

Variational Analysis in Spectral Decomposition Systems*

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Abstract. This work is concerned with variational analysis of so-called spectral functions and spectral sets of matrices that only depend on eigenvalues of the matrix. Based on our previous work [7] on convex analysis of such functions, we consider the question in the abstract framework of spectral decomposition systems, which covers a wide range of previously studied settings, including eigenvalue decomposition of Hermitian matrices and singular value decomposition of rectangular matrices, and allows deriving new results in more general settings such as normal decomposition systems and signed singular value decompositions. The main results characterize Fréchet and limiting normal cones to spectral sets as well as Fréchet, limiting, and Clarke subdifferentials of spectral functions in terms of the reduced functions. For the latter, we also characterize Fréchet differentiability. Finally, we obtain a generalization of Lidskiĭ's theorem on the spectrum of additive perturbations of Hermitian matrices to arbitrary spectral decomposition systems.

Keywords. Fréchet normal cone; limiting normal cone; Fréchet subdifferential; limiting subdifferential; Fréchet differentiability; Clarke subdifferential; Lidskiĭ's theorem; spectral decomposition system; spectral function; spectral set;

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1. Introduction

Many practically relevant optimization problems are naturally posed in terms of matrices instead of vectors; prominent examples include non-negative matrix factorization [23], matrix completion [8], low-rank approximation [18, 39], or operator learning [26]. Particularly – but not only – in the last example, one is actually interested in optimizing over (finite-dimensional) linear operators and not their particular matrix representations. This implies that the functions to be minimized should be invariant under basis changes. It is a well-known fact from linear algebra that, under appropriate assumptions, such functions are fully characterized by their dependence on the eigenvalues (or singular values) of its argument; the most well-known example is probably the nuclear norm of a matrix. Correspondingly, examples of such *spectral functions* are ubiquitous in applications such as robust matrix estimation [4], signal processing [9], conic programming [12], semi-definite programming [24], nonlinear elasticity [50], and brain network analysis [55]. Along the same lines, such optimization problems can include constraints in terms of *spectral sets* that are defined in terms of eigenvalues or singular values; the most common example is the set of positive (semi-)definite matrices.

While many of these problems can be formulated as convex problems, this is not always the case; by way of example we only mention low-rank matrix completion via Schatten p -norm minimization for $0 < p < 1$ [38] or mathematical programs with semidefinite cone complementarity constraints (SDCMPCC) [15]. Here the main question is about the characterization of the fundamental objects of variational analysis and geometry such as (Clarke, Fréchet, limiting) subdifferentials or normal cones of the spectral function or set in terms of the invariant function or set of the eigenvalues (or singular values) in order to obtain sharp necessary optimality conditions. The central challenge in this is the fact that the invariant function only depends on the *set* of eigenvalues but not their ordering. The same issue already arises in studying the Fréchet differentiability and characterizing the Fréchet derivative of spectral functions.

Correspondingly, variational analysis of convex and nonconvex spectral optimization problems has been studied in a variety of different settings. For symmetric functions of eigenvalues, [28] analyzed Fréchet differentiability and characterized Clarke subdifferentials while [31] treated Fréchet, limiting, horizon, and Clarke subdifferentials. Later, [16] provided a simplified approach for these results. Higher-order differentiability was studied in [2], where also other related type of functions such as radial functions and isotropic functions were considered. Following the same patterns as in [31], the series of works [34, 35] studied Fréchet, limiting, horizon, and Clarke subdifferentials of signed-symmetric functions of singular values of rectangular matrices. These results have also been generalized to the setting of Euclidean Jordan algebras (see [Example 2.10](#) below). Specifically, [52] analyzed (higher-order) differentiability analyticity, and semismoothness of Löwner's operator and spectral functions, [1] studied convex analysis and differentiability of (non-convex) spectral functions. On the other hand, following the pattern developed by Lewis in [28, 31] for the case of symmetric matrices, [37] characterized Fréchet, limiting, horizon, and Clarke subdifferentials. See also [51] for these results in the setting of a specific Euclidean Jordan algebra.

However, each of these works treated a specific setting in isolation. Building on our earlier work [7] on convex analysis of spectral functions, we therefore aim to bring together these results on Fréchet, limiting, and Clarke subdifferentials as well as Fréchet differentiability in a general framework that covers all these settings and – more importantly – allows deriving results more easily for settings and objects not covered so far. In a nutshell, we work in a *spectral decomposition system* consisting of

- (i) a family of *spectral decompositions* that generalize constructing a matrix with given eigenvalues (e.g., via a basis of eigenvectors);

- (ii) a *spectral mapping* that generalizes computing the eigenvalues from a given matrix;
- (iii) an *ordering mapping* that generalizes sorting eigenvalues in decreasing order;

that satisfy some natural compatibility conditions such as a generalization of von Neumann’s trace inequality; see [Definition 2.1](#) for a precise definition. To be able to treat limiting subdifferentials, we also require – in addition to our previous work – a closedness condition on the family of spectral decompositions ([Assumption 2.2](#)). This definition covers all previously considered settings in uniform generality (applying, for instance, in each case to matrices over the real, complex, or quaternion fields). Moreover, our results appear to be new in the setting of normal decomposition system [\[29\]](#), which settles the open question set forth in the discussion in the paragraph following [\[33, Theorem 7.2\]](#), as well as for the setting of signed singular values as studied in [\[13\]](#) and [\[50\]](#) (which treated only the case of convex spectral functions). Our general approach allows bypassing the matrix-dependent proof techniques of existing works via a geometric approach common in variational analysis: First we leverage the results in the first part [\[7\]](#) to establish characterizations of normal cones to spectral sets, which are of interest in their own right. We then transfer these results in the usual way via normal cones to epigraphs (with the help of “product space spectral decompositions”, see [Example 2.14](#)) to the subdifferentials of interest. Along the way, we give a full characterization of the Fréchet differentiability of spectral functions. Our characterization of the Clarke subdifferential also yields a generalized Lidskiĭ’s theorem on the effect of additive perturbations on the spectrum (see [Theorem 5.1](#)) that brings together the classical Lidskiĭ’s theorem [\[36\]](#), the version for rectangular matrices, the version for Euclidean Jordan algebras [\[25\]](#), and the version for Eaton triples [\[21\]](#) (in particular, the Lie-theoretic majorization result of [\[5\]](#)).

This work is organized as follows. In the next [Section 2](#), we recall the precise definition of spectral decomposition systems and illustrate how it covers previously studied settings. We also recall from [\[7\]](#) various basic properties that are fundamental to the analysis in this work. [Section 3](#) is then concerned with variational geometry of spectral sets, where we give full characterizations of the Fréchet and limiting normal cones ([Proposition 3.4](#)). In [Section 4](#), we exploit these results to obtain full characterizations of the Fréchet and limiting subdifferential of spectral functions in terms of their invariant functions ([Theorem 4.1](#)). The former in particular yields a full characterization of Fréchet differentiability and of Fréchet derivatives ([Corollary 4.8](#)). We also obtain from this a representation of the Clarke subdifferential of spectral functions ([Proposition 4.11](#)). In the final [Section 5](#), this is then used to derive a generalization of Lidskiĭ’s theorem in spectral decomposition systems ([Theorem 5.1](#)).

2. Spectral decomposition systems

Let $\mathcal{H}$ be a Euclidean space, that is, a finite-dimensional real Hilbert space. The scalar product and its associated norm are denoted by $\langle \cdot | \cdot \rangle$ and $\|\cdot\|$, respectively. (We will use $\|\cdot\|_{\mathcal{H}}$ when there is a potential ambiguity.) The space of linear operators from $\mathcal{H}$ to a Euclidean space $\mathcal{G}$ is denoted by $\mathcal{L}(\mathcal{H}, \mathcal{G})$ and we equip it with the topology induced by the norm

$$(\forall L \in \mathcal{L}(\mathcal{H}, \mathcal{G})) \quad \|L\|_{\mathcal{L}(\mathcal{H}, \mathcal{G})} = \max_{\substack{x \in \mathcal{H} \\ \|x\| \leq 1}} \|Lx\|, \quad (2.1)$$

and we write $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$. (Since we are working in finite-dimensional spaces, this topology coincides with the topology of pointwise convergence.) Group operations are written multiplicatively. Now let G be a group acting on $\mathcal{H}$, where we use $\cdot$ to denote group action. Additionally:

- The *orbit* of an element $x \in \mathcal{H}$ is $G \cdot x = \{g \cdot x \mid g \in G\}$.
- Given a nonempty set $\mathcal{U}$, a mapping $\Phi: \mathcal{H} \rightarrow \mathcal{U}$ is *G-invariant* if $(\forall x \in \mathcal{H})(\forall g \in G) \Phi(g \cdot x) = \Phi(x)$.
- A subset C of $\mathcal{H}$ is *G-invariant* if its indicator function

$$\iota_C: \mathcal{H} \rightarrow]-\infty, +\infty]: x \mapsto \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise,} \end{cases} \quad (2.2)$$

is *G-invariant* or, equivalently, $(\forall x \in C)(\forall g \in G) g \cdot x \in C$.

- We say that G *acts on $\mathcal{H}$ by linear isometries* if, for every $g \in G$, the mapping $\mathcal{H} \rightarrow \mathcal{H}: x \mapsto g \cdot x$ is a linear isometry.

2.1. Definitions, assumptions, and characterization

We are now ready to state the formal definition of the abstract framework for this work.

Definition 2.1 (spectral decomposition system [7, Definition 2.1]). A *spectral decomposition system* for a Euclidean space $\mathfrak{H}$ is a tuple $\mathfrak{S} = (\mathcal{X}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$, where $\mathcal{X}$ is a Euclidean space, S is a group which acts on $\mathcal{X}$ by linear isometries, γ is a mapping from $\mathfrak{H}$ to $\mathcal{X}$, and $(\Lambda_a)_{a \in \mathcal{A}}$ is a family of linear isometries from $\mathcal{X}$ to $\mathfrak{H}$ such that the following are satisfied:

[A] There exists an S -invariant mapping $\tau: \mathcal{X} \rightarrow \mathcal{X}$ such that

$$\left[(\forall x \in \mathcal{X}) \tau(x) \in S \cdot x \right] \quad \text{and} \quad \left[(\forall a \in \mathcal{A}) \gamma \circ \Lambda_a = \tau \right]. \quad (2.3)$$

[B] $(\forall X \in \mathfrak{H})(\exists a \in \mathcal{A}) X = \Lambda_a \gamma(X)$.

[C] $(\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H}) \langle X | Y \rangle \leq \langle \gamma(X) | \gamma(Y) \rangle$.

In which case:

- The mapping γ is called the *spectral mapping* of the system $\mathfrak{S}$.
- The mapping τ in property [A] is called the *spectral-induced ordering mapping* of the system $\mathfrak{S}$.
- We set

$$(\forall X \in \mathfrak{H}) \quad \mathcal{A}_X = \{a \in \mathcal{A} \mid X = \Lambda_a \gamma(X)\}. \quad (2.4)$$

(Property [B] guarantees that the sets $(\mathcal{A}_X)_{X \in \mathfrak{H}}$ are nonempty.)

- Given $X \in \mathfrak{H}$, the vector $\gamma(X)$ is called the *spectrum* of X with respect to $\mathfrak{S}$ and, for every $a \in \mathcal{A}_X$, the identity

$$X = \Lambda_a \gamma(X) \quad (2.5)$$

is called a *spectral decomposition* of X with respect to $\mathfrak{S}$.

Our results will be formulated under the following assumption, which extends the assumptions in [7] with a closedness condition on $\{\Lambda_a\}_{a \in \mathcal{A}}$ that will be essential in several limiting arguments in this paper. As we shall demonstrate in Section 2.2, this assumption is satisfied in all previously considered examples.

Assumption 2.2. $\mathfrak{H}$ is a Euclidean space and $\mathfrak{S} = (\mathcal{X}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ is a spectral decomposition system for $\mathfrak{H}$. Moreover, the set $\{\Lambda_a\}_{a \in \mathcal{A}}$ is closed in $\mathcal{L}(\mathcal{X}, \mathfrak{H})$.

We now formally define the classes of functions and sets that will be studied in this paper.

Definition 2.3 (spectral function and spectral set). Let $\mathfrak{H}$ be a Euclidean space, and let $(\mathcal{X}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ be a spectral decomposition system for $\mathfrak{H}$.

(i) A function $\Phi: \mathfrak{H} \rightarrow [-\infty, +\infty]$ is said to be a *spectral function* if

$$(\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H}) \quad \gamma(X) = \gamma(Y) \quad \Rightarrow \quad \Phi(X) = \Phi(Y). \quad (2.6)$$

(ii) A set $\mathcal{D} \subset \mathfrak{H}$ is said to be a *spectral set* if its indicator function $\iota_{\mathcal{D}}$ is a spectral function or, equivalently,

$$(\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H}) \quad \begin{cases} \gamma(X) = \gamma(Y) \\ X \in \mathcal{D} \end{cases} \quad \Rightarrow \quad Y \in \mathcal{D}. \quad (2.7)$$

The next result characterizes spectral functions precisely as those of the form

$$\Phi = \varphi \circ \gamma, \quad \text{where } \varphi: \mathcal{X} \rightarrow [-\infty, +\infty] \text{ is } S\text{-invariant}, \quad (2.8)$$

which thus motivates the study of variational analytic properties and objects attached to Φ in terms of those of φ .

Proposition 2.4 ([7, Proposition 4.1]). Let $\mathfrak{H}$ be a Euclidean space, and let $(\mathcal{X}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ be a spectral decomposition system for $\mathfrak{H}$. Then a function $\Phi: \mathfrak{H} \rightarrow [-\infty, +\infty]$ is a spectral function if and only if there exists an S -invariant function $\varphi: \mathcal{X} \rightarrow [-\infty, +\infty]$ such that $\Phi = \varphi \circ \gamma$, in which case, φ is uniquely determined by

$$(\forall a \in \mathcal{A}) \quad \varphi = \Phi \circ \Lambda_a, \quad (2.9)$$

and we call φ the invariant function associated with Φ .

By specializing [Proposition 2.4](#) to indicator functions, we obtain a characterization of spectral sets precisely as those of the form

$$\mathcal{D} = \gamma^{-1}(D), \quad \text{where } D \subset \mathcal{X} \text{ is } S\text{-invariant}. \quad (2.10)$$

Corollary 2.5 ([7, Corollary 4.3]). Let $\mathfrak{H}$ be a Euclidean space, and let $(\mathcal{X}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ be a spectral decomposition system for $\mathfrak{H}$. Then a set $\mathcal{D} \subset \mathfrak{H}$ is a spectral set if and only if there exists an S -invariant set $D \subset \mathcal{X}$ such that $\mathcal{D} = \gamma^{-1}(D)$, in which case D is uniquely determined by

$$(\forall a \in \mathcal{A}) \quad D = \Lambda_a^{-1}(\mathcal{D}), \quad (2.11)$$

and we call D the invariant set associated with $\mathcal{D}$.

2.2. Examples

We now illustrate the versatility of spectral decomposition systems. The examples here are mostly drawn from [7], where we first introduced this notion and to which we refer the reader for additional background and details. In addition, we verify that the additional closedness assumption required in this work is satisfied for these examples. To that end, let us introduce some notation.

Notation 2.6. Let M and N be strictly positive integers.

- $\mathbb{K}$ denotes one of the following: the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, or the skew-field $\mathbb{H}$ of Hamiltonian quaternions (we refer the reader to [49] for background on quaternions).
- The canonical involution on $\mathbb{K}$ is $\xi \mapsto \bar{\xi}$, which fixes only the elements of $\mathbb{R}$.
- $\mathbb{K}^{M \times N}$ stands for the real vector space of $M \times N$ matrices with entries in $\mathbb{K}$.
- The conjugate transpose of $X = [\xi_{ij}]_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} \in \mathbb{K}^{M \times N}$ is $X^* = [\bar{\xi}_{ji}]_{\substack{1 \leq j \leq N \\ 1 \leq i \leq M}} \in \mathbb{K}^{N \times M}$.
- The trace of a matrix $X \in \mathbb{K}^{N \times N}$ is denoted by $\text{tra } X$.
- $\mathcal{H}^N(\mathbb{K}) = \{X \in \mathbb{K}^{N \times N} \mid X = X^*\}$ is the vector subspace of $\mathbb{K}^{N \times N}$ which consists of Hermitian matrices.
- $\mathcal{U}^N(\mathbb{K}) = \{U \in \mathbb{K}^{N \times N} \mid U^*U = \text{Id}\}$ is the multiplicative group of unitary matrices.
- $\text{SO}^N = \{U \in \mathcal{U}^N(\mathbb{R}) \mid \det U = 1\}$ is the special orthogonal group.
- $\mathcal{P}_{\pm}^N$ denotes the multiplicative group of $N \times N$ *signed permutation matrices*, that is, matrices which contain exactly one nonzero entry in every row and every column, and that entry is -1 or 1 .
- $\mathcal{P}^N$ is the group of permutation matrices, that is, the subgroup of $\mathcal{P}_{\pm}^N$ which consists of matrices with entries in $\{0, 1\}$.
- For every $x = (\xi_i)_{1 \leq i \leq N} \in \mathbb{R}^N$, $x^{\downarrow}$ denotes the rearrangement vector of x with entries listed in decreasing order, and $|x|^{\downarrow}$ denotes the rearrangement vector of $(|\xi_i|)_{1 \leq i \leq N}$ with entries listed in decreasing order.
- $\mathcal{O}(\mathcal{H})$ stands for the orthogonal group of a Euclidean space $\mathcal{H}$, that is, the set of linear isometries from $\mathcal{H}$ to $\mathcal{H}$ equipped with the composition operation.

Our first example shows that every Euclidean space admits a spectral decomposition system.

Example 2.7. Let $2 \leq N \in \mathbb{N}$, set

$$e = (1, 0, \dots, 0) \in \mathbb{R}^N \quad \text{and} \quad (\forall U \in \mathcal{U}^N(\mathbb{R})) \quad \Lambda_U: \mathbb{R} \rightarrow \mathbb{R}^N: \xi \mapsto \xi U e, \quad (2.12)$$

and let the multiplicative group $\{-1, 1\}$ act on $\mathbb{R}$ via multiplication. Then $(\mathbb{R}, \{-1, 1\}, \|\cdot\|_2, (\Lambda_U)_{U \in \mathcal{U}^N(\mathbb{R})})$ is a spectral decomposition system for $\mathbb{R}^N$, and the set $\{\Lambda_U\}_{U \in \mathcal{U}^N(\mathbb{R})}$ is closed in $\mathcal{L}(\mathbb{R}, \mathbb{R}^N)$.

Proof. The former assertion is proved in [7, Example 2.3], while the latter follows from the closedness of $\mathcal{U}^N(\mathbb{R})$ in $\mathbb{R}^{N \times N}$ for the topology of pointwise convergence. $\square$

Our next example shows that the normal decomposition system framework of [29] can be viewed as a spectral decomposition system. This subsumes, in particular, the Lie-theoretic framework of [5, 32, 53].

Example 2.8 (normal decomposition system). Let $(\mathfrak{H}, G, \gamma)$ be a *normal decomposition system* in the sense of [29, Definition 2.1], that is, $\mathfrak{H}$ is a Euclidean space, G is a closed subgroup of $O(\mathfrak{H})$, and $\gamma: \mathfrak{H} \rightarrow \mathfrak{H}$ is a G -invariant mapping (here G acts on $\mathfrak{H}$ via the canonical action $(g, X) \mapsto g(X)$) such that

$$\begin{cases} (\forall X \in \mathfrak{H})(\exists g \in G) & X = g(\gamma(X)) \\ (\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H}) & \langle X | Y \rangle \leq \langle \gamma(X) | \gamma(Y) \rangle. \end{cases} \quad (2.13)$$

Let $\mathfrak{X}$ be a vector subspace of $\mathfrak{H}$ which contains range γ , and set

$$S = \{g|_{\mathfrak{X}} \mid g \in G \text{ such that } g(\mathfrak{X}) = \mathfrak{X}\} \quad \text{and} \quad (\forall g \in G) \Lambda_g: \mathfrak{X} \rightarrow \mathfrak{H}: X \mapsto g(X). \quad (2.14)$$

Moreover, let S act on $\mathfrak{X}$ via $(s, X) \mapsto s(X)$. Suppose that

$$(\forall X \in \mathfrak{X})(\exists s \in S) \quad X = s(\gamma(X)). \quad (2.15)$$

Then $\mathfrak{S} = (\mathfrak{X}, S, \gamma, (\Lambda_g)_{g \in G})$ is a spectral decomposition system for $\mathfrak{H}$, and the set $\{\Lambda_g\}_{g \in G}$ is closed in $\mathcal{L}(\mathfrak{X}, \mathfrak{H})$.

Proof. For the former assertion, note that property [A] in Definition 2.1 is fulfilled with $\tau = \gamma|_{\mathfrak{X}}$. The later assertion follows from the closedness of G in $O(\mathfrak{H})$. $\square$

A framework equivalent to normal decomposition systems is the notion of an Eaton triple [17], which arose in probability theory and group majorization.

Example 2.9 (Eaton triple). Let $(\mathfrak{H}, G, \mathcal{K})$ be an *Eaton triple* [17, Chapter 6], that is, $\mathfrak{H}$ is a Euclidean space, G is a closed subgroup of $O(\mathfrak{H})$, and $\mathcal{K}$ is a closed convex cone in $\mathfrak{H}$ such that

$$(\forall X \in \mathfrak{H})(\exists g \in G) \quad g(X) \in \mathcal{K}, \quad (2.16)$$

and

$$(\forall X \in \mathcal{K})(\forall Y \in \mathcal{K}) \quad \max_{g \in G} \langle X | g(Y) \rangle = \langle X | Y \rangle. \quad (2.17)$$

As shown on [43, p. 14], for every $X \in \mathfrak{H}$, the intersection $\mathcal{K} \cap (G \cdot X)$ is a singleton, which we denote by $\gamma(X)$. This thus defines a G -invariant mapping

$$\gamma: \mathfrak{H} \rightarrow \mathfrak{H}, \quad (2.18)$$

where G acts on $\mathfrak{H}$ via $(g, X) \mapsto g(X)$. Now let $\mathfrak{X}$ be a vector subspace of $\mathfrak{H}$ which contains range γ , set

$$S = \{g|_{\mathfrak{X}} \mid g \in G \text{ such that } g(\mathfrak{X}) = \mathfrak{X}\} \quad \text{and} \quad (\forall g \in G) \Lambda_g: \mathfrak{X} \rightarrow \mathfrak{H}: X \mapsto g(X), \quad (2.19)$$

and let S act on $\mathfrak{X}$ via $(s, X) \mapsto s(X)$. Suppose that

$$(\forall X \in \mathfrak{X})(\exists s \in S) \quad X = s(\gamma(X)). \quad (2.20)$$

Then $\mathfrak{S} = (\mathfrak{X}, S, \gamma, (\Lambda_g)_{g \in G})$ is a spectral decomposition system for $\mathfrak{H}$, and the set $\{\Lambda_g\}_{g \in G}$ is closed in $\mathcal{L}(\mathfrak{X}, \mathfrak{H})$.

Proof. We claim that $(\mathfrak{H}, G, \gamma)$ is a normal decomposition system. To this end, it is enough to verify the inequality

$$(\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H}) \quad \langle X | Y \rangle \leq \langle \gamma(X) | \gamma(Y) \rangle. \quad (2.21)$$

Take X and Y in $\mathfrak{H}$, and let g and h be in G such that $\gamma(X) = g(X)$ and $\gamma(Y) = h(Y)$. Since $\{\gamma(X), \gamma(Y)\} \subset \mathcal{K}$ by construction, we derive from (2.17) that

$$\langle \gamma(X) | \gamma(Y) \rangle \geq \langle \gamma(X) | (g^{-1} \circ h)(\gamma(Y)) \rangle = \langle g(\gamma(X)) | h(\gamma(Y)) \rangle = \langle X | Y \rangle. \quad (2.22)$$

Consequently, we obtain the conclusion by invoking [Example 2.8](#). $\square$

The next example demonstrates that our notion of a spectral decomposition system encompasses the Euclidean Jordan algebra framework of [1, 25, 37, 52], which in turn captures the space $H^N(\mathbb{K})$ of Hermitian matrices (see [Example 2.11](#)). It was shown in [44] that, in general, Euclidean Jordan algebras cannot be embedded into a normal decomposition system of [Example 2.8](#), demonstrating that these are distinct notions.

Example 2.10 (Euclidean Jordan algebra). Let $\mathfrak{H}$ be a *Euclidean Jordan algebra*, that is, $\mathfrak{H}$ is a finite-dimensional real vector space which is endowed with a bilinear form

$$\mathfrak{H} \times \mathfrak{H} \rightarrow \mathfrak{H}: (X, Y) \mapsto X \otimes Y \quad (2.23)$$

such that the following are satisfied:

- [A] $(\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H}) \quad X \otimes Y = Y \otimes X$ and $X \otimes ((X \otimes X) \otimes Y) = (X \otimes X) \otimes (X \otimes Y)$.
- [B] There exists a scalar product $(\cdot | \cdot)$ on $\mathfrak{H}$ such that $(\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H})(\forall Z \in \mathfrak{H}) \quad (X \otimes Y | Z) = (X | Y \otimes Z)$;

see [19] for background and complements on Euclidean Jordan algebras. We equip $\mathfrak{H}$ with the scalar product

$$(\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H}) \quad \langle X | Y \rangle = \text{Tra}(X \otimes Y), \quad (2.24)$$

where $\text{Tra} X$ is the trace in $\mathfrak{H}$ of an element $X \in \mathfrak{H}$ (see [19, Section II.2]). Denote by E the identity element of $\mathfrak{H}$ and by N the rank of $\mathfrak{H}$, and let P^N act on $\mathbb{R}^N$ via matrix-vector multiplication. Next, a *Jordan frame* of $\mathfrak{H}$ is a family $(A_i)_{1 \leq i \leq N}$ in $\mathfrak{H}^N \setminus \{0\}$ such that

$$\left\{ \begin{array}{l} (\forall i \in \{1, \dots, N\})(\forall j \in \{1, \dots, N\}) \quad A_i \otimes A_j = \begin{cases} A_i & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \\ \sum_{i=1}^N A_i = E. \end{array} \right. \quad (2.25)$$

The spectral decomposition theorem for Euclidean Jordan algebras [19, Theorem III.1.2] states that, for every $X \in \mathfrak{H}$, there exist a unique vector $(\lambda_1(X), \dots, \lambda_N(X)) \in \mathbb{R}^N$, the entries of which are called the *eigenvalues* of X , and a Jordan frame $(A_i)_{1 \leq i \leq N}$ such that

$$\lambda_1(X) \geq \dots \geq \lambda_N(X) \quad \text{and} \quad X = \sum_{i=1}^N \lambda_i(X) A_i. \quad (2.26)$$

We thus obtain a mapping

$$\lambda: \mathfrak{H} \rightarrow \mathbb{R}^N: X \mapsto (\lambda_1(X), \dots, \lambda_N(X)). \quad (2.27)$$

Further, denote by $\mathcal{A}$ the set of Jordan frames of $\mathfrak{H}$ and set

$$(\forall \mathfrak{a} = (A_i)_{1 \leq i \leq N} \in \mathcal{A}) \quad \Lambda_{\mathfrak{a}}: \mathbb{R}^N \rightarrow \mathfrak{H}: x = (\xi_i)_{1 \leq i \leq N} \mapsto \sum_{i=1}^N \xi_i A_i. \quad (2.28)$$

Then $\mathfrak{S} = (\mathbb{R}^N, \mathbb{P}^N, \lambda, (\Lambda_{\mathfrak{a}})_{\mathfrak{a} \in \mathcal{A}})$ is a spectral decomposition system for $\mathfrak{H}$, and the set $\{\Lambda_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathcal{A}}$ is closed in $\mathcal{L}(\mathbb{R}^N, \mathfrak{H})$.

Proof. The first assertion is proved in [7, Example 2.5]. To establish the closedness of $\{\Lambda_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathcal{A}}$ in $\mathcal{L}(\mathbb{R}^N, \mathfrak{H})$, let $(\mathfrak{a}_n)_{n \in \mathbb{N}}$ be a sequence of Jordan frames of $\mathfrak{H}$ such that $(\Lambda_{\mathfrak{a}_n})_{n \in \mathbb{N}}$ converges pointwise to some $L \in \mathcal{L}(\mathbb{R}^N, \mathfrak{H})$. We must show that there exists a Jordan frame $\mathfrak{a} \in \mathcal{A}$ such that $L = \Lambda_{\mathfrak{a}}$. To do so, denote by $(e_i)_{1 \leq i \leq N}$ the canonical basis of $\mathbb{R}^N$, set

$$(\forall i \in \{1, \dots, N\}) \quad A_i = L e_i. \quad (2.29)$$

Additionally, for every $n \in \mathbb{N}$, we write $\mathfrak{a}_n = (A_{i,n})_{1 \leq i \leq N}$. We deduce from (2.28) that

$$(\forall i \in \{1, \dots, N\}) \quad A_{i,n} = \Lambda_{\mathfrak{a}_n} e_i \rightarrow L e_i = A_i. \quad (2.30)$$

On the other hand, for every $n \in \mathbb{N}$, since $(A_{i,n})_{1 \leq i \leq N}$ is a Jordan frame of $\mathfrak{H}$, we have

$$\begin{cases} (\forall i \in \{1, \dots, N\}) (\forall j \in \{1, \dots, N\}) & A_{i,n} \otimes A_{j,n} = \begin{cases} A_{i,n} & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \\ \sum_{i=1}^N A_{i,n} = E. \end{cases} \quad (2.31)$$

Hence, since the bilinear form (2.23) is continuous (note that $\mathfrak{H}$ is finite-dimensional), letting $n \rightarrow +\infty$ shows that $(A_i)_{1 \leq i \leq N}$ satisfies (2.25) and it is therefore a Jordan frame of $\mathfrak{H}$. Furthermore, we derive from the linearity of L that

$$(\forall x = (\xi_i)_{1 \leq i \leq N} \in \mathbb{R}^N) \quad Lx = L \left(\sum_{i=1}^N \xi_i e_i \right) = \sum_{i=1}^N \xi_i L e_i = \sum_{i=1}^N \xi_i A_i. \quad (2.32)$$

Consequently, upon setting $\mathfrak{a} = (A_i)_{1 \leq i \leq N} \in \mathcal{A}$, we conclude that $L = \Lambda_{\mathfrak{a}}$. $\square$

Specializing Example 2.10 to the case of Hermitian matrices (see [19, Section V.2]) yields at once the following example.

Example 2.11 (eigenvalue decomposition). Let $2 \leq N \in \mathbb{N}$. We equip $H^N(\mathbb{K})$ with the scalar product

$$\langle \cdot | \cdot \rangle: (X, Y) \mapsto \operatorname{Re} \operatorname{tra}(XY) \quad (2.33)$$

and let $\mathbb{P}^N$ act on $\mathbb{R}^N$ via matrix-vector multiplication. For every $X \in H^N(\mathbb{K})$, we denote by $\lambda(X) = (\lambda_1(X), \dots, \lambda_N(X))$ the vector of the N (not necessarily distinct) eigenvalues of X listed in decreasing order (see [49, Theorem 5.3.6(c)] for the quaternion case). Additionally, set

$$(\forall U \in U^N(\mathbb{K})) \quad \Lambda_U: \mathbb{R}^N \rightarrow H^N(\mathbb{K}): x \mapsto U(\operatorname{Diag} x)U^*. \quad (2.34)$$

Then $\mathfrak{S} = (\mathbb{R}^N, \mathbb{P}^N, \lambda, (\Lambda_U)_{U \in U^N(\mathbb{K})})$ is a spectral decomposition system for $H^N(\mathbb{K})$, and the set $\{\Lambda_U\}_{U \in U^N(\mathbb{K})}$ is closed in $\mathcal{L}(\mathbb{R}^N, H^N(\mathbb{K}))$.

The next example concerns singular values of rectangular matrices.

Example 2.12 (singular value decomposition). Let M and N be strictly positive integers and set $m = \min\{M, N\}$. Let $\mathfrak{H}$ be the Euclidean space obtained by equipping $\mathbb{K}^{M \times N}$ with the scalar product

$$(X, Y) \mapsto \operatorname{Re} \operatorname{tra}(X^* Y), \quad (2.35)$$

and let $P_{\pm}^m$ act on $\mathbb{R}^m$ via matrix-vector multiplication. Given a matrix $X \in \mathfrak{H}$, the vector in $\mathbb{R}_{+}^m$ formed by the m (not necessarily distinct) singular values of X , with the convention that they are listed in decreasing order, is denoted by $(\sigma_1(X), \dots, \sigma_m(X))$; see [49, Proposition 3.2.5(f)] for singular value decomposition of matrices in $\mathbb{H}^{M \times N}$. This thus defines a mapping

$$\sigma: \mathfrak{H} \rightarrow \mathbb{R}^m: X \mapsto (\sigma_1(X), \dots, \sigma_m(X)). \quad (2.36)$$

Further, set $\mathcal{A} = U^M(\mathbb{K}) \times U^N(\mathbb{K})$ and

$$(\forall a = (U, V) \in \mathcal{A}) \quad \Lambda_a: \mathbb{R}^m \rightarrow \mathfrak{H}: x \mapsto U(\operatorname{Diag} x)V^*, \quad (2.37)$$

where the operator $\operatorname{Diag}: \mathbb{R}^m \rightarrow \mathfrak{H}$ maps a vector $(\xi_i)_{1 \leq i \leq m}$ to the diagonal matrix in $\mathfrak{H}$ of which the diagonal entries are $\xi_1, \dots, \xi_m$. Then $\mathfrak{S} = (\mathbb{R}^m, P_{\pm}^m, \sigma, (\Lambda_a)_{a \in \mathcal{A}})$ is a spectral decomposition system for $\mathfrak{H}$, and the set $\{\Lambda_a\}_{a \in \mathcal{A}}$ is closed in $\mathcal{L}(\mathbb{R}^m, \mathfrak{H})$.

Proof. The former assertion is established in [7, Example 2.7], while the latter follows from the closedness of $U^m(\mathbb{K})$ in $\mathbb{K}^{m \times m}$. $\square$

Our next example is a framework that arose in the study of isotropic stored energy functions in nonlinear elasticity [50] and the study of existence of a matrix with prescribed singular values and main diagonal elements [54]. Several convex analysis results in this setting were established in [13].

Example 2.13 (signed singular value decomposition). Let $2 \leq N \in \mathbb{N}$ and let $\mathfrak{H}$ be the Euclidean space obtained by equipping $\mathbb{R}^{N \times N}$ with the scalar product

$$(X, Y) \mapsto \operatorname{tra}(X^T Y), \quad (2.38)$$

let S be the subgroup of $P_{\pm}^N$ which consists of all matrices with an even number of entries equal to -1 , and let S act on $\mathbb{R}^N$ via matrix-vector multiplication. As in Example 2.12, $\sigma(X) = (\sigma_1(X), \dots, \sigma_N(X))$ designates the vector of the N singular values of a matrix $X \in \mathfrak{H}$, with the convention that $\sigma_1(X) \geq \dots \geq \sigma_N(X)$. Define a mapping

$$\gamma: \mathfrak{H} \rightarrow \mathbb{R}^N: X \mapsto (\gamma_1(X), \dots, \gamma_N(X)) \quad (2.39)$$

by

$$(\forall X \in \mathfrak{H})(\forall i \in \{1, \dots, N\}) \quad \gamma_i(X) = \begin{cases} \sigma_i(X), & \text{if } 1 \leq i \leq N-1; \\ \sigma_N(X) \operatorname{sign}(\det X), & \text{if } i = N. \end{cases} \quad (2.40)$$

Finally, set $\mathcal{A} = \operatorname{SO}^N \times \operatorname{SO}^N$ and

$$(\forall a = (U, V) \in \mathcal{A}) \quad \Lambda_a: \mathbb{R}^N \rightarrow \mathfrak{H}: x \mapsto U(\operatorname{Diag} x)V^T. \quad (2.41)$$

Then $\mathfrak{S} = (\mathbb{R}^N, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ is a spectral decomposition system for $\mathfrak{H}$, and the set $\{\Lambda_a\}_{a \in \mathcal{A}}$ is closed in $\mathcal{L}(\mathbb{R}^N, \mathfrak{H})$.

Proof. It was shown in [7, Example 2.8] that $\mathfrak{S}$ is a spectral decomposition system for $\mathfrak{X}$. Furthermore, the closedness of $\{\Lambda_a\}_{a \in \mathcal{A}}$ follows from that of SO^N in $\mathbb{R}^{N \times N}$. $\square$

The final example concerns spectral decomposition systems of product spaces and is instrumental to the development of Section 4.

Example 2.14. Suppose that Assumption 2.2 is in force. Let S act on the direct sum $\mathcal{X} \oplus \mathbb{R}$ via

$$(s, (x, \xi)) \mapsto (s \cdot x, \xi), \quad (2.42)$$

and define

$$\begin{cases} \gamma: \mathfrak{X} \oplus \mathbb{R} \rightarrow \mathcal{X} \oplus \mathbb{R}: (X, \xi) \mapsto (\gamma(X), \xi) \\ (\forall a \in \mathcal{A}) \Lambda_a: \mathcal{X} \oplus \mathbb{R} \rightarrow \mathfrak{X} \oplus \mathbb{R}: (x, \xi) \mapsto (\Lambda_a x, \xi). \end{cases} \quad (2.43)$$

Then $(\mathcal{X} \oplus \mathbb{R}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ is a spectral decomposition system for $\mathfrak{X} \oplus \mathbb{R}$, and the set $\{\Lambda_a\}_{a \in \mathcal{A}}$ is closed in $\mathcal{L}(\mathcal{X} \oplus \mathbb{R}, \mathfrak{X} \oplus \mathbb{R})$.

2.3. Fundamental properties

We now gather several essential properties of spectral decomposition systems that will be used throughout the paper.

Proposition 2.15. Suppose that Assumption 2.2 is in force. Then the following hold:

- (i) γ is nonexpansive, that is, $(\forall X \in \mathfrak{X})(\forall Y \in \mathfrak{X}) \|\gamma(X) - \gamma(Y)\| \leq \|X - Y\|$.
- (ii) A function $\varphi: \mathcal{X} \rightarrow [-\infty, +\infty]$ is S -invariant if and only if $(\forall a \in \mathcal{A}) \varphi \circ \gamma \circ \Lambda_a = \varphi$.

Proof. See [7, Propositions 3.5 (iv) and 4.4 (i)], respectively. $\square$

Given a Euclidean space $\mathcal{H}$ and a set $D \subset \mathcal{H}$, the closure of D is denoted by $\overline{D}$, the distance function to D is

$$d_D: \mathcal{H} \rightarrow \mathbb{R}: x \mapsto \inf_{y \in D} \|x - y\|, \quad (2.44)$$

and the projector onto D is

$$\text{Proj}_D: \mathcal{H} \rightarrow 2^{\mathcal{H}}: x \mapsto \{y \in D \mid \|x - y\| = d_D(x)\}, \quad (2.45)$$

where $2^{\mathcal{H}}$ designates the power set of $\mathcal{H}$.

Proposition 2.16. Suppose that Assumption 2.2 is in force and let D be a nonempty S -invariant subset of $\mathcal{X}$. Then the following hold:

- (i) $\overline{\gamma^{-1}(D)} = \gamma^{-1}(\overline{D})$.
- (ii) d_D is S -invariant and $(\forall a \in \mathcal{A}) d_D \circ \gamma \circ \Lambda_a = d_D$.
- (iii) $d_{\gamma^{-1}(D)} = d_D \circ \gamma$.
- (iv) For every $X \in \mathfrak{X}$ and every $Z \in \mathfrak{X}$, $Z \in \text{Proj}_{\gamma^{-1}(D)} X$ if and only if $\gamma(Z) \in \text{Proj}_D \gamma(X)$ and there exists $a \in \mathcal{A}$ such that $X = \Lambda_a \gamma(X)$ and $Z = \Lambda_a \gamma(Z)$.
- (v) For every $X \in \mathfrak{X}$, $\text{Proj}_{\gamma^{-1}(D)} X = \{\Lambda_a z \mid z \in \text{Proj}_D \gamma(X) \text{ and } a \in \mathcal{A}_X\}$.

Proof. (i): This is [7, Corollary 4.8 (ii)].

(ii): For every $x \in X$ and every $s \in S$, since $y \mapsto s^{-1} \cdot y$ is an isometry and D is S -invariant,

$$d_D(s \cdot x) = \inf_{y \in D} \|s \cdot x - y\| = \inf_{y \in D} \|x - s^{-1} \cdot y\| = \inf_{y \in D} \|x - y\| = d_D(x). \quad (2.46)$$

Thus d_D is S -invariant. The latter claim follows from Proposition 2.15 (ii).

(iii), (iv), and (v): It results from [7, Proposition 3.5 (iii)] that

$$\left(\frac{\|\cdot\|_X^2}{2} \right) \circ \gamma = \frac{\|\cdot\|_{\mathfrak{H}}^2}{2}. \quad (2.47)$$

Therefore, these claims follow respectively from items (i), (ii), and (iii) of [7, Corollary 6.4] applied to the S -invariant Legendre function $\psi = \frac{1}{2} \|\cdot\|_X^2$ (note that S acts on X by linear isometries). $\square$

Proposition 2.17. *Suppose that Assumption 2.2 is in force. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{H}$ that converges to some $X \in \mathfrak{H}$ and, for every $n \in \mathbb{N}$, let $\alpha_n \in \mathcal{A}_{X_n}$. Suppose that there exists $\alpha \in \mathcal{A}$ such that $\Lambda_{\alpha_n} \rightarrow \Lambda_\alpha$. Then $\alpha \in \mathcal{A}_X$.*

Proof. Since γ is nonexpansive (Proposition 2.15 (i)), we get $\gamma(X_n) \rightarrow \gamma(X)$. Thus $X = \lim X_n = \lim \Lambda_{\alpha_n} \gamma(X_n) = \Lambda_\alpha \gamma(X)$, which confirms that $\alpha \in \mathcal{A}_X$. $\square$

Finally, we point out an important consequence of Assumption 2.2 that will be employed repeatedly in the subsequent proofs.

Proposition 2.18. *Suppose that Assumption 2.2 is in force. Then the set $\{\Lambda_\alpha\}_{\alpha \in \mathcal{A}}$ is compact in $\mathcal{L}(X, \mathfrak{H})$.*

Proof. This follows from the closedness of $\{\Lambda_\alpha\}_{\alpha \in \mathcal{A}}$ and the fact that $(\forall \alpha \in \mathcal{A}) \|\Lambda_\alpha\|_{\mathcal{L}(X, \mathfrak{H})} = 1$. $\square$

3. Variational geometry of spectral sets

Consider Assumption 2.2. A main result in this paper is Theorem 4.1, which establishes the following relationship between the generalized subdifferentials of a spectral function and those of the associated invariant function:

$$(\forall X \in \mathfrak{H}) \quad \partial_\#(\varphi \circ \gamma)(X) = \{\Lambda_\alpha y \mid y \in \partial_\# \varphi(\gamma(X)) \text{ and } \alpha \in \mathcal{A}_X\}. \quad (3.1)$$

Here, $\varphi: X \rightarrow [-\infty, +\infty]$ is an S -invariant function, and $\partial_\#$ stands for either the Fréchet or the limiting subdifferential operator. Let us trace the path leading to this result. A common strategy in variational analysis for establishing a result for functions is to first treat the special case of sets, and then apply it to epigraphs to obtain the general case. Following this path, we establish in this section the identity

$$(\forall X \in \mathfrak{H}) \quad N_\#(X; \gamma^{-1}(D)) = \{\Lambda_\alpha y \mid y \in N_F(\gamma(X); D) \text{ and } \alpha \in \mathcal{A}_X\}, \quad (3.2)$$

where $\emptyset \neq D \subset X$ is S -invariant, and $N_\#$ signifies either the Fréchet or the limiting normal cone operator. In Section 4.1, we will apply this result to epigraphs – within the context of the “product” spectral decomposition system from Example 2.14 – to obtain the desired expressions (3.1).

We proceed by recalling relevant notions from variational analysis following the standard references [42, 48], to which we refer for background and further details. Let $\mathcal{H}$ be a Euclidean space and let D be a nonempty subset of $\mathcal{H}$. The Fréchet normal cone operator of D is

$$N_F: \mathcal{H} \rightarrow 2^{\mathcal{H}}: x \mapsto \begin{cases} \left\{ y \in \mathcal{H} \mid \limsup_{\substack{z \rightarrow x \\ z \in D \setminus \{x\}}} \frac{\langle z - x, y \rangle}{\|z - x\|} \leq 0 \right\} & \text{if } x \in D, \\ \emptyset & \text{otherwise.} \end{cases} \quad (3.3)$$

Evidently,

$$d_D = d_{\overline{D}} \quad \text{and} \quad (\forall x \in D) \quad N_F(x; D) = N_F(x; \overline{D}). \quad (3.4)$$

The limiting normal cone to D at a point $x \in D$, in symbols $N_L(x; D)$, is the set of all $y \in \mathcal{H}$ for which there exists a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ in $\text{gra } N_F$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$; here, $\text{gra } M = \{(x, y) \in \mathcal{H} \times \mathcal{H} \mid y \in Mx\}$ denotes the graph of a set-valued operator $M: \mathcal{H} \rightarrow 2^{\mathcal{H}}$. In addition, we set $(\forall x \in \mathcal{H} \setminus D) \quad N_L(x; D) = \emptyset$.

Next, we collect two preliminary results that will be frequently employed.

Lemma 3.1. *Let $\mathcal{H}$ be a Euclidean space, let D be a nonempty subset of $\mathcal{H}$, and let $x \in D$. Then the following hold:*

(i) *Let $y \in N_F(x; D)$. Then there exist sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$ such that*

$$x_n \rightarrow x, \quad y_n \rightarrow y, \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad y_n \in \text{cone}(x_n - \text{Proj}_{\overline{D}} x_n), \quad (3.5)$$

where $(\forall C \in 2^{\mathcal{H}}) \quad \text{cone } C = \bigcup_{\alpha \in]0, +\infty[} \alpha C$.

(ii) *Let $y \in \mathcal{H}$. Then*

$$y \in N_F(x; D) \quad \Leftrightarrow \quad \lim_{\alpha \downarrow 0} \frac{d_D(x + \alpha y)}{\alpha} = \|y\|. \quad (3.6)$$

Proof. (i): Combine (3.4) and [42, Theorem 1.6].

(ii): We adapt and simplify the proof of [16, Lemma 2.1 (i)] for this special case. Since $x \in D$, we infer from (3.4) that it suffices to establish the equivalence

$$y \in N_F(x; \overline{D}) \quad \Leftrightarrow \quad \lim_{\alpha \downarrow 0} \frac{d_{\overline{D}}(x + \alpha y)}{\alpha} = \|y\|. \quad (3.7)$$

For every $\alpha \in]0, +\infty[$, because $\overline{D}$ is closed and $\mathcal{H}$ is finite-dimensional, there exists $z_\alpha \in \overline{D}$ such that $\|x + \alpha y - z_\alpha\| = d_{\overline{D}}(x + \alpha y)$. In turn, since $x \in D$, we have

$$(\forall \alpha \in]0, +\infty[) \quad \|x + \alpha y - z_\alpha\| = d_{\overline{D}}(x + \alpha y) = \inf_{z \in D} \|x + \alpha y - z\| \leq \alpha \|y\| \quad (3.8)$$

Hence

$$\limsup_{\alpha \downarrow 0} \frac{d_{\overline{D}}(x + \alpha y)}{\alpha} \leq \|y\| \quad (3.9)$$

and

$$\lim_{\alpha \downarrow 0} z_\alpha = x. \quad (3.10)$$

Now suppose that $y \in N_F(x; \overline{D})$ and let $\varepsilon \in]0, 1[$. Then there exists $\delta \in]0, 1[$ such that

$$(\forall z \in \overline{D} \cap B(x; \delta)) \quad \langle z - x | y \rangle \leq \varepsilon \|z - x\|. \quad (3.11)$$

We deduce from (3.10) that there exists $\beta \in]0, 1[$ such that $(\forall \alpha \in]0, \beta[) \ z_\alpha \in \overline{D} \cap B(x; \delta)$. Therefore

$$\begin{aligned} (\forall \alpha \in]0, \beta[) \quad d_D^2(x + \alpha y) &= \|x - z_\alpha\|^2 - 2\alpha \langle z_\alpha - x | y \rangle + \alpha^2 \|y\|^2 \\ &\geq \|x - z_\alpha\|^2 - 2\alpha \varepsilon \|x - z_\alpha\| + \alpha^2 \|y\|^2 \\ &= (\|x - z_\alpha\| - \alpha \varepsilon)^2 + \alpha^2 (\|y\|^2 - \varepsilon^2). \\ &\geq \alpha^2 (\|y\|^2 - \varepsilon^2). \end{aligned} \quad (3.12)$$

Thus

$$\liminf_{\alpha \downarrow 0} \frac{d_D^2(x + \alpha y)}{\alpha^2} \geq \|y\|^2 - \varepsilon^2. \quad (3.13)$$

Since $\varepsilon \in]0, 1[$ was arbitrarily chosen, it follows that

$$\liminf_{\alpha \downarrow 0} \frac{d_D^2(x + \alpha y)}{\alpha^2} \geq \|y\|^2. \quad (3.14)$$

Combining with (3.9), we obtain $\lim_{\alpha \downarrow 0} d_D(x + \alpha y)/\alpha = \|y\|$. Conversely, suppose that $\lim_{\alpha \downarrow 0} d_D(x + \alpha y)/\alpha = \|y\|$ but that $y \notin N_F(x; \overline{D})$. Then, by definition of Fréchet normal cones, there exists $\kappa \in]0, 1[$ and a sequence $(w_n)_{n \in \mathbb{N}}$ in $\overline{D} \setminus \{x\}$ such that $w_n \rightarrow x$ and that $(\forall n \in \mathbb{N}) \ \langle w_n - x | y \rangle \geq \kappa \|w_n - x\|$. Set $(\forall n \in \mathbb{N}) \ \alpha_n = \|w_n - x\|/\kappa$. Then $\alpha_n \rightarrow 0$ and, since $\{w_n\}_{n \in \mathbb{N}} \subset \overline{D}$, we derive that

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad \frac{d_D^2(x + \alpha_n y)}{\alpha_n^2} &\leq \frac{1}{\alpha_n^2} (\|x - w_n\|^2 - 2\alpha_n \langle w_n - x | y \rangle + \alpha_n^2 \|y\|^2) \\ &\leq \kappa^2 - \frac{2\kappa \|w_n - x\|}{\alpha_n} + \|y\|^2 \\ &= -\kappa^2 + \|y\|^2. \end{aligned} \quad (3.15)$$

Letting $n \rightarrow +\infty$ yields a contradiction. $\square$

The inclusion $\supset$ in (3.2) follows at once from the following result.

Proposition 3.2. *Suppose that Assumption 2.2 is in force. Let D be a nonempty S -invariant subset of X , let x and y be in X , and let $a \in \mathcal{A}$. Then $y \in N_F(x; D)$ if and only if $\Lambda_a y \in N_F(\Lambda_a x; \gamma^{-1}(D))$.*

Proof. Since [7, Corollary 4.3] asserts that $D = \Lambda_a^{-1}(\gamma^{-1}(D))$, we deduce that $x \in D \Leftrightarrow \Lambda_a x \in \gamma^{-1}(D)$. On the other hand, using items (iii) and (ii) in Proposition 2.16, we obtain

$$(\forall \alpha \in \mathbb{R}_+) \quad d_{\gamma^{-1}(D)}(\Lambda_a x + \alpha \Lambda_a y) = (d_D \circ \gamma \circ \Lambda_a)(x + \alpha y) = d_D(x + \alpha y). \quad (3.16)$$

Altogether, since Λ_a is a linear isometry, we derive from Lemma 3.1 (ii) that

$$\begin{aligned} y \in N_F(x; D) &\Leftrightarrow x \in D \text{ and } \lim_{\alpha \downarrow 0} \frac{d_D(x + \alpha y)}{\alpha} = \|y\| \\ &\Leftrightarrow \Lambda_a x \in \gamma^{-1}(D) \text{ and } \lim_{\alpha \downarrow 0} \frac{d_{\gamma^{-1}(D)}(\Lambda_a x + \alpha \Lambda_a y)}{\alpha} = \|\Lambda_a y\| \\ &\Leftrightarrow \Lambda_a y \in N_F(\Lambda_a x; \gamma^{-1}(D)), \end{aligned} \quad (3.17)$$

as claimed. $\square$

We now establish a key identity that underpins the proof of the inclusion $\subset$ in (3.2). Beyond this context, it may also be of independent interest, and we therefore state it as a standalone result.

Proposition 3.3. *Suppose that Assumption 2.2 is in force. Let D be a nonempty S -invariant subset of $\mathcal{X}$ and let $X \in \mathfrak{S}$. Then*

$$\text{cone}(X - \text{Proj}_{\gamma^{-1}(D)} X) = \{\Lambda_a y \mid y \in \text{cone}(\gamma(X) - \text{Proj}_D \gamma(X)) \text{ and } a \in \mathcal{A}_X\}. \quad (3.18)$$

Proof. Thanks to the linearity of the operators $(\Lambda_a)_{a \in \mathcal{A}}$, it suffices to show that

$$X - \text{Proj}_{\gamma^{-1}(D)} X = \{\Lambda_a y \mid y \in \gamma(X) - \text{Proj}_D \gamma(X) \text{ and } a \in \mathcal{A}_X\}. \quad (3.19)$$

Let $Y \in \mathfrak{S}$ and suppose first that $Y \in X - \text{Proj}_{\gamma^{-1}(D)} X$, that is, $X - Y \in \text{Proj}_{\gamma^{-1}(D)} X$. Proposition 2.16 (iv) asserts that

$$\gamma(X - Y) \in \text{Proj}_D \gamma(X) \quad (3.20)$$

and that there exists $a \in \mathcal{A}$ for which

$$X = \Lambda_a \gamma(X) \quad \text{and} \quad X - Y = \Lambda_a \gamma(X - Y). \quad (3.21)$$

Since $\Lambda_a^* \circ \Lambda_a = \text{Id}_X$, it follows that

$$\gamma(X - Y) = \Lambda_a^*(X - Y) = \Lambda_a^* X - \Lambda_a^* Y = \Lambda_a^*(\Lambda_a \gamma(X)) - \Lambda_a^* Y = \gamma(X) - \Lambda_a^* Y. \quad (3.22)$$

In turn, invoking (3.21) once more, we obtain $X - Y = \Lambda_a(\gamma(X) - \Lambda_a^* Y) = X - \Lambda_a(\Lambda_a^* Y)$, which leads to

$$Y = \Lambda_a(\Lambda_a^* Y). \quad (3.23)$$

On the other hand, we deduce from (3.20) and (3.22) that $\Lambda_a^* Y \in \gamma(X) - \text{Proj}_D \gamma(X)$. We have thus established the inclusion $\subset$ in (3.19). Conversely, suppose that $Y = \Lambda_a y$ for some $y \in \gamma(X) - \text{Proj}_D \gamma(X)$ and some $a \in \mathcal{A}_X$, and let $z \in \text{Proj}_D \gamma(X)$ be such that $y = \gamma(X) - z$. Proposition 2.16 (v) implies that $\Lambda_a z \in \text{Proj}_{\gamma^{-1}(D)} X$ and, therefore,

$$Y = \Lambda_a(\gamma(X) - z) = X - \Lambda_a z \in X - \text{Proj}_{\gamma^{-1}(D)} X, \quad (3.24)$$

which completes the proof. $\square$

We are now in a position to establish (3.2).

Proposition 3.4. *Suppose that Assumption 2.2 is in force. Let D be a nonempty S -invariant subset of $\mathcal{X}$ and let $X \in \mathfrak{S}$. Then the following hold:*

- (i) $N_F(X; \gamma^{-1}(D)) = \{\Lambda_a y \mid y \in N_F(\gamma(X); D) \text{ and } a \in \mathcal{A}_X\}.$
- (ii) $N_L(X; \gamma^{-1}(D)) = \{\Lambda_a y \mid y \in N_L(\gamma(X); D) \text{ and } a \in \mathcal{A}_X\}.$

Proof. We assume henceforth that

$$X \in \gamma^{-1}(D) \quad (3.25)$$

since otherwise all the cones are empty and the assertions are thus trivial. Additionally, note that $\overline{D}$ is also S -invariant. Now let $Y \in \mathfrak{S}$.

(i): The inclusion $\supset$ follows from [Proposition 3.2](#) applied to $x = \gamma(X)$. To establish the converse, suppose that $Y \in N_F(X; \gamma^{-1}(D))$. Then [Lemma 3.1 \(i\)](#) states that there exist sequences $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ in $\mathfrak{X}$ such that

$$X_n \rightarrow X, \quad Y_n \rightarrow Y, \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad Y_n \in \text{cone}(X_n - \text{Proj}_{\gamma^{-1}(D)} X_n). \quad (3.26)$$

In turn, we learn from [Proposition 2.16 \(i\)](#) and [Proposition 3.3](#) (applied to $\overline{D}$) that

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad Y_n &\in \text{cone}(X_n - \text{Proj}_{\gamma^{-1}(\overline{D})} X_n) \\ &= \{ \Lambda_a y \mid y \in \text{cone}(\gamma(X_n) - \text{Proj}_{\overline{D}} \gamma(X_n)) \text{ and } a \in \mathcal{A}_{X_n} \}. \end{aligned} \quad (3.27)$$

Hence, we obtain sequences $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$ and $(a_n)_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n \in \text{cone}(\gamma(X_n) - \text{Proj}_{\overline{D}} \gamma(X_n)) \\ a_n \in \mathcal{A}_{X_n} \\ Y_n = \Lambda_{a_n} y_n. \end{cases} \quad (3.28)$$

Since the operators $(\Lambda_a)_{a \in \mathcal{A}}$ are linear isometries, (3.26) implies that $\sup_{n \in \mathbb{N}} \|y_n\| = \sup_{n \in \mathbb{N}} \|Y_n\| < +\infty$. Thus, we deduce from [Proposition 2.18](#) that there exist a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$, together with $y \in \mathcal{X}$ and $a \in \mathcal{A}$ such that

$$y_{k_n} \rightarrow y \quad \text{and} \quad \Lambda_{a_{k_n}} \rightarrow \Lambda_a. \quad (3.29)$$

We derive from (3.26) and (3.28) that $Y = \lim Y_{k_n} = \lim \Lambda_{a_{k_n}} y_{k_n} = \Lambda_a y$. At the same time, invoking (3.26), (3.28), and (3.29), we deduce from [Proposition 2.17](#) that $a \in \mathcal{A}_X$. Therefore

$$\Lambda_a y = Y \in N_F(X; \gamma^{-1}(D)) = N_F(\Lambda_a \gamma(X); \gamma^{-1}(D)). \quad (3.30)$$

Consequently, [Proposition 3.2](#) yields $y \in N_F(\gamma(X); D)$.

(ii): Assume that $Y \in N_L(X; \gamma^{-1}(D))$ and let $(X_n, Y_n)_{n \in \mathbb{N}}$ be a sequence in $\text{gra } N_F(\cdot; \gamma^{-1}(D))$ such that $X_n \rightarrow X$ and $Y_n \rightarrow Y$. It results from (i) that there exist sequences $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$ and $(a_n)_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that

$$(\forall n \in \mathbb{N}) \quad Y_n = \Lambda_{a_n} y_n, \quad y_n \in N_F(\gamma(X_n); D), \quad \text{and} \quad a_n \in \mathcal{A}_{X_n}. \quad (3.31)$$

Note that $\sup_{n \in \mathbb{N}} \|y_n\| = \sup_{n \in \mathbb{N}} \|Y_n\| < +\infty$. Combining with [Proposition 2.18](#), we obtain a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$, a point $y \in \mathcal{X}$, and an element $a \in \mathcal{A}$ such that

$$y_{k_n} \rightarrow y \quad \text{and} \quad \Lambda_{a_{k_n}} \rightarrow \Lambda_a. \quad (3.32)$$

Moreover, the nonexpansiveness of γ ([Proposition 2.15 \(i\)](#)) gives $\gamma(X_{k_n}) \rightarrow \gamma(X)$. Therefore, by (3.31) and (3.32), $y \in N_L(\gamma(X); D)$. At the same time, we have $Y = \lim Y_{k_n} = \lim \Lambda_{a_{k_n}} y_{k_n} = \Lambda_a y$, while [Proposition 2.17](#) entails that $a \in \mathcal{A}_X$. Finally, to establish the reverse inclusion, assume that $Y = \Lambda_{\mathfrak{b}} v$ for some $v \in N_L(\gamma(X); D)$ and some $\mathfrak{b} \in \mathcal{A}_X$. Let $(z_n, v_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H} \times \mathcal{H}$ such that $z_n \rightarrow \gamma(X)$, $v_n \rightarrow v$, and $(\forall n \in \mathbb{N}) \quad v_n \in N_F(z_n; D)$. [Proposition 3.2](#) entails that $(\forall n \in \mathbb{N}) \quad \Lambda_{\mathfrak{b}} v_n \in N_F(\Lambda_{\mathfrak{b}} z_n; \gamma^{-1}(D))$. Hence, since the continuity of $\Lambda_{\mathfrak{b}}$ gives $\Lambda_{\mathfrak{b}} z_n \rightarrow \Lambda_{\mathfrak{b}} \gamma(X) = X$ and $\Lambda_{\mathfrak{b}} v_n \rightarrow \Lambda_{\mathfrak{b}} v = Y$, we conclude that $Y \in N_L(X; \gamma^{-1}(D))$. $\square$

4. Generalized subgradients of spectral functions

The goal of this section is to relate Fréchet, limiting, and Clarke subdifferentials and Fréchet differentiability of an invariant function to those of the induced spectral function.

4.1. Fréchet and limiting subdifferentials of spectral functions

We start by recalling the notions of Fréchet and limiting subdifferentials. Let $\mathcal{H}$ be a Euclidean space and let $f: \mathcal{H} \rightarrow [-\infty, +\infty]$. The Fréchet subdifferential of f is

$$\partial_{\mathbb{F}} f: \mathcal{H} \rightarrow 2^{\mathcal{H}}: x \mapsto \begin{cases} \left\{ y \in \mathcal{H} \mid \liminf_{\substack{z \rightarrow x \\ z \neq x}} \frac{f(z) - f(x) - \langle z - x, y \rangle}{\|z - x\|} \geq 0 \right\} & \text{if } f(x) \in \mathbb{R}, \\ \emptyset & \text{otherwise.} \end{cases} \quad (4.1)$$

The limiting subdifferential of f at a point $x \in \mathcal{H}$ such that $f(x) \in \mathbb{R}$, in symbols $\partial_{\mathbb{L}} f(x)$, is the set of all $y \in \mathcal{H}$ for which there exists a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ in $\text{gra } \partial_{\mathbb{F}} f$ such that $x_n \rightarrow x$, $f(x_n) \rightarrow f(x)$, and $y_n \rightarrow y$. Further, for every $x \in \mathcal{H}$ such that $f(x) \in \{\pm\infty\}$, we set $\partial_{\mathbb{L}} f(x) = \emptyset$. For every $\# \in \{\mathbb{F}, \mathbb{L}\}$ and every $x \in \mathcal{H}$ such that $f(x) \in \mathbb{R}$, we have

$$\partial_{\#} f(x) = \{y \in \mathcal{H} \mid (y, -1) \in N_{\#}((x, f(x)); \text{epi } f)\}, \quad (4.2)$$

where $\text{epi } f = \{(y, \eta) \in \mathcal{H} \times \mathbb{R} \mid f(y) \leq \eta\}$ is the epigraph of f ; see [48, Theorem 8.9].

Having assembled the necessary tools in Section 3, we can now prove (3.1). Our key observation is the identity (4.2) and that, in the spectral decomposition system $(\mathcal{X} \oplus \mathbb{R}, \mathbb{S}, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ for $\mathfrak{H} \oplus \mathbb{R}$ constructed in Example 2.14, given an $\mathbb{S}$ -invariant function $\varphi: \mathcal{X} \rightarrow [-\infty, +\infty]$, $\text{epi } \varphi$ is an $\mathbb{S}$ -invariant subset of $\mathcal{X} \oplus \mathbb{R}$ and

$$\begin{aligned} \gamma^{-1}(\text{epi } \varphi) &= \{(X, \xi) \in \mathfrak{H} \oplus \mathbb{R} \mid \gamma(X, \xi) \in \text{epi } \varphi\} \\ &= \{(X, \xi) \in \mathfrak{H} \oplus \mathbb{R} \mid (\gamma(X), \xi) \in \text{epi } \varphi\} \\ &= \{(X, \xi) \in \mathfrak{H} \oplus \mathbb{R} \mid \varphi(\gamma(X)) \leq \xi\} \\ &= \text{epi}(\varphi \circ \gamma). \end{aligned} \quad (4.3)$$

As a byproduct of our analysis, we also establish the implication

$$(\forall x \in \mathcal{X})(\forall y \in \mathcal{X})(\forall a \in \mathcal{A}) \quad y \in \partial_{\#} \varphi(x) \implies \Lambda_a y \in \partial_{\#}(\varphi \circ \gamma)(\Lambda_a x), \quad (4.4)$$

which may be viewed as a strengthening of the inclusion $\supset$ in (3.1). It reveals an invariance property of the subdifferential operators under the “action” of the operators $(\Lambda_a)_{a \in \mathcal{A}}$; see Remark 4.7 for a detailed discussion. In particular, we will leverage this to establish in Corollary 4.8 that the Fréchet differentiability of an invariant function transfers to the corresponding spectral function.

Theorem 4.1. *Suppose that Assumption 2.2 is in force. Let $\varphi: \mathcal{X} \rightarrow [-\infty, +\infty]$ be $\mathbb{S}$ -invariant and let $X \in \mathfrak{X}$. Then the following hold:*

- (i) $\partial_{\mathbb{F}}(\varphi \circ \gamma)(X) = \{\Lambda_a y \mid y \in \partial_{\mathbb{F}} \varphi(\gamma(X)) \text{ and } a \in \mathcal{A}_X\}.$
- (ii) $\partial_{\mathbb{L}}(\varphi \circ \gamma)(X) = \{\Lambda_a y \mid y \in \partial_{\mathbb{L}} \varphi(\gamma(X)) \text{ and } a \in \mathcal{A}_X\}.$

Proof. We work with the spectral decomposition system $(\mathcal{X} \oplus \mathbb{R}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ of [Example 2.14](#). It will be convenient to set

$$x = \gamma(X) \quad \text{and} \quad \xi = \varphi(\gamma(X)). \quad (4.5)$$

Note that $\gamma(X, \xi) = (x, \xi)$. Moreover, since the assertions trivially hold when $\xi \in \{\pm\infty\}$, we assume henceforth that $\xi \in \mathbb{R}$.

(i): We derive from (4.2), (4.3), and [Proposition 3.4\(i\)](#) (applied to the system $(\mathcal{X} \oplus \mathbb{R}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ and the S -invariant set $\text{epi } \varphi$) that

$$\begin{aligned} (\forall Y \in \mathfrak{H}) \quad Y \in \partial_{\mathbb{F}}(\varphi \circ \gamma)(X) & \Leftrightarrow (Y, -1) \in N_{\mathbb{F}}((X, \xi); \text{epi}(\varphi \circ \gamma)) = N_{\mathbb{F}}((X, \xi); \gamma^{-1}(\text{epi } \varphi)) \\ & \Leftrightarrow (\exists (y, \eta) \in \mathcal{X} \oplus \mathbb{R})(\exists a \in \mathcal{A}) \begin{cases} (y, \eta) \in N_{\mathbb{F}}(\gamma(X, \xi); \text{epi } \varphi) \\ (X, \xi) = \Lambda_a \gamma(X, \xi) = (\Lambda_a x, \xi) \\ (Y, -1) = \Lambda_a (y, \eta) = (\Lambda_a y, \eta) \end{cases} \\ & \Leftrightarrow (\exists y \in \mathcal{X})(\exists a \in \mathcal{A}) \begin{cases} (y, -1) \in N_{\mathbb{F}}((x, \xi); \text{epi } \varphi) \\ X = \Lambda_a x \text{ and } Y = \Lambda_a y \end{cases} \\ & \Leftrightarrow (\exists y \in \partial_{\mathbb{F}} \varphi(x))(\exists a \in \mathcal{A}_X) \quad Y = \Lambda_a y, \end{aligned} \quad (4.6)$$

as desired.

(ii): Argue as in (i) and use [Proposition 3.4\(ii\)](#) instead of [Proposition 3.4\(i\)](#). $\square$

Remark 4.2. In the case of convex subdifferentials, it was established in [7, Proposition 5.5 (i)] that, under [Assumption 2.2](#) and for every proper S -invariant function $\varphi: \mathcal{X} \rightarrow]-\infty, +\infty]$, we have

$$(\forall X \in \mathfrak{H})(\forall Y \in \mathfrak{H}) \quad Y \in \partial(\varphi \circ \gamma)(X) \quad \Leftrightarrow \quad \begin{cases} \gamma(Y) \in \partial \varphi(\gamma(X)) \\ (\exists a \in \mathcal{A}) \quad X = \Lambda_a \gamma(X) \text{ and } Y = \Lambda_a \gamma(Y), \end{cases} \quad (4.7)$$

where ∂ stands for the convex subdifferential operator. By contrast, one cannot assert that $Y \in \partial_{\mathbb{F}}(\varphi \circ \gamma)(X) \Rightarrow \gamma(Y) \in \partial \varphi(\gamma(X))$. To see this, consider the setting of [Example 2.11](#) with $\mathbb{K} = \mathbb{R}$ and $N = 2$, and the following function φ considered in [14, Example 3.2]:

$$\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}: (\xi_1, \xi_2) \mapsto \xi_1 \xi_2. \quad (4.8)$$

Then $\varphi \circ \lambda$ is Fréchet differentiable at $X = \text{Diag}(1, 2)$ with $\nabla(\varphi \circ \lambda)(X) = \text{Diag}(2, 1)$. On the other hand, $\nabla \varphi(\lambda(X)) = \nabla \varphi(2, 1) = (1, 2) \neq \lambda(\text{Diag}(2, 1))$.

Remark 4.3. [Theorem 4.1](#) unifies several well-known results describing the subdifferentials of a spectral function through those of the associated invariant function and the spectral mapping. More precisely, in the settings of [Examples 2.10, 2.11, and 2.12](#), [Theorem 4.1](#) reduces to [37, Theorem 17] (see [51, Theorems 8.5 and 9.1] for special cases), [31, Theorem 6] (see also [16, Theorem 4.2]), and [34, Theorem 7.1], respectively. To the best of our knowledge, [Theorem 4.1](#) is new in the settings of [Examples 2.8 and 2.13](#); see also the discussion in the paragraph following [33, Theorem 7.2].

As an application of [Theorem 4.1](#), we establish a generalization of the so-called *commutation principle* established in [20, 47].

Corollary 4.4. Suppose that [Assumption 2.2](#) is in force. Let $\varphi: \mathcal{X} \rightarrow]-\infty, +\infty]$ be proper and S -invariant, let $\Psi: \mathfrak{H} \rightarrow \mathbb{R}$ be Fréchet differentiable, and let $X \in \mathfrak{H}$ be such that $\gamma(X) \in \text{dom } \varphi$. Suppose that X is a local minimizer of $\varphi \circ \gamma + \Psi$. Then there exist $y \in \partial_{\mathbb{F}}\varphi(\gamma(X))$ and $\alpha \in \mathcal{A}_X$ such that $-\nabla\Psi(X) = \Lambda_\alpha y$.

Proof. Since $X \in \text{dom}(\varphi \circ \gamma + \Psi)$, we derive from the sum rule ([42, Propositions 1.114 and 1.107(i)]) that $0 \in \partial_{\mathbb{F}}(\varphi \circ \gamma + \Psi)(X) = \partial_{\mathbb{F}}(\varphi \circ \gamma)(X) + \nabla\Psi(X)$. Now apply [Theorem 4.1 \(i\)](#). $\square$

Remark 4.5. [Corollary 4.4](#) brings together and extends two commutation principles found in the literature, while providing a stronger conclusion even in those particular settings:

- (i) Consider the normal decomposition system framework [Example 2.8](#). By specializing [Corollary 4.4](#) to the case where $\mathfrak{X} = \mathfrak{H}$, we obtain an extension of [20, Theorem 1.3] (see also [7, Example 5.7]) to the general nonconvex setting.
- (ii) In the Euclidean Jordan algebra framework of [Example 2.10](#), the conclusion of [Corollary 4.4](#) reads: There exist $y = (\eta_i)_{1 \leq i \leq N} \in \partial_{\mathbb{F}}\varphi(\lambda(X))$ and a Jordan frame $\alpha = (A_i)_{1 \leq i \leq N} \in \mathcal{A}$ such that

$$X = \sum_{i=1}^N \lambda_i(X) A_i \quad \text{and} \quad -\nabla\Psi(X) = \Lambda_\alpha y = \sum_{i=1}^N \eta_i A_i. \quad (4.9)$$

At the same time, for every i and j in $\{1, \dots, N\}$, the operators $L_i: \mathfrak{H} \rightarrow \mathfrak{H}: Z \mapsto A_i \otimes Z$ and $L_j: \mathfrak{H} \rightarrow \mathfrak{H}: Z \mapsto A_j \otimes Z$ satisfy $L_i \circ L_j = L_j \circ L_i$ [19, Lemma IV.1.3]. In turn, we deduce that

$$\begin{aligned} (\forall Z \in \mathfrak{H}) \quad \nabla\Psi(X) \otimes (X \otimes Z) &= - \sum_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}} \lambda_i(X) \eta_j A_j \otimes (A_i \otimes Z) \\ &= - \sum_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}} \lambda_i(X) \eta_j A_i \otimes (A_j \otimes Z) \\ &= X \otimes (\nabla\Psi(X) \otimes Z). \end{aligned} \quad (4.10)$$

We thus recover [47, Theorem 7], which states that the operators $Z \mapsto \nabla\Psi(X) \otimes Z$ and $Z \mapsto X \otimes Z$ commute (with respect to composition).

We end this subsection with a strengthening of the inclusions $\supset$ in [Theorem 4.1](#). In essence, given prior knowledge of subgradients of an invariant function at a point $x \in \mathcal{X}$, it allows to construct those of the induced spectral function at the points $\{\Lambda_\alpha x\}_{\alpha \in \mathcal{A}}$.

Proposition 4.6. Suppose that [Assumption 2.2](#) is in force. Let $\varphi: \mathcal{X} \rightarrow [-\infty, +\infty]$ be S -invariant, let x and y be in $\mathcal{X}$, and let $\alpha \in \mathcal{A}$. Then the following hold:

- (i) $y \in \partial_{\mathbb{F}}\varphi(x)$ if and only if $\Lambda_\alpha y \in \partial_{\mathbb{F}}(\varphi \circ \gamma)(\Lambda_\alpha x)$.
- (ii) Suppose that $y \in \partial_{\mathbb{L}}\varphi(x)$. Then $\Lambda_\alpha y \in \partial_{\mathbb{L}}(\varphi \circ \gamma)(\Lambda_\alpha x)$.

Proof. (i): As in the proof of [Theorem 4.1](#), we consider the spectral decomposition system $(\mathcal{X} \oplus \mathbb{R}, S, \gamma, (\Lambda_\alpha)_{\alpha \in \mathcal{A}})$ for $\mathfrak{H} \oplus \mathbb{R}$ constructed in [Example 2.14](#). Then $\text{epi } \varphi$ is an S -invariant subset of $\mathcal{X} \oplus \mathbb{R}$ and $\gamma^{-1}(\text{epi } \varphi) = \text{epi}(\varphi \circ \gamma)$; see (4.3). We derive from (4.2), [Proposition 3.2](#) (applied to

$(\mathcal{X} \oplus \mathbb{R}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ and the S -invariant set $\text{epi } \varphi$, and the identity $\varphi = \varphi \circ \gamma \circ \Lambda_a$ (Proposition 2.15 (ii)) that

$$\begin{aligned}
 y \in \partial_F \varphi(x) &\Leftrightarrow \varphi(x) \in \mathbb{R} \text{ and } (y, -1) \in N_F((x, \varphi(x)); \text{epi } \varphi) \\
 &\Leftrightarrow \varphi(x) \in \mathbb{R} \text{ and } \Lambda_a(y, -1) \in N_F(\Lambda_a(x, \varphi(x)); \gamma^{-1}(\text{epi } \varphi)) \\
 &\Leftrightarrow \varphi(x) \in \mathbb{R} \text{ and } (\Lambda_a y, -1) \in N_F((\Lambda_a x, \varphi(x)); \text{epi}(\varphi \circ \gamma)) \\
 &\Leftrightarrow (\varphi \circ \gamma)(\Lambda_a x) \in \mathbb{R} \text{ and } (\Lambda_a y, -1) \in N_F((\Lambda_a x, (\varphi \circ \gamma)(\Lambda_a x)); \text{epi}(\varphi \circ \gamma)) \\
 &\Leftrightarrow \Lambda_a y \in \partial_F(\varphi \circ \gamma)(\Lambda_a x),
 \end{aligned} \tag{4.11}$$

as announced.

(ii): Use the definition of limiting subdifferentials, (i), and the identity $\varphi \circ \gamma \circ \Lambda_a = \varphi$ (Proposition 2.15 (ii)). $\square$

Remark 4.7. Here are several noteworthy instances of Proposition 4.6 (i) found in the literature.

- (i) Consider the setting of Example 2.11, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, and let $\varphi: \mathbb{R}^N \rightarrow [-\infty, +\infty]$ be P^N -invariant, that is, permutation-invariant. Then Proposition 4.6 (i) yields

$$\begin{aligned}
 (\forall x \in \mathbb{R}^N)(\forall y \in \mathbb{R}^N)(\forall U \in \mathcal{U}^N(\mathbb{K})) \\
 y \in \partial_F \varphi(x) \quad \Rightarrow \quad U(\text{Diag } y)U^* \in \partial_F(\varphi \circ \lambda)(U(\text{Diag } x)U^*),
 \end{aligned} \tag{4.12}$$

which is precisely [31, Theorem 5]. In [31], this implication serves as a key step in deriving the subdifferential formula (3.1) within the setting of Example 2.11, and its proof relies on a technically involved result concerning directional derivatives of the eigenvalue mapping λ .

- (ii) Likewise, in the setting of Example 2.12, Proposition 4.6 (i) yields [34, Theorem 6.10]. As in the case of [31], this implication is a key ingredient in the derivation of the corresponding subdifferential formula in [34], and its proof draws on technically involved properties of the singular value mapping.
- (iii) In the Euclidean Jordan algebra framework of Example 2.10, Proposition 4.6 (i) yields [37, Proposition 15], which – in line with the approaches of [31, 34] – plays a key role in the derivation of (3.1) within that setting.

4.2. Fréchet differentiability of spectral functions

The results of Section 4.1 allow us to fully characterize the Fréchet differentiability of a spectral function through that of the associated invariant function. Recall that, given a Euclidean space $\mathcal{H}$, a function $f: \mathcal{H} \rightarrow [-\infty, +\infty]$ is Fréchet differentiable at a point $x \in \mathcal{H}$ in which $f(x) \in \mathbb{R}$ if there exists a unique point in $\mathcal{H}$, denoted by $\nabla f(x)$, such that

$$\lim_{\substack{z \rightarrow x \\ z \neq x}} \frac{|f(z) - f(x) - \langle z - x | \nabla f(x) \rangle|}{\|z - x\|} = 0; \tag{4.13}$$

moreover, a characterization of Fréchet differentiability in terms of Fréchet subdifferentiability is

$$f \text{ is Fréchet differentiable at } x \quad \Leftrightarrow \quad \partial_F(\pm f)(x) \neq \emptyset; \tag{4.14}$$

see, e.g., [42, Proposition 1.87].

Corollary 4.8. Suppose that [Assumption 2.2](#) is in force and let $\varphi: \mathcal{X} \rightarrow [-\infty, +\infty]$ be S -invariant.

- (i) Let $x \in \mathcal{X}$ and $a \in \mathcal{A}$. Suppose that $\varphi(x) \in \mathbb{R}$. Then $\varphi \circ \gamma$ is Fréchet differentiable at $\Lambda_a x$ if and only if φ is Fréchet differentiable at x , in which case

$$\nabla(\varphi \circ \gamma)(\Lambda_a x) = \Lambda_a(\nabla\varphi(x)). \quad (4.15)$$

- (ii) Let $X \in \mathfrak{S}$ and suppose that $\varphi(\gamma(X)) \in \mathbb{R}$. Then $\varphi \circ \gamma$ is Fréchet differentiable at X if and only if φ is Fréchet differentiable at $\gamma(X)$, in which case

$$(\forall a \in \mathcal{A}_X) \quad \nabla(\varphi \circ \gamma)(X) = \Lambda_a(\nabla\varphi(\gamma(X))). \quad (4.16)$$

Proof. (i): If $\varphi \circ \gamma$ is Fréchet differentiable at $\Lambda_a x$, then the Fréchet differentiability of φ at x follows from the identity $\varphi = (\varphi \circ \gamma) \circ \Lambda_a$ ([Proposition 2.15 \(ii\)](#)) and the chain rule. Now assume that φ is Fréchet differentiable at x . Then $\pm \nabla\varphi(x) \in \partial_{\mathbb{F}}(\pm\varphi)(x)$ [[48, Exercise 8.8\(a\)](#)]. Hence, applying [Proposition 4.6 \(i\)](#) to the functions $\pm\varphi$ and using the linearity of Λ_a , we obtain $\pm \Lambda_a(\nabla\varphi(x)) \in \partial_{\mathbb{F}}(\pm\varphi \circ \gamma)(\Lambda_a x)$. Thus, we conclude via [\(4.14\)](#) that $\varphi \circ \gamma$ is Fréchet differentiable at $\Lambda_a x$ and that $\nabla(\varphi \circ \gamma)(\Lambda_a x) = \Lambda_a(\nabla\varphi(x))$.

(ii): Apply (i) with $x = \gamma(X)$. $\square$

Remark 4.9. Let us relate [Corollary 4.8](#) to existing results on the Fréchet differentiability of spectral functions.

- (i) In the Euclidean Jordan algebra framework of [Example 2.10](#), we recover at once [[1, Theorem 38](#)] and [[52, Theorem 4.1](#)] from [Corollary 4.8 \(ii\)](#).
- (ii) In the case of Hermitian matrices ([Example 2.11](#)), [Corollary 4.8 \(i\)](#) yields [[28, Theorem 2.4](#) and [Corollary 2.5](#)], while [Corollary 4.8 \(ii\)](#) yields [[28, Theorem 1.1](#)].

4.3. Clarke subdifferentials of spectral functions

For locally Lipschitz continuous functions, the Clarke subdifferential (or generalized gradient) is especially useful in optimization due to its strong calculus and (at least in finite dimensions) explicit characterization. In particular, it serves as a unifying framework both for convex and for strictly differentiable functionals and covers, e.g., the composition of nonsmooth convex functionals and continuously differentiable operators common in nonsmooth optimal control of differential equations; see [[10, Chapter 2](#)]. In finite dimensions, the Clarke subdifferential also furnishes generalized derivatives that can be used in superlinearly convergent semismooth Newton methods; see [[40, 45, 46](#)] as well as [[11, Chapter 14](#)].

We first recall the definition. Let $\mathcal{H}$ be a Euclidean space and let $f: \mathcal{H} \rightarrow [-\infty, +\infty]$ be locally Lipschitz continuous near a point $x \in \mathcal{H}$, that is, $f(x) \in \mathbb{R}$ and there exist $\varepsilon \in]0, 1[$ and $\kappa \in \mathbb{R}_+$ such that

$$(\forall y \in B(x; \varepsilon)) (\forall z \in B(x; \varepsilon)) \quad |f(y) - f(z)| \leq \kappa \|y - z\|, \quad (4.17)$$

where $B(x; \varepsilon) = \{y \in \mathcal{H} \mid \|x - y\| \leq \varepsilon\}$. Given $u \in \mathcal{H}$, the Clarke generalized directional derivative of f at x in the direction u is

$$f^\circ(x; u) = \limsup_{\substack{z \rightarrow x \\ \alpha \downarrow 0}} \frac{f(z + \alpha u) - f(z)}{\alpha}. \quad (4.18)$$

In turn, the Clarke subdifferential of f at x is

$$\partial_C f(x) = \{y \in \mathcal{H} \mid (\forall u \in \mathcal{H}) \langle u \mid y \rangle \leq f^\circ(x; u)\}, \quad (4.19)$$

which is always nonempty, compact, and convex [10, Proposition 2.1.2 (a)]. The following characterization of Clarke subdifferentials will be crucial in our analysis of this section and Section 5.

Lemma 4.10. *Let $\mathcal{H}$ be a Euclidean space, let $f: \mathcal{H} \rightarrow [-\infty, +\infty]$ be locally Lipschitz continuous near a point $x \in \mathcal{H}$, and denote by S_f the set of points at which f is Fréchet differentiable. Then the following hold:*

- (i) $S_f \neq \emptyset$.
- (ii) *Let $\Omega \subset \mathcal{H}$ be a Lebesgue null set, i.e., a Lebesgue measurable subset of $\mathcal{H}$ of Lebesgue measure 0, and let D be the set of all $y \in \mathcal{H}$ for which there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $S_f \setminus \Omega$ such that $x_n \rightarrow x$ and $\nabla f(x_n) \rightarrow y$. Then $\partial_C f(x) = \text{conv } D \neq \emptyset$.*

Proof. (i): This is a consequence of Rademacher's theorem.

(ii): See [10, Theorem 2.5.1]. $\square$

We can now state and prove the main result of this subsection.

Proposition 4.11. *Suppose that Assumption 2.2 is in force, and let $\varphi: \mathcal{X} \rightarrow [-\infty, +\infty]$ be S -invariant and locally Lipschitz continuous near the spectrum $\gamma(X)$ of a point $X \in \mathfrak{S}$. Then*

$$\partial_C(\varphi \circ \gamma)(X) = \text{conv}\{\Lambda_\alpha y \mid y \in \partial_C \varphi(\gamma(X)) \text{ and } \alpha \in \mathcal{A}_X\}. \quad (4.20)$$

Proof. Denote by D the set of all $y \in \mathcal{X}$ for which there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$ such that

$$\begin{cases} x_n \rightarrow \gamma(X) \\ \text{for every } n \in \mathbb{N}, \varphi \text{ is Fréchet differentiable at } x_n \\ \nabla \varphi(x_n) \rightarrow y. \end{cases} \quad (4.21)$$

In the light of Lemma 4.10, $\partial_C \varphi(\gamma(X)) = \text{conv } D \neq \emptyset$, which leads to

$$\text{conv}\{\Lambda_\theta v \mid v \in D \text{ and } \theta \in \mathcal{A}_X\} = \text{conv}\{\Lambda_\theta v \mid v \in \partial_C \varphi(\gamma(X)) \text{ and } \theta \in \mathcal{A}_X\}. \quad (4.22)$$

The nonexpansiveness of γ (Proposition 2.15 (i)) entails that $\varphi \circ \gamma$ is locally Lipschitz continuous near X . Now let $Y \in \mathfrak{S}$ be a point for which there exists a sequence $(X_n)_{n \in \mathbb{N}}$ in $\mathfrak{S}$ such that

$$\begin{cases} X_n \rightarrow X \\ \text{for every } n \in \mathbb{N}, \varphi \circ \gamma \text{ is Fréchet differentiable at } X_n \\ \nabla(\varphi \circ \gamma)(X_n) \rightarrow Y \end{cases} \quad (4.23)$$

(such a point exists thanks to Lemma 4.10 (ii)). Since γ is nonexpansive,

$$\gamma(X_n) \rightarrow \gamma(X). \quad (4.24)$$

Next, for every $n \in \mathbb{N}$, take $\alpha_n \in \mathcal{A}_{X_n}$. By Corollary 4.8 (ii),

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \varphi \text{ is Fréchet differentiable at } \gamma(X_n) \\ \nabla(\varphi \circ \gamma)(X_n) = \Lambda_{\alpha_n}(\nabla \varphi(\gamma(X_n))). \end{cases} \quad (4.25)$$

Hence, because $(\Lambda_{a_n})_{n \in \mathbb{N}}$ are linear isometries,

$$\sup_{n \in \mathbb{N}} \|\nabla \varphi(\gamma(X_n))\| = \sup_{n \in \mathbb{N}} \|\nabla(\varphi \circ \gamma)(X_n)\| < +\infty. \quad (4.26)$$

Therefore, appealing to [Proposition 2.18](#), we obtain a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$, a point $y \in X$, and an element $a \in \mathcal{A}$ such that

$$\nabla \varphi(\gamma(X_{k_n})) \rightarrow y \quad \text{and} \quad \Lambda_{a_{k_n}} \rightarrow \Lambda_a. \quad (4.27)$$

In turn, we derive from [\(4.23\)](#) and [\(4.25\)](#) that

$$Y = \lim \nabla(\varphi \circ \gamma)(X_{k_n}) = \lim \Lambda_{a_{k_n}}(\nabla \varphi(\gamma(X_{k_n}))) = \Lambda_a y. \quad (4.28)$$

At the same time, combining [\(4.24\)](#), [\(4.25\)](#), and [\(4.27\)](#) yields $y \in D$, while [\(4.23\)](#) and [Proposition 2.17](#) ensure that $a \in \mathcal{A}_X$. Thus, we deduce from [Lemma 4.10 \(ii\)](#) and [\(4.22\)](#) that

$$\begin{aligned} \partial_{\mathbb{C}}(\varphi \circ \gamma)(X) &\subset \text{conv}\{\Lambda_{\mathfrak{b}} v \mid v \in D \text{ and } \mathfrak{b} \in \mathcal{A}_X\} \\ &= \text{conv}\{\Lambda_{\mathfrak{b}} v \mid v \in \partial_{\mathbb{C}} \varphi(\gamma(X)) \text{ and } \mathfrak{b} \in \mathcal{A}_X\}. \end{aligned} \quad (4.29)$$

We now establish the converse inclusion. Toward this end, take $v \in D$ and $\mathfrak{b} \in \mathcal{A}_X$, and let $(z_n)_{n \in \mathbb{N}}$ be a sequence in X such that

$$\begin{cases} z_n \rightarrow \gamma(X) \\ \text{for every } n \in \mathbb{N}, \varphi \text{ is Fréchet differentiable at } z_n \\ \nabla \varphi(z_n) \rightarrow v. \end{cases} \quad (4.30)$$

Then $\Lambda_{\mathfrak{b}} z_n \rightarrow \Lambda_{\mathfrak{b}} \gamma(X) = X$ and $\Lambda_{\mathfrak{b}}(\nabla \varphi(z_n)) \rightarrow \Lambda_{\mathfrak{b}} v$. On the other hand, for every $n \in \mathbb{N}$, [Corollary 4.8 \(i\)](#) asserts that $\varphi \circ \gamma$ is Fréchet differentiable at $\Lambda_{\mathfrak{b}} z_n$ with $\nabla(\varphi \circ \gamma)(\Lambda_{\mathfrak{b}} z_n) = \Lambda_{\mathfrak{b}}(\nabla \varphi(z_n))$. Thus, [Lemma 4.10 \(ii\)](#) implies that $\Lambda_{\mathfrak{b}} v \in \partial_{\mathbb{C}}(\varphi \circ \gamma)(X)$, and we conclude the proof by invoking [\(4.22\)](#) and the convexity of $\partial_{\mathbb{C}}(\varphi \circ \gamma)(X)$. $\square$

Remark 4.12. In [Proposition 4.11](#), we do not know whether the convex hull operation on the right-hand side of [\(4.20\)](#) can be omitted. However, this is indeed the case for the particular instances of spectral decomposition systems in [Example 2.10](#), [Example 2.11](#), [Example 2.12](#) (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), and was established in [\[37, Theorem 21\(ii\)\]](#), [\[31, Theorem 8\]](#) (see also [\[28, Theorem 1.4\]](#)), and [\[34, Theorems 3.7 and 4.2\]](#), respectively.

5. A generalization of Lidskiĭ's theorem

A pivotal result in the perturbation analysis of Hermitian matrices is Lidskiĭ's theorem [\[36\]](#) which quantifies how eigenvalues vary under additive perturbations. More precisely, using the notation of [Example 2.11](#), it asserts that

$$(\forall X \in H^N(\mathbb{C})) (\forall Y \in H^N(\mathbb{C})) \quad \lambda(X + Y) - \lambda(X) \in \text{conv}(P^N \cdot \lambda(Y)); \quad (5.1)$$

see also [\[6, Corollary III.4.2 and Theorem II.1.10\]](#). This type of majorization has been established beyond the Hermitian setting to encompass a range of algebraic frameworks, including Lie-theoretic framework [\[5\]](#), Eaton triples [\[21, Theorem 6.4\]](#), singular values of rectangular matrices [\[22, Theorem 3.4.5\]](#), and eigenvalues in Euclidean Jordan algebras [\[25, Theorem 5.1\]](#). However, as illustrated earlier, these settings can be viewed as spectral decomposition systems. This thus motivates the following generalization of Lidskiĭ's theorem, which brings together these results under the umbrella of our framework.

Theorem 5.1. Suppose that [Assumption 2.2](#) is in force and, in addition, that S is a finite group. Then

$$(\forall X \in \mathfrak{S})(\forall Y \in \mathfrak{S}) \quad \gamma(X + Y) - \gamma(X) \in \text{conv}(S \cdot \gamma(Y)). \quad (5.2)$$

Proof. We employ the techniques of [30]. Let τ be the spectral-induced ordering mapping of the system $(\mathcal{X}, S, \gamma, (\Lambda_a)_{a \in \mathcal{A}})$ (see property [A] in [Definition 2.1](#)), and set

$$K = \text{range } \tau. \quad (5.3)$$

Then K is a closed convex cone in $\mathcal{X}$ [7, Proposition 3.5 (ii)]. Furthermore, we get from (2.3) that

$$\mathcal{X} = \bigcup_{s \in S} s \cdot K, \quad (5.4)$$

where we adopt the notation

$$(\forall D \in 2^{\mathcal{X}})(\forall s \in S) \quad s \cdot D = \{s \cdot x \mid x \in D\}. \quad (5.5)$$

Hence, applying [3, Lemma 1.44 (i)] to the finite family $(s \cdot K)_{s \in S}$ of closed subsets of $\mathcal{X}$, we obtain

$$\overline{\bigcup_{s \in S} \text{int}(s \cdot K)} = \overline{\text{int} \bigcup_{s \in S} s \cdot K} = \mathcal{X}. \quad (5.6)$$

Since, for every $s \in S$, the mapping $x \mapsto s \cdot x$ is a homeomorphism, we must have

$$\text{int } K \neq \emptyset. \quad (5.7)$$

In turn, since [7, Proposition 3.3 (iv) and (i)] give

$$(\forall x \in K)(\forall y \in K) \quad \max_{s \in S} \langle s \cdot x \mid y \rangle = \langle x \mid y \rangle, \quad (5.8)$$

we infer from [43, Lemma 2.1] (applied to the finite subgroup $\{x \mapsto s \cdot x \mid s \in S\}$ of $O(\mathcal{X})$ and the set K) that

$$(\forall s \in S)(\forall t \in S) \quad (s \cdot \text{int } K) \cap (t \cdot K) \neq \emptyset \quad \Rightarrow \quad [(\forall x \in K) \ s \cdot x = t \cdot x]. \quad (5.9)$$

This and (5.4) yield

$$\mathcal{X} \setminus \bigcup_{s \in S} \text{int}(s \cdot K) = \bigcap_{s \in S} (\mathcal{X} \setminus \text{int}(s \cdot K)) = \bigcup_{s \in S} \text{bdry}(s \cdot K), \quad (5.10)$$

where $\text{bdry } D$ denotes the boundary of a subset D of $\mathcal{X}$. At the same time, for every $s \in S$, [27, Theorem 1] implies that $\text{bdry}(s \cdot K)$ is a Lebesgue null set. Thus, since S is finite,

$$\mathcal{X} \setminus \bigcup_{s \in S} \text{int}(s \cdot K) \text{ is a Lebesgue null set.} \quad (5.11)$$

Now let X and Y be in $\mathfrak{S}$. In the light of [7, Proposition 3.8], the desired assertion is equivalent to

$$(\forall z \in \mathcal{X}) \quad \langle \gamma(X + Y) - \gamma(X) \mid z \rangle \leq \langle \gamma(Y) \mid \tau(z) \rangle. \quad (5.12)$$

To prove this, take $z \in \mathcal{X}$ and define an S -invariant function $\varphi: \mathcal{X} \rightarrow \mathbb{R}$ via

$$(\forall x \in \mathcal{X}) \quad \varphi(x) = \langle \tau(x) \mid z \rangle. \quad (5.13)$$

It follows from [7, Proposition 3.3 (v)] that φ is Lipschitz continuous on $\mathcal{X}$. Therefore, since γ is non-expansive, $\varphi \circ \gamma$ is Lipschitz continuous on $\mathfrak{S}$. To proceed further, define

$$D = \bigcup_{s \in S} \text{int}(s \cdot K). \quad (5.14)$$

We claim that

$$\varphi \text{ is Fréchet differentiable on } D \text{ with } (\forall x \in D) \nabla \varphi(x) \in S \cdot z. \quad (5.15)$$

Toward this end, fix temporarily $s \in S$ and set $D_s = \text{int}(s \cdot K)$. Since $x \mapsto s \cdot x$ is a homeomorphism, D_s is open and $D_s = s \cdot \text{int } K$. Let us verify that

$$(\forall x \in D_s) \quad x = s \cdot \tau(x). \quad (5.16)$$

Indeed, fix temporarily $x \in D_s$ and let $t \in S$ be such that $x = t \cdot \tau(x)$. Then $x \in (s \cdot \text{int } K) \cap (t \cdot K)$ and it thus results from (5.9) that $s \cdot \tau(x) = t \cdot \tau(x) = x$. In turn, it follows from (5.16) and [7, Lemma 3.2] that

$$(\forall x \in D_s) \quad \varphi(x) = \langle s^{-1} \cdot x \mid z \rangle = \langle x \mid s \cdot z \rangle. \quad (5.17)$$

Hence, (5.15) holds. On the other hand, the set $\text{conv}(S \cdot z)$ is compact as the convex hull of the finite set $S \cdot z$. Therefore, combining (5.11) and (5.15), we deduce from Lemma 4.10 (ii) that

$$(\forall x \in \mathcal{X}) \quad \partial_C \varphi(x) \subset \text{conv}(S \cdot z). \quad (5.18)$$

Next, we learn from Lebourg's mean value theorem Lemma 4.10 (ii) that there exist $V \in \{(1 - \alpha)X + \alpha Y \mid \alpha \in [0, 1]\}$ and $W \in \partial_C(\varphi \circ \gamma)(V)$ such that

$$\langle \gamma(X + Y) - \gamma(X) \mid z \rangle = (\varphi \circ \gamma)(X + Y) - (\varphi \circ \gamma)(X) = \langle W \mid Y \rangle, \quad (5.19)$$

where the first identity follows from [7, Proposition 3.5 (i)]. However, Proposition 4.11 asserts that there exist finite families $(w_i)_{i \in I}$ in $\partial_C \varphi(\gamma(V))$, $(\alpha_i)_{i \in I}$ in $\mathcal{A}_V$, and $(\alpha_i)_{i \in I}$ in $]0, 1]$ such that

$$\sum_{i \in I} \alpha_i = 1 \quad \text{and} \quad W = \sum_{i \in I} \alpha_i \Lambda_{\alpha_i} w_i. \quad (5.20)$$

By (5.18) and (2.3), $\{\tau(w_i)\}_{i \in I} \subset \text{conv}(S \cdot z)$, and we thus derive from [7, Propositions 3.8 and 3.5 (i)] that

$$(\forall i \in I) \quad \langle \tau(w_i) \mid \gamma(Y) \rangle \leq \langle \tau(z) \mid \tau(\gamma(Y)) \rangle = \langle \tau(z) \mid \gamma(Y) \rangle. \quad (5.21)$$

Hence, by properties [C] and [A] in Definition 2.1,

$$\begin{aligned} \langle \gamma(X + Y) - \gamma(X) \mid z \rangle &= \sum_{i \in I} \alpha_i \langle \Lambda_{\alpha_i} w_i \mid Y \rangle \\ &\leq \sum_{i \in I} \alpha_i \langle \gamma(\Lambda_{\alpha_i} w_i) \mid \gamma(Y) \rangle \\ &= \sum_{i \in I} \alpha_i \langle \tau(w_i) \mid \gamma(Y) \rangle \\ &\leq \sum_{i \in I} \alpha_i \langle \tau(z) \mid \gamma(Y) \rangle \\ &= \langle \gamma(Y) \mid \tau(z) \rangle, \end{aligned} \quad (5.22)$$

which completes the proof. $\square$

Remark 5.2. We now substantiate the claim made at the beginning of this section that [Theorem 5.1](#) unifies several existing results.

- (i) In the context of Hermitian matrices of [Example 2.11](#), [Theorem 5.1](#) reads

$$(\forall X \in H^N(\mathbb{K})) (\forall Y \in H^N(\mathbb{K})) \quad \lambda(X + Y) - \lambda(X) \in \text{conv}(P^N \cdot \lambda(Y)), \quad (5.23)$$

and we thus recover Lidskiĭ's theorem [36].

- (ii) In the Euclidean Jordan algebra framework of [Example 2.10](#), [Theorem 5.1](#) reduces to [25, Theorem 5.1], that is,

$$(\forall X \in \mathfrak{H}) (\forall Y \in \mathfrak{H}) \quad \lambda(X + Y) - \lambda(X) \in \text{conv}(P^N \cdot \lambda(Y)). \quad (5.24)$$

- (iii) By specializing [Theorem 5.1](#) to [Example 2.12](#) with $\mathbb{K} = \mathbb{C}$, we obtain the version of Lidskiĭ's theorem for singular values [6, Theorem IV.3.4 and Exercise II.2.10], that is,

$$(\forall X \in \mathbb{C}^{M \times N}) (\forall Y \in \mathbb{C}^{M \times N}) \quad \sigma(X + Y) - \sigma(X) \in \text{conv}(P_{\pm}^m \cdot \sigma(Y)). \quad (5.25)$$

- (iv) Consider the Eaton triple framework of [Example 2.9](#) and suppose additionally that

$$\mathcal{X} = \mathcal{K} - \mathcal{K}. \quad (5.26)$$

We show that S is a finite group. On the one hand, (2.20) and the very definition of γ ensure that $(\forall X \in \mathcal{X}) \mathcal{K} \cap (S \cdot X) \neq \emptyset$. On the other hand, since $\tau = \gamma|_{\mathcal{X}}$ is the spectral-induced ordering mapping of the spectral decomposition system $\mathfrak{S}$ and since $\text{range } \tau = \mathcal{K}$, we deduce from [7, Proposition 3.3 (iv) and (i)] that $(\forall X \in \mathcal{K}) (\forall Y \in \mathcal{K}) \max_{s \in S} \langle X | s(Y) \rangle = \langle X | Y \rangle$. Hence, [43, Theorem 3.2] (applied to the Eaton triple $(\mathfrak{H}, G, \mathcal{K})$ and the reduced triple $(\mathcal{X}, S, \mathcal{K})$) implies that S is a finite group. Thus, [Theorem 5.1](#) is applicable and reduces to [21, Theorem 6.4] in this context. This result encompasses, in particular, the Lie-theoretic majorization result due to Berezin and Gel'fand [5].

6. Conclusion

Continuing from our previous work [7], we have derived results on variational geometry and analysis in the abstract framework of spectral decomposition systems that covers a wide range of related settings such as eigenvalue or singular value decompositions of real, complex, and quaternion matrices, or Euclidean Jordan algebras. Specifically, we have derived representations of Fréchet and limiting normal cones to spectral sets and, based on that, Fréchet and limiting subdifferentials of spectral functions. These results further allowed deriving characterizations of Fréchet derivatives and Clarke generalized gradients. Using the latter, we generalized Lidskiĭ's theorem on the spectrum of additive perturbations of Hermitian matrices to arbitrary spectral decomposition systems.

This work can be extended in a number of directions. First, the representations of generalized subdifferentials can be used to obtain explicit necessary optimality conditions for concrete matrix optimization problems such as low-rank matrix completion via nonconvex Schatten p -norm penalization in a suitably general setting. In the context of variational analysis, an important open question is on characterizations of Lipschitz-like properties of solution mappings such as metric regularity or subregularity, cf. [11, Chapter 27] and the literature cited therein. Such properties can then be used to generalize results on second-order variational analysis of spectral functions such as [41].

References

- [1] M. Baes, Convexity and differentiability properties of spectral functions and spectral mappings on Euclidean Jordan algebras, *Linear Algebra Appl.* 422 (2007), 664–700, doi:[10.1016/j.laa.2006.11.025](https://doi.org/10.1016/j.laa.2006.11.025).
- [2] J.M. Ball, Differentiability properties of symmetric and isotropic functions, *Duke Math. J.* 51 (1984), 699–728, doi:[10.1215/s0012-7094-84-05134-2](https://doi.org/10.1215/s0012-7094-84-05134-2).
- [3] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, Cham, Switzerland, 2nd edition, 2017, doi:[10.1007/978-3-319-48311-5](https://doi.org/10.1007/978-3-319-48311-5).
- [4] A. Benfenati, E. Chouzenoux, and J. C. Pesquet, Proximal approaches for matrix optimization problems: Application to robust precision matrix estimation, *Signal Process.* 169 (2020), 107417, doi:[10.1016/j.sigpro.2019.107417](https://doi.org/10.1016/j.sigpro.2019.107417).
- [5] F. A. Berezin and I. M. Gel'fand, Some remarks on the theory of spherical functions on symmetric Riemannian manifolds, *Tr. Moskov. Mat. Obšč.* 5 (1956), 311–351. English translation in *Amer. Math. Soc. Transl.* (2) 21 (1962), 193–238.
- [6] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997, doi:[10.1007/978-1-4612-0653-8](https://doi.org/10.1007/978-1-4612-0653-8).
- [7] H. T. Bui, M. N. Bui, and C. Clason, Convex analysis in spectral decomposition systems, 2025, [arXiv:2503.14981](https://arxiv.org/abs/2503.14981).
- [8] J. F. Cai, E. J. Candès, and Z. Shen, A singular value thresholding algorithm for matrix completion, *SIAM J. Optim.* 20 (2010), 1956–1982, doi:[10.1137/080738970](https://doi.org/10.1137/080738970).
- [9] T. S. T. Chan and Y. H. Yang, Complex and quaternionic principal component pursuit and its application to audio separation, *IEEE Signal Process. Lett.* 23 (2016), 287–291, doi:[10.1109/lsp.2016.2514845](https://doi.org/10.1109/lsp.2016.2514845).
- [10] F. H. Clarke, *Optimization and Nonsmooth Analysis*, SIAM, 1990, doi:[10.1137/1.9781611971309](https://doi.org/10.1137/1.9781611971309).
- [11] C. Clason and T. Valkonen, Introduction to Nonsmooth Analysis and Optimization, 2024, [arXiv:2001.00216](https://arxiv.org/abs/2001.00216).
- [12] C. Coey, L. Kapelevich, and J. P. Vielma, Conic optimization with spectral functions on Euclidean Jordan algebras, *Math. Oper. Res.* 48 (2023), 1906–1933, doi:[10.1287/moor.2022.1324](https://doi.org/10.1287/moor.2022.1324).
- [13] B. Dacorogna and P. Maréchal, Convex $SO(N) \times SO(n)$ -invariant functions and refinements of von Neumann's inequality, *Ann. Fac. Sci. Toulouse Math.* (6) 16 (2007), 71–89, doi:[10.5802/afst.1139](https://doi.org/10.5802/afst.1139).
- [14] A. Daniilidis, A. Lewis, J. Malick, and H. Sendov, Prox-regularity of spectral functions and spectral sets, *J. Convex Anal.* 15 (2008), 547–560, <https://www.heldermann.de/JCA/JCA15/JCA153/jca15039.htm>.
- [15] C. Ding, D. Sun, and J. J. Ye, First order optimality conditions for mathematical programs with semidefinite cone complementarity constraints, *Math. Program.* A147 (2014), 539–579, doi:[10.1007/s10107-013-0735-z](https://doi.org/10.1007/s10107-013-0735-z).

-
- [16] D. Drusvyatskiy and C. Paquette, Variational analysis of spectral functions simplified, *J. Convex Anal.* 25 (2018), 119–134, <https://www.heldermann.de/JCA/JCA25/JCA251/jca25008.htm>.
 - [17] M. L. Eaton, *Lectures on Topics in Probability Inequalities*, Stichting Mathematisch Centrum, Amsterdam, 1987.
 - [18] C. Eckart and G. Young, The approximation of one matrix by another of lower rank, *Psychometrika* 1 (1936), 211–218, doi:10.1007/bf02288367.
 - [19] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford University Press, Oxford, 1994, doi:10.1093/oso/9780198534778.001.0001.
 - [20] M. S. Gowda and J. Jeong, Commutation principles in Euclidean Jordan algebras and normal decomposition systems, *SIAM J. Optim.* 27 (2017), 1390–1402, doi:10.1137/16m1071006.
 - [21] W. C. Hill, *G-Invariant Norm, an Extension of Berezin–Gel’fand’s Theorem via Nonsmooth Analysis and Applications*, PhD thesis, Auburn University, 2001.
 - [22] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991, doi:10.1017/cbo9780511840371.
 - [23] P. O. Hoyer, Non-negative matrix factorization with sparseness constraints, *J. Mach. Learn. Res.* 5 (2003/04), 1457–1469, <https://www.jmlr.org/papers/volume5/hoyer04a/hoyer04a.pdf>.
 - [24] H. Hu, R. Sotirov, and H. Wolkowicz, Facial reduction for symmetry reduced semidefinite and doubly nonnegative programs, *Math. Program.* A200 (2023), 475–529, doi:10.1007/s10107-022-01890-9.
 - [25] J. Jeong, Y. M. Jung, and Y. Lim, Weak majorization, doubly substochastic maps, and some related inequalities in Euclidean Jordan algebras, *Linear Algebra Appl.* 597 (2020), 133–154, doi:10.1016/j.laa.2020.03.028.
 - [26] N. Kovachki, Z. Li, B. Liu, K. Azizzadenesheli, K. Bhattacharya, A. Stuart, and A. Anandkumar, Neural operator: learning maps between function spaces with applications to PDEs, *J. Mach. Learn. Res.* 24 (2023), 1–97, <https://www.jmlr.org/papers/volume24/21-1524/21-1524.pdf>.
 - [27] R. Lang, A note on the measurability of convex sets, *Arch. Math.* 47 (1986), 90–92, doi:10.1007/bf01202504.
 - [28] A. S. Lewis, Derivatives of spectral functions, *Math. Oper. Res.* 21 (1996), 576–588, doi:10.1287/moor.21.3.576.
 - [29] A. S. Lewis, Group invariance and convex matrix analysis, *SIAM J. Matrix Anal. Appl.* 17 (1996), 927–949, doi:10.1137/s0895479895283173.
 - [30] A. S. Lewis, Lidskii’s theorem via nonsmooth analysis, *SIAM J. Matrix Anal. Appl.* 21 (1999), 379–381, doi:10.1137/s0895479898338676.
 - [31] A. S. Lewis, Nonsmooth analysis of eigenvalues, *Math. Program.* 84 (1999), 1–24, doi:10.1007/s10107980004a.
 - [32] A. S. Lewis, Convex analysis on Cartan subspaces, *Nonlinear Anal.* 42 (2000), 813–820, doi:10.1016/s0362-546x(99)00126-1.

- [33] A. S. Lewis, The mathematics of eigenvalue optimization, *Math. Program.* B97 (2003), 155–176, doi:10.1007/s10107-003-0441-3.
- [34] A. S. Lewis and H. S. Sendov, Nonsmooth analysis of singular values. Part I: Theory, *Set-Valued Var. Anal.* 13 (2005), 213–241, doi:10.1007/s11228-004-7197-7.
- [35] A. S. Lewis and H. S. Sendov, Nonsmooth analysis of singular values. Part II: Applications, *Set-Valued Var. Anal.* 13 (2005), 243–264, doi:10.1007/s11228-004-7198-6.
- [36] V. B. Lidskiĭ, On the proper values of the sum and product of symmetric matrices, *Dokl. Akad. Nauk SSSR* 75 (1950), 769–772.
- [37] B. F. Lourenço and A. Takeda, Generalized subdifferentials of spectral functions over Euclidean Jordan algebras, *SIAM J. Optim.* 30 (2020), 3387–3414, doi:10.1137/19m1245001.
- [38] G. Marjanovic and V. Solo, On l_q optimization and matrix completion, *IEEE Trans. Signal Process.* 60 (2012), 5714–5724, doi:10.1109/tsp.2012.2212015.
- [39] I. Markovsky, Structured low-rank approximation and its applications, *Automatica J. IFAC* 44 (2008), 891–909, doi:10.1016/j.automatica.2007.09.011.
- [40] R. Mifflin, Semismooth and semiconvex functions in constrained optimization, *SIAM J. Control Optim.* 15 (1977), 959–972, doi:10.1137/0315061.
- [41] A. Mohammadi and E. Sarabi, Parabolic regularity of spectral functions, *Math. Oper. Res.* 50 (2025), 2017–2046, doi:10.1287/moor.2023.0010.
- [42] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I*, Springer, Heidelberg, 2006, doi:10.1007/3-540-31247-1.
- [43] M. Niezgoda, Group majorization and Schur type inequalities, *Linear Algebra Appl.* 268 (1998), 9–30, doi:10.1016/s0024-3795(97)89322-6.
- [44] M. Orlitzky, Proscribed normal decompositions of Euclidean Jordan algebras, *J. Convex Anal.* 29 (2022), 755–766, <https://www.heldermann.de/JCA/JCA29/JCA293/jca29043.htm>.
- [45] L. Qi, Convergence analysis of some algorithms for solving nonsmooth equations, *Math. Oper. Res.* 18 (1993), 227–244, doi:10.1287/moor.18.1.227.
- [46] L. Qi and J. Sun, A nonsmooth version of Newton’s method, *Math. Program.* A58 (1993), 353–367, doi:10.1007/bf01581275.
- [47] H. Ramírez, A. Seeger, and D. Sossa, Commutation principle for variational problems on Euclidean Jordan algebras, *SIAM J. Optim.* 23 (2013), 687–694, doi:10.1137/120879397.
- [48] R. T. Rockafellar and R. J. B. Wets, *Variational Analysis*, Springer, Heidelberg, 2009, doi:10.1007/978-3-642-02431-3.
- [49] L. Rodman, *Topics in Quaternion Linear Algebra*, Princeton University Press, Princeton, NJ, 2014, doi:10.1515/9781400852741.
- [50] P. Rosakis, Characterization of convex isotropic functions, *J. Elasticity* 49 (1998), 257–267, doi:10.1023/a:1007468902439.

-
- [51] H. S. Sendov, Nonsmooth analysis of Lorentz invariant functions, *SIAM J. Optim.* 18 (2007), 1106–1127, doi:[10.1137/060658370](https://doi.org/10.1137/060658370).
 - [52] D. Sun and J. Sun, Löwner’s operator and spectral functions in Euclidean Jordan algebras, *Math. Oper. Res.* 33 (2008), 421–445, doi:[10.1287/moor.1070.0300](https://doi.org/10.1287/moor.1070.0300).
 - [53] T. Y. Tam, A unified extension of two results of Ky Fan on the sum of matrices, *Proc. Amer. Math. Soc.* 126 (1998), 2607–2614, doi:[10.1090/s0002-9939-98-04770-4](https://doi.org/10.1090/s0002-9939-98-04770-4).
 - [54] R. C. Thompson, Singular values, diagonal elements, and convexity, *SIAM J. Appl. Math.* 32 (1977), 39–63, doi:[10.1137/0132003](https://doi.org/10.1137/0132003).
 - [55] S. Yang, Z. Lu, X. Shen, P. Wonka, and J. Ye, Fused multiple graphical lasso, *SIAM J. Optim.* 25 (2015), 916–943, doi:[10.1137/130936397](https://doi.org/10.1137/130936397).