. Sci., 15 (1992), pp. 559–570.
- [17] M. V. KLIBANOV AND F. SANTOSA, *A computational quasi-reversibility method for Cauchy problems for Laplace’s equation*, SIAM J. Appl. Math., 51 (1991), pp. 1653–1675.
- [18] M. V. KLIBANOV AND A. TIMONOV, *Carleman estimates for coefficient inverse problems and numerical applications*, VSP, Utrecht, 2004.
- [19] V. KOMORNIK, *Exact controllability and stabilization: the multiplier method*, Masson, Paris, 1994.
- [20] R. A. KRUGER, W. L. KISER, K. D. MILLER, H. E. REYNOLDS, D. R. REINECKE, G. A. KRUGER, AND P. J. HOFACKER, *Thermoacoustic CT: imaging principles*, Proc. SPIE, 3916 (2000), pp. 150–159.
- [21] R. A. KRUGER, K. K. KOPECKY, A. M. AISEN, D. R. REINECKE, G. A. KRUGER, AND W. L. K. JR., *Thermoacoustic CT with radio waves: a medical imaging paradigm.*, Radiology, 211 (1999), pp. 275–278.
- [22] R. A. KRUGER, K. STANTZ, AND W. L. KISER, *Thermoacoustic CT of the breast*, Proc. SPIE, 4682 (2002), pp. 521–525.
- [23] G. KU AND L. V. WANG, *Scanning thermoacoustic tomography in biological tissue.*, Med Phys, 27 (2000), pp. 1195–1202.
- [24] R. LATTÈS AND J.-L. LIONS, *The method of quasi-reversibility. Applications to partial differential equations*, American Elsevier Publishing Co., Inc., New York, 1969.
- [25] M. LAVRENT’EV, V. ROMANOV, AND S. SHISHATSKIJ, *Ill-posed problems of mathematical physics and analysis*, American Mathematical Society (AMS), Providence, R.I., 1986.
- [26] S. LI AND M. YAMAMOTO, *Carleman estimate for Maxwell’s equations in anisotropic media and the observability inequality*, Journal of Physics: Conference Series, 12 (2005), pp. 110–115.
- [27] P. LIU, *Image reconstruction from photoacoustic pressure signals*, in Proc. SPIE Vol. 2681, p. 285–296, Laser-Tissue Interaction VII, Steven L. Jacques; Ed., S. L. Jacques, ed., May 1996, pp. 285–296.
- [28] I. J. SCHOENBERG, *Contributions to the problem of approximation of equidistant data by analytic functions. Part B. On the problem of osculatory interpolation. A second class of analytic approximation formulae*, Quart. Appl. Math., 4 (1946), pp. 112–141.
- [29] M. H. SCHULTZ, *Approximation theory of multivariate spline functions in Sobolev spaces*, SIAM J. Numer. Anal., 6 (1969), pp. 570–582.
- [30] L. L. SCHUMAKER, *Spline functions: basic theory*, John Wiley & Sons Inc., New York, 1981.
- [31] A. H. VERMEULEN, R. H. BARTELS, AND G. R. HEPPLER, *Integrating products of B-splines*, SIAM J. Sci. Statist. Comput., 13 (1992), pp. 1025–1038.
- [32] V. VLADIMIROV, *Equations of mathematical physics*, Marcel Dekker, New York, 1971.
- [33] J. WLOKA, *Partial differential equations*, Cambridge University Press, Cambridge, 1987.
- [34] M. XU, Y. XU, AND L. V. WANG, *Time-domain reconstruction algorithms and numerical simulations for thermoacoustic tomography in various geometries.*, IEEE Trans Biomed Eng, 50 (2003), pp. 1086–1099.
- [35] Y. XU, D. FENG, AND L. V. WANG, *Exact frequency-domain reconstruction for thermoacoustic tomography—I: Planar geometry.*, IEEE Trans Med Imaging, 21 (2002), pp. 823–828.