PARABOLIC CONTROL PROBLEMS IN MEASURE SPACES WITH SPARSE SOLUTIONS

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August 9, 2012

Optimal control problems in measure spaces lead to controls that have small support, which is desirable, e.g., in the context of optimal actuator placement. For problems governed by parabolic partial differential equations, well-posedness is guaranteed in the space of square-integrable measure-valued functions, which leads to controls with a spatial sparsity structure. A conforming approximation framework allows deriving numerically accessible optimality conditions as well as convergence rates. In particular, although the state is discretized, the control problem can still be formulated and solved in the measure space. Numerical examples illustrate the structural features of the optimal controls.

1 INTRODUCTION

This paper is concerned with the analysis and approximation of the optimal control problem

$$\text{(P)} \qquad \min_{u\in L^2(I,\mathcal{M}(\Omega))} J(u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega_T)} + \alpha\|u\|_{L^2(\mathcal{M})},$$

where $I = [0,T]$ and y is the unique solution to the initial–boundary value problem

$$(1.1) \qquad \begin{cases} \partial_t y - \Delta y &= u \quad \text{in } \Omega_T = \Omega\times(0,T),\\ y &= 0 \quad \text{on } \Sigma_T = \Gamma\times(0,T),\\ y(x,0) &= y_0 \quad \text{in } \Omega \end{cases}$$

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for given $y_0 \in L^2(\Omega)$. We assume that $\alpha > 0$, $y_d \in L^2(\Omega_T)$ and Ω is a bounded domain in $\mathbb{R}^n$, $1 \leqslant n \leqslant 3$, which is supposed to either be convex or have a $C^{1,1}$ boundary Γ. Hereafter $\mathcal{M}(\Omega)$ denotes the space of regular Borel measures in Ω and $\|u\|_{L^2(\mathcal{M})}$ denotes the norm of u in the space $L^2(I, \mathcal{M}(\Omega))$; see section 2 below for details.

Formulating the control problem in a measure space is motivated by the observation that the resulting optimal controls possess sparsity properties (i.e., have small support), which is desirable in many applications such as optimal sensor or actuator placement; see [Clason and Kunisch 2011; Casas, Clason, et al. 2012] in the context of elliptic equations. Although similar features can be achieved using L^1 control costs, the corresponding control problem in general does not admit a solution in the absence of further regularization because L^1 spaces lack the necessary compactness properties. For parabolic problems, the situation is even more delicate since (1.1) is not well-posed for right hand sides in $\mathcal{M}(\Omega_T)$ (which would require $C(\Omega_T)$ regularity for the adjoint equation; see Definition 2.1 below). This leads to considering controls in $L^2(I, \mathcal{M}(\Omega))$. The associated norm $\|u\|_{L^2(\mathcal{M})}$ for the control is a natural one from the point of view of well-posedness of the state equation (1.1) and allows for sparsity in space. The numerical results will illustrate precisely this property of our formulation. The spatio-temporal coupling of the corresponding control cost, however, presents a challenge for deriving numerically useful optimality conditions.

Besides the analysis of the control problem (P), the main focus of this paper consists in providing an approximation framework which, in spite of the difficulties due to the measure space setting, leads to implementable schemes for which a priori error estimates can be provided. We show that the optimal measure controls can be approximated efficiently by linear combinations of Dirac measures in space which are piecewise constant in time. We point out that even after discretization, the control problem is formulated and solved in the measure space.

Let us mention some related works. A similar approximation framework for elliptic control problems in measure spaces was proposed in [Casas, Clason, et al. 2012]. Differently from the elliptic case, parabolic control problems with sparsity-promoting constraints have received very little attention. In [Casas and Zuazua 2012], the approximate control of $y(T)$ by measures $u \in \mathcal{M}([t_0, t_1] \times \Omega)$ with $0 < t_0 < t_1 < T$ is discussed (using the smoothing property of the heat equation to ensure $y(T) \in L^2(\Omega)$); finite-dimensional approximation and numerical solution are not addressed. Although not specifically concerned with parabolic equations, the approach of [Herzog et al. 2012] covers control problems with $L^1(\Omega, L^2([0, T]))$ control costs (together with additional pointwise control constraints). The resulting optimal controls have *directional sparsity*, i.e., their support is constant in time. In contrast, we will show that solutions to (P) have a non-separable sparsity structure.

This paper is organized as follows. In the next section, we discuss the functional analytic setting of the control problem and analyze well-posedness of the state equation. Section 3 is concerned with existence of and optimality conditions for solutions to (P), the latter implying a sparsity property of the optimal controls. The proposed approximation framework is the subject of section 4, where we introduce the discretization (§ 4.1) and show convergence of solutions to the discretized state equation (§ 4.2) and to the discrete optimal control problem (§ 4.3). Convergence rates are derived in section 5. Section 6 addresses the numerical solution of the discrete control problem, for which we derive a reformulated optimality system that is amenable to solution by a semismooth Newton method. (The continuous counterpart of this optimality system is sketched in Appendix A.) Finally, section 7 illustrates the structure of the optimal controls with some numerical examples.

2 FUNCTION SPACES AND WELL-POSEDNESS OF THE STATE EQUATION

In this section we first define the control space and give some of its properties. Then, we turn to the analysis of the state equation.

2.1 CONTROL SPACE

We denote by $C_0(\Omega)$ the space of continuous functions in $\bar{\Omega}$ vanishing on $\Gamma = \partial\Omega$, endowed with the supremum norm $\|\cdot\|_\infty$. Its topological dual is identified with the space of regular Borel measures in Ω, denoted by $\mathcal{M}(\Omega)$. Moreover, we have

$$\|u\|_{\mathcal{M}} = \sup\left\{\int_\Omega z\,du : z \in C_0(\Omega) \text{ and } \|z\|_\infty \leqslant 1\right\} = |u|(\Omega),$$

where $|u|$ denotes the total variation measure.

Associated to the interval $I = [0, T]$ we define the spaces $L^2(I, C_0(\Omega))$ and $L^2(I, \mathcal{M}(\Omega))$, where $L^2(I, C_0(\Omega))$ is the space of measurable functions $z : [0, T] \to C_0(\Omega)$ for which the associated norm given by

$$\|z\|_{L^2(C_0)} = \left(\int_0^T \|z(t)\|_\infty^2\,dt\right)^{1/2}$$

is finite. Due to the fact that $C_0(\Omega)$ is a separable Banach space, $L^2(I, C_0(\Omega))$ is also a separable Banach space; see e.g. [Warga 1972, Theorem I.5.18].

As a consequence of the non-separability of $\mathcal{M}(\Omega)$, the definition of the space $L^2(I, \mathcal{M}(\Omega))$ is more delicate. Indeed, we need to distinguish between weakly and strongly measurable functions $u : [0, T] \to \mathcal{M}(\Omega)$. Hereafter we denote by $L^2(I, \mathcal{M}(\Omega))$ the space of weakly measurable functions u for which the norm

$$\|u\|_{L^2(\mathcal{M})} = \left(\int_0^T \|u(t)\|_{\mathcal{M}}^2\,dt\right)^{1/2}$$

is finite. This choice makes $L^2(I, \mathcal{M}(\Omega))$ a Banach space and guarantees that it can be identified with the dual of $L^2(I, C_0(\Omega))$, where the duality relation is given by

$$\langle u, z\rangle_{L^2(\mathcal{M}), L^2(C_0)} = \int_0^T \langle u(t), z(t)\rangle \, dt,$$

with $\langle \cdot, \cdot \rangle$ denoting the duality between $\mathcal{M}(\Omega)$ and $C_0(\Omega)$. The reader is referred to [Edwards 1965, section 8.14.1 and Proposition 8.15.3] for the different notions of measurability and [Edwards 1965, Theorem 8.20.3] for the duality identification. (The distinction between weak and strong measurability is not required for the space $L^2(I, C_0(\Omega))$ because $C_0(\Omega)$ is separable and hence both notions are equivalent; see [Edwards 1965, Theorem 8.15.2].)

2.2 ANALYSIS OF THE STATE EQUATION

Given $1 < p < \infty$, we denote by $W_0^{1,p}(\Omega)$ the Sobolev space of functions of $L^p(\Omega)$ with distributional derivatives in $L^p(\Omega)$ and having a zero trace on Γ and we set $W^{-1,p'}(\Omega)$ to be the dual of $W_0^{1,p}(\Omega)$, where $1/p' + 1/p = 1$. These spaces are reflexive and separable, and hence the spaces $L^2(I, W_0^{1,p}(\Omega))$ formed by the measurable functions $y : [0, T] \to W_0^{1,p}(\Omega)$ for which the norm

$$\|y\|_{L^2(W_0^{1,p})} = \left(\int_0^T \|y(t)\|^2_{W_0^{1,p}} \, dt\right)^{1/2}$$

is finite, are separable and reflexive Banach spaces whose dual is identified with $L^2(I, W^{-1,p'}(\Omega))$; see [Edwards 1965, Theorem 8.25.5].

The notion of solution to the state equation makes use of the following space of test functions

$$\mathcal{Z} = \{z \in H^{2,1}(\Omega_T) : z = 0 \text{ on } \Sigma_T \text{ and } z(T) = 0 \text{ in } \Omega\},$$

where

$$H^{2,1}(\Omega_T) = \left\{z \in L^2(\Omega_T) : \partial_t z, \frac{\partial^{|\beta|} z}{\partial x^\beta} \in L^2(\Omega_T), \text{ with } \beta \in \mathbb{N}^n, \ |\beta| \leqslant 2\right\}$$

is endowed with the graph norm. By the Rellich–Kondrachov theorem, $\mathcal{Z}$ embeds compactly into $L^2(I, C_0(\Omega))$.

Definition 2.1. *We say that $y \in L^2(\Omega_T)$ is a solution to equation (1.1) if*

$$\int_{\Omega_T} y(-\partial_t z - \Delta z) \, dx \, dt = \int_0^T \langle u(t), z(t)\rangle \, dt + \int_\Omega y_0(x) z(x, 0) \, dx, \quad \forall z \in \mathcal{Z}. \tag{2.1}$$

Theorem 2.2. *For all* $(u, y_0) \in L^2(I, \mathcal{M}(\Omega)) \times L^2(\Omega)$ *the equation* (1.1) *has a unique solution* y. *Moreover,* $y \in L^2(I, W_0^{1,p}(\Omega))$ *for every* $p \in [1, \frac{n}{n-1})$ *and there exist constants* C_p *such that*

$$\|y\|_{L^2(W_0^{1,p})} \leqslant C_p \left(\|u\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)}\right). \tag{2.2}$$

Proof. We adapt the proof of [Casas 1997]. Let $\{u_k\}_k$ be a sequence in $C(\bar{\Omega}_T)$ satisfying

$$u_k \stackrel{*}{\rightharpoonup} u \ \text{ in } L^2(I, \mathcal{M}(\Omega)) \ \text{ and } \ \|u_k\|_{L^2(L^1)} \leqslant \|u\|_{L^2(\mathcal{M})}. \tag{2.3}$$

Let $y_k \in L^2(I, H_0^1(\Omega))$ denote the variational solution to

$$\begin{cases} \partial_t y_k - \Delta y_k &= u_k & \text{in } \Omega_T, \\ y_k &= 0 & \text{on } \Sigma_T, \\ y_k(x, 0) &= y_0(x) & \text{in } \Omega. \end{cases} \tag{2.4}$$

For $\psi = (\psi_0, \dots, \psi_n) \in \mathcal{D}(\Omega_T)^{n+1}$ we denote by $z \in \mathcal{Z}$ the solution to

$$\begin{cases} -\partial_t z - \Delta z &= \psi_0 - \sum_{j=1}^n \partial_{x_j} \psi_j & \text{in } \Omega_T, \\ z &= 0 & \text{on } \Sigma_T, \\ z(x, T) &= 0 & \text{in } \Omega. \end{cases} \tag{2.5}$$

From the last two equations we get for any $1 < p < \frac{n}{n-1}$

$$\begin{aligned} \int_{\Omega_T} (\psi_0 y_k + \sum_{j=1}^n \psi_j \partial_{x_j} y_k) \,dx\,dt &= \int_{\Omega_T} u_k z \,dx\,dt + \int_\Omega y_0(x) z(x, 0) \,dx \\ &\leqslant \|u_k\|_{L^2(L^1)} \|z\|_{L^2(W_0^{1,p'})} + \|y_0\|_{L^2(\Omega)} \|z(0)\|_{L^2(\Omega)}. \end{aligned}$$

In the following estimate we use maximal regularity of the heat equation in an essential way. If Ω is convex, its boundary is of Lipschitz class, and hence there exists a $\hat{p}$ with $\hat{p} > 4$ if $n = 2$ and $\hat{p} > 3$ when $n = 3$ such that $\Delta : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is an isomorphism for each $\hat{p}' < p < \hat{p}$, where $1/\hat{p}' + 1/\hat{p} = 1$; see [Jerison and Kenig 1995]. (If $n = 1$ or if Ω has a $C^{1,1}$ boundary, $\Delta : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is an isomorphism for every $1 < p < +\infty$.) In particular, combining [Haller–Dintelmann and Rehberg 2009, Theorem 5.4] and (2.3), we obtain for every $\hat{p}' < p < \frac{n}{n-1} < \hat{p}$ the existence of a constant $\hat{C}_p$ such that

$$\begin{aligned} \int_{\Omega_T} y_k \left(\psi_0 - \sum_{j=1}^n \partial_{x_j} \psi_j\right) dx\,dt &= \int_{\Omega_T} (\psi_0 y_k + \sum_{j=1}^n \psi_j \partial_{x_j} y_k) \,dx\,dt \\ &\leqslant \hat{C}_p \left(\|y_0\|_{L^2(\Omega)} + \|u\|_{L^2(\mathcal{M})}\right) \sum_{j=0}^n \|\psi_j\|_{L^2(L^{p'})}. \end{aligned}$$

From the density of $\{\psi_0 - \sum_{j=1}^n \partial_{x_j}\psi_j : \psi \in \mathcal{D}(\Omega_T)^{n+1}\}$ in $L^2(I, W^{-1,p'}(\Omega))$ and the duality identification $L^2(I, W_0^{1,p}(\Omega))^* = L^2(I, W^{-1,p'}(\Omega))$, we deduce the boundedness of $\{y_k\}_{k=1}^\infty$ in $L^2(I, W_0^{1,p}(\Omega))$ and the existence of a constant C_p such that

$$\|y_k\|_{L^2(W_0^{1,p})} \leqslant C_p \left(\|u\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)}\right). \tag{2.6}$$

Using the reflexivity of $L^2(I, W_0^{1,p}(\Omega))$, we can obtain a subsequence, denoted in the same way, and an element $y \in L^2(I, W_0^{1,p}(\Omega))$ such that $y_k \rightharpoonup y$ in $L^2(I, W_0^{1,p}(\Omega))$.

For $\psi_0 \in L^2(\Omega_T)$ arbitrary and $z \in \mathcal{Z}$ solution to (2.5) for $\psi_i = 0$, $1 \leqslant j \leqslant n$, it follows from (2.4) and (2.5) that

$$\int_{\Omega_T} y_k(-\partial_t z - \Delta z)\,dx\,dt = \int_{\Omega_T} y_k\psi_0\,dx\,dt = \int_{\Omega_T} u_k z\,dx\,dt + \int_\Omega y_0(x)z(x,0)\,dx.$$

Passing to the limit in this identity and in (2.6), we obtain (2.1) and (2.2). Using the fact that $\partial_t + \Delta$ is an isomorphism from $\mathcal{Z}$ to $L^2(\Omega_T)$ and (2.1), we conclude the uniqueness of $y \in W_0^{1,p}(\Omega)$.

Finally, independence of y with respect to p follows from the existence of a solution y in $L^2(I, W_0^{1,p}(\Omega))$ for every $\hat{p}' < p < \frac{n}{n-1}$ and its uniqueness in $L^2(\Omega_T)$, since $W_0^{1,p_1}(\Omega) \subset W_0^{1,p_2}(\Omega)$ for $p_1 > p_2$. □

Remark 2.3.

(i) The solution to (1.1) belongs to $L^2(I, W_0^{1,p}(\Omega))$ for every $\hat{p} \leqslant p < \frac{n}{n-1}$, and from the equation (1.1) we know that $\partial_t y \in L^2(I, W^{-1,p}(\Omega))$. Observe that $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ for $p \geqslant p_0 := \max\{\hat{p}', \frac{2n}{n+2}\}$, with $\hat{p}$ as in the proof of Theorem 2.2, and hence $y \in L^2(\Omega_T)$. As a consequence, we deduce that $y \in C(I, L^2(\Omega))$; see [Showalter 1997, Proposition III.1.2].

(ii) Under our regularity conditions, an equivalent definition for the solution to equation (1.1) is the following. A function $y \in L^2(I, W_0^{1,p}(\Omega))$ with $p_0 < p < \frac{n}{n-1}$ is called a solution to (1.1) if

$$\begin{aligned} -\int_0^T \langle y(t), \partial_t z(t)\rangle_{W_0^{1,p}, W^{-1,p'}}\,dt + \int_{\Omega_T} \nabla y \nabla z\,dx\,dt \\ = \int_0^T \langle u(t), z(t)\rangle\,dt + \int_\Omega y_0(x)z(x,0)\,dx \end{aligned}$$

for all $z \in L^2(I, W_0^{1,p'}(\Omega))$ such that $\partial_t z \in L^2(I, W^{-1,p'}(\Omega))$ (which implies $z(\cdot, 0) \in L^2(\Omega)$; see (i)) and $z(T) = 0$. This follows from (2.1) and the density of $\mathcal{Z}$ in this new space of test functions. Theorem 2.2 remains valid with this definition if we only assume for Ω to have a Lipschitz boundary. This is the regularity of Ω required to have the maximal parabolic regularity; see [Haller-Dintelmann and Rehberg 2009]. We have chosen the above definition because it is more convenient for the numerical analysis to be developed later in this paper.

(iii) The preceding theorem as well as the rest of the results given in this paper are valid if we replace the heat operator in (1.1) by a more general parabolic operator $\partial_t + A$ that enjoys maximal parabolic regularity.

We finish this section by proving a continuity result of the states with respect to the controls.

Theorem 2.4. *Let* $\{u_k\}_{k=1}^{\infty} \subset L^2(I, \mathcal{M}(\Omega))$ *be a sequence such that* $u_k \stackrel{*}{\rightharpoonup} u$ *in* $L^2(I, \mathcal{M}(\Omega))$*. If* y_k *and* y *denote the states associated to* u_k *and* u*, respectively, then* $\|y_k - y\|_{L^2(\Omega_T)} \to 0$*.*

Proof. For every k, let $z_k \in \mathcal{Z}$ satisfy

$$\begin{cases} -\partial_t z_k - \Delta z_k &= y - y_k \quad \text{in } \Omega_T, \\ z_k &= 0 \quad \text{on } \Sigma_T, \\ z_k(x, T) &= 0 \quad \text{in } \Omega. \end{cases}$$

Then, from Definition 2.1 and using the boundedness of $\{u_k\}_{k=1}^{\infty}$ in $L^2(I, \mathcal{M}(\Omega))$, we have

$$\begin{aligned} \|y - y_k\|_{L^2(\Omega_T)}^2 &= \int_{\Omega_T} (y - y_k)(-\partial_t z_k - \Delta z_k)\, dx\, dt = \int_0^T \langle u(t) - u_k(t), z_k(t) \rangle\, dt \\ &\leqslant \|u - u_k\|_{L^2(\mathcal{M})} \|z_k\|_{L^2(C_0)} \leqslant C \|z_k\|_{L^2(C_0)}. \end{aligned}$$

From Theorem 2.2, we know that $y_k \rightharpoonup y$ in $L^2(\Omega_T)$, therefore $z_k \rightharpoonup 0$ in $H^{2,1}(\Omega_T)$. Since the embedding $H^{2,1}(\Omega_T) \subset L^2(I, C_0(\Omega))$ is compact, we get that $\|z_k\|_{L^2(C_0)} \to 0$. This convergence and the above inequality conclude the proof. □

3 ANALYSIS OF THE CONTROL PROBLEM

In this section we establish existence of an optimal control and derive the optimality conditions.

Proposition 3.1. *The control problem* (P) *has a unique solution* $\bar{u}$*.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be a minimizing sequence, which is thus bounded in the space $L^2(I, \mathcal{M}(\Omega))$. Since the predual $L^2(I, C_0(\Omega))$ is separable, there exists a subsequence, denoted in the same way, converging weakly-$\star$ to some $\bar{u} \in L^2(I, \mathcal{M}(\Omega))$. From Theorem 2.4 we get that $y(u_k) \to y(\bar{u})$ strongly in $L^2(\Omega_T)$. Hence, the weakly-$\star$ lower semicontinuity of the norm $\|\cdot\|_{L^2(\mathcal{M})}$ implies that $\bar{u}$ is a solution. The uniqueness is a consequence of the strict convexity of J, which follows from the injectivity of the control-to-state mapping. □

Hereafter $\bar{u}$ will denote the solution to (P) and $\bar{y}$ the associated state. Now, we give the first order optimality conditions, which are necessary and sufficient due to the convexity of (P).

Theorem 3.2. *There exists a unique element* $\bar\varphi \in H^{2,1}(\Omega_T)$ *satisfying*

$$
\begin{cases} -\partial_t\bar\varphi - \Delta\bar\varphi &= \bar y - y_d \quad \text{in } \Omega_T,\\ \bar\varphi &= 0 \quad \text{on } \Sigma_T,\\ \bar\varphi(x,T) &= 0 \quad \text{in } \Omega, \end{cases} \tag{3.1}
$$

such that

$$
\int_0^T \langle \bar u(t), \bar\varphi(t)\rangle \, dt + \alpha\|\bar u\|_{L^2(\mathcal{M})} = 0, \tag{3.2}
$$

$$
\|\bar\varphi\|_{L^2(C_0)} \begin{cases} = \alpha & \text{if } \bar u \neq 0,\\ \leqslant \alpha & \text{if } \bar u = 0. \end{cases} \tag{3.3}
$$

Proof. Let us introduce $j(u) = \|u\|_{L^2(\mathcal{M})}$ and $F(u) = \frac{1}{2}\|y(u) - y_d\|^2_{L^2(\Omega_T)}$, so that $J(u) = F(u) + \alpha j(u)$. By the differentiability of F and the convexity of j we obtain

$$
F'(\bar u)(u - \bar u) + \alpha j(u) - \alpha j(\bar u) \geqslant 0 \quad \forall u \in L^2(I, \mathcal{M}(\Omega)),
$$

and hence

$$
\int_{\Omega_T} (\bar y - y_d)(y(u) - \bar y)\, dx\, dt + \alpha j(u) - \alpha j(\bar u) \geqslant 0.
$$

Utilizing the adjoint equation (3.1) and the state equation (2.1), we deduce from the above inequality

$$
\int_0^T \langle u(t) - \bar u(t), \bar\varphi(t)\rangle \, dt + \alpha j(u) - \alpha j(\bar u) \geqslant 0 \quad \forall u \in L^2(I, \mathcal{M}(\Omega)). \tag{3.4}
$$

Taking $u = 2\bar u$ and $u = \frac{1}{2}\bar u$, respectively, in (3.4) we obtain (3.2). On the other hand, setting $u = \bar u - v$ in (3.4), it follows that

$$
\int_0^T \langle v(t), \bar\varphi(t)\rangle \, dt \leqslant \alpha(j(\bar u - v) - j(\bar u)) \leqslant \alpha\|v\|_{L^2(\mathcal{M})} \quad \forall v \in L^2(I, \mathcal{M}(\Omega)). \tag{3.5}
$$

By the duality $L^2(I, \mathcal{M}(\Omega)) = L^2(I, C_0(\Omega))^*$ we have that

$$
\|\bar\varphi\|_{L^2(C_0)} = \max_{\|v\|_{L^2(\mathcal{M})} \leqslant 1} \int_0^T \langle v(t), \bar\varphi(t)\rangle \, dt \leqslant \alpha. \tag{3.6}
$$

Then, (3.3) is an immediate consequence of (3.2) and (3.6). $\square$

From now on, we will assume that the optimal control $\bar u \neq 0$. By using (3.2) and (3.3) we can prove some sparsity property for $\bar u$. Let us consider the Jordan decomposition $\bar u(t) = \bar u^+(t) - \bar u^-(t)$ for almost every $t \in I$. Then we have the following theorem.

Theorem 3.3. *For almost every* $t \in I$ *the following embeddings hold*

$$\mathit{Supp}(\bar{u}^+(t)) \subset \{x \in \Omega : \bar{\varphi}(x,t) = -\|\bar{\varphi}(t)\|_\infty\}, \tag{3.7}$$

$$\mathit{Supp}(\bar{u}^-(t)) \subset \{x \in \Omega : \bar{\varphi}(x,t) = +\|\bar{\varphi}(t)\|_\infty\}. \tag{3.8}$$

Proof. Since $\bar{\varphi} : I \times \bar{\Omega} \to \mathbb{R}$ is a Caratheodory function, there exists a measurable selection $t \in I \mapsto x_t \in \bar{\Omega}$ such that $\bar{\varphi}(x_t, t) = \|\bar{\varphi}(t)\|_\infty$; see [Ekeland and Témam 1999, Chapter 8, Theorem 1.2]. Now, we define the element $v \in L^2(I, \mathcal{M}(\Omega))$ by $v(t) = \mathrm{sign}(\bar{\varphi}(x_t))\|u(t)\|_{\mathcal{M}}\delta_{x_t}$. We have to check that $v : I \to \mathcal{M}(\Omega)$ is weakly measurable. To this end the only delicate point is the weak measurability of $t \in I \mapsto \delta_{x_t} \in \mathcal{M}(\Omega)$. This follows from the measurability of the mapping $t \mapsto x_t$ and the continuity of $x \in \bar{\Omega} \mapsto \delta_x \in \mathcal{M}(\Omega)$ when $\mathcal{M}(\Omega)$ is endowed with the weak-$\star$ topology. By definition of v we get

$$\langle v(t), \bar{\varphi}(t)\rangle = \|\bar{u}(t)\|_{\mathcal{M}}\|\bar{\varphi}(t)\|_\infty \geqslant -\langle \bar{u}(t), \bar{\varphi}(t)\rangle \tag{3.9}$$

and

$$\begin{aligned} \|v\|_{L^2(\mathcal{M})} &= \left(\int_0^T \|\bar{u}(t)\|_{\mathcal{M}}^2 \|\delta_{x_t}\|_{\mathcal{M}}^2 \, dt\right)^{1/2} = \left(\int_0^T \|\bar{u}(t)\|_{\mathcal{M}}^2 \, dt\right)^{1/2} \\ &= \|\bar{u}\|_{L^2(\mathcal{M})}. \end{aligned} \tag{3.10}$$

From (3.2), (3.9), (3.5) and (3.10) we obtain

$$\alpha\|\bar{u}\|_{L^2(\mathcal{M})} = -\int_0^T \langle \bar{u}(t), \bar{\varphi}(t)\rangle \, dt \leqslant \int_0^T \langle v(t), \bar{\varphi}(t)\rangle \, dt \leqslant \alpha\|v\|_{L^2(\mathcal{M})} = \alpha\|\bar{u}\|_{L^2(\mathcal{M})}.$$

As a consequence of these inequalities and (3.9) we conclude that

$$\|\bar{u}(t)\|_{\mathcal{M}}\|\bar{\varphi}(t)\|_\infty = -\langle \bar{u}(t), \bar{\varphi}(t)\rangle \quad \text{for a. e. } t \in I. \tag{3.11}$$

Finally, (3.7) and (3.8) follow from (3.11) and Lemma 3.4 below applied to $\mu = -\bar{u}(t)$. □

Lemma 3.4. *Let* $\mu \in \mathcal{M}(\Omega)$ *and* $z \in C_0(\Omega)$*, both of them not zero, be such that*

$$\langle \mu, z\rangle = \|\mu\|_{\mathcal{M}}\|z\|_\infty, \tag{3.12}$$

and let $\mu = \mu^+ - \mu^-$ *be the Jordan decomposition of* μ*. Then we have*

$$\mathit{Supp}(\mu^+) \subset \Omega_+ = \{x \in \Omega : z(x) = +\|z\|_\infty\}, \tag{3.13}$$

$$\mathit{Supp}(\mu^-) \subset \Omega_- = \{x \in \Omega : z(x) = -\|z\|_\infty\}. \tag{3.14}$$

Proof. We will prove (3.13), the proof of (3.14) being analogous. First we observe that due to (3.12) we obtain for all measures $\nu \in \mathcal{M}(\Omega)$ with $\|\nu\|_{\mathcal{M}} \leqslant \|\mu\|_{\mathcal{M}}$ that

$$\langle \nu, z\rangle \leqslant \|\nu\|_{\mathcal{M}}\|z\|_\infty \leqslant \|\mu\|_{\mathcal{M}}\|z\|_\infty = \langle \mu, z\rangle. \tag{3.15}$$

We have as well that

$$\langle \mu, z\rangle = \langle \mu^+, z^+\rangle + \langle \mu^-, z^-\rangle - \langle \mu^+, z^-\rangle - \langle \mu^-, z^+\rangle \leqslant \langle \mu^+, z^+\rangle + \langle \mu^-, z^-\rangle.$$

Moreover, the inequality is strict unless μ^+ and μ^- are concentrated at the set of points $x \in \Omega$ where $z(x) \geqslant 0$ and $z(x) \leqslant 0$, respectively. Let us define the sets

$$A_+ = \{x \in \Omega : z(x) \geqslant 0\} \text{ and } A_- = \{x \in \Omega : z(x) \leqslant 0\}$$

and the measures $\nu^+ = \mu^+ \, |_{A_+}$, $\nu^- = \mu^- \, |_{A_-}$ and $\nu = \nu^+ - \nu^-$. Then we have that $\|\nu\|_{\mathcal{M}} \leqslant \|\mu\|_{\mathcal{M}}$ and $\langle \nu, z\rangle > \langle \mu, z\rangle$ if $\mathrm{Supp}(\mu^+) \not\subset A_+$ or $\mathrm{Supp}(\mu^-) \not\subset A_-$. Because of (3.15) we conclude that $\mathrm{Supp}(\mu^+) \subset A_+$ and $\mathrm{Supp}(\mu^-) \subset A_-$. Now we distinguish two cases in the proof of (3.13) depending on whether the norm bound is attained from above.

Case 1: $\max_{x\in\bar\Omega} z(x) < \|z\|_\infty$. In this case we prove that $\mu^+ = 0$. Indeed, let $x_0 \in \Omega$ such that $z(x_0) = -\|z\|_\infty$ and define $\nu = -\mu^+(\Omega)\delta_{x_0} - \mu^-$. Then it is obvious that $\|\nu\|_{\mathcal{M}} = \|\mu\|_{\mathcal{M}}$. If $\mu^+ \neq 0$, since the support of μ^+ is in A_+ and $\max_{x\in\bar\Omega} z(x) < \|z\|_\infty$, we have that

$$\langle \nu, z\rangle = \|z\|_\infty \mu^+(\Omega) - \langle \mu^-, z\rangle > \langle \mu^+, z\rangle - \langle \mu^-, z\rangle = \langle \mu, z\rangle,$$

which contradicts (3.15). Then, (3.13) holds.

Case 2: $\max_{x\in\bar\Omega} z(x) = \|z\|_\infty$. Let $x_0 \in \Omega$ be such that $z(x_0) = \|z\|_\infty$. We argue by contradiction and assume that $\mu^+(S) > 0$ where

$$S = \{x \in \Omega : 0 \leqslant z(x) < \|z\|_\infty\}.$$

We take $\nu = \mu^+(\Omega)\delta_{x_0} - \mu^-$ and once again

$$\|\nu\|_{\mathcal{M}} = \|\mu\|_{\mathcal{M}} \text{ and } \langle \nu, z\rangle = \mu^+(\Omega)\|z\|_\infty - \langle \mu^-, z\rangle > \langle \mu, z\rangle,$$

since $\mu^+(S) > 0$. Again this contradicts (3.15). Therefore, $\mu^+(S) = 0$ and hence (3.13) follows from the inclusion $\mathrm{Supp}(\mu^+) \subset A_+$. □

Corollary 3.5. *There exists $\bar\alpha > 0$ such that $\bar u = 0$ for every $\alpha > \bar\alpha$.*

Proof. Let us denote by J_α the cost functional associated to the parameter α. Similarly, let $(u_\alpha, y_\alpha, \varphi_\alpha)$ denote the solution to the corresponding optimality system. For each $\alpha > 0$ we have the inequalities

$$\frac{1}{2}\|y_\alpha - y_d\|^2_{L^2(\Omega_T)} \leqslant J_\alpha(u_\alpha) \leqslant J_\alpha(0) = \frac{1}{2}\|\hat y_0 - y_d\|^2_{L^2(\Omega_T)},$$

where $\hat{y}_0$ denotes the uncontrolled state, i.e., the solution to (1.1) with $u = 0$. Consequently, $\|y_\alpha - y_d\|_{L^2(\Omega_T)} \leqslant \|\hat{y}_0 - y_d\|_{L^2(\Omega_T)}$ holds for every $\alpha > 0$. From the adjoint state equation (3.1) and the embedding of $H^{2,1}(\Omega_T) \hookrightarrow L^2(I, C(\bar{\Omega}))$, we deduce the existence of a constant $C > 0$ such that

$$\|\varphi_\alpha\|_{L^2(C_0)} \leqslant C' \|\bar{\varphi}\|_{H^{2,1}} \leqslant C\|y_\alpha - y_d\|_{L^2(\Omega_T)} \leqslant C\|\hat{y}_0 - y_d\|_{L^2(\Omega_T)}.$$

Setting $\bar{\alpha} = C\|\hat{y}_0 - y_d\|_{L^2(\Omega_T)}$, we obtain from the above inequality and (3.3) that $u_\alpha = 0$ for every $\alpha > \bar{\alpha}$. □

4 APPROXIMATION OF THE CONTROL PROBLEM

We consider a dG(0)cG(1) discontinuous Galerkin approximation of the state equation (1.1) (i.e., piecewise constant in time and linear nodal basis finite elements in space; see, e.g., [Thomée 2006]). Associated with a parameter h we consider a family of triangulations $\{\mathcal{K}_h\}_{h>0}$ of $\bar{\Omega}$. To every element $K \in \mathcal{K}_h$ we assign two parameters $\rho(K)$ and $\vartheta(K)$, where $\rho(K)$ denotes the diameter of K and $\vartheta(K)$ is the diameter of the biggest ball contained in K. The size of the grid is given by $h = \max_{K\in\mathcal{K}_h} \rho(K)$. We will denote by $\{x_j\}_{j=1}^{N_h}$ the interior nodes of the triangulation $\mathcal{K}_h$. In this section Ω will be assumed to be convex. In addition, the following usual regularity assumptions on the triangulation are assumed.

(i) There exist two positive constants ρ_Ω and ϑ_Ω such that

$$\frac{h}{\rho(K)} \leqslant \rho_\Omega \quad \text{and} \quad \frac{\rho(K)}{\vartheta(K)} \leqslant \vartheta_\Omega$$

hold for every $K \in \mathcal{K}_h$ and all $h > 0$.

(ii) Let us set $\overline{\Omega}_h = \bigcup_{K\in\mathcal{K}_h} K$ with Ω_h and Γ_h being its interior and boundary, respectively. We assume that the vertices of $\mathcal{K}_h$ placed on the boundary Γ_h are also points of Γ and there exists a constant $C_\Gamma > 0$ such that $\operatorname{dist}(x, \Gamma) \leqslant C_\Gamma h^2$ for every $x \in \Gamma_h$. This always holds if Γ is a C^2 boundary. In the case of polygonal or polyhedral domains, it is reasonable to assume that the triangulation satisfies that $\Gamma_h = \Gamma$. From this assumption we know [Raviart and Thomas 1983, section 5.2] that

$$|\Omega \setminus \Omega_h| \leqslant Ch^2, \tag{4.1}$$

where $|\cdot|$ denotes the Lebesgue measure.

We also introduce a temporal grid $0 = t_0 < t_1 < \ldots < t_{N_\tau} = T$ with $\tau_k = t_k - t_{k-1}$ and set $\tau = \max_{1\leqslant k\leqslant N_\tau} \tau_k$. We assume that there exist $\rho_T > 0$, $C_{\Omega,T} > 0$ and $c_{\Omega,T} > 0$ independent of h and τ such that

$$\tau \leqslant \rho_T \tau_k, \text{ for } 1 \leqslant k \leqslant N_\tau \quad \text{and} \quad c_{\Omega,T} h^{\max\{n,2\}} \leqslant \tau \leqslant C_{\Omega,T} h^{\max\{n,2\}}. \tag{4.2}$$

We will use the notation $\sigma = (\tau, h)$ and $\Omega_{hT} = \Omega_h \times (0, T)$.

We first discuss the spatial discretization, which follows [Casas, Clason, et al. 2012]. Associated to the interior nodes $\{x_j\}_{j=1}^{N_h}$ of $\mathcal{K}_h$ we consider the spaces

$$U_h = \left\{ u_h \in \mathcal{M}(\Omega) : u_h = \sum_{j=1}^{N_h} u_j \delta_{x_j},\ \text{where } \{u_j\}_{j=1}^{N_h} \subset \mathbb{R} \right\}$$

and

$$Y_h = \left\{ y_h \in C_0(\Omega) : y_h = \sum_{j=1}^{N_h} y_j e_j,\ \text{where } \{y_j\}_{j=1}^{N_h} \subset \mathbb{R} \right\},$$

where $\{e_j\}_{j=1}^{N_h}$ is the nodal basis formed by the continuous piecewise linear functions such that $e_j(x_i) = \delta_{ij}$ for every $1 \leqslant i, j \leqslant N_h$. Such functions attain their maximum and minimum at one of the nodes, and thus for all $y_h \in Y_h$,

$$\|y_h\|_\infty = \max_{1 \leqslant j \leqslant N_h} |y_j| = |\vec{y}_h|_\infty,$$

where we have identified y_h with the vector $\vec{y}_h = (y_1, \dots, y_{N_h})^T \in \mathbb{R}^{N_h}$ of its expansion coefficients, and $|\cdot|_p$ denotes the usual p-norm in $\mathbb{R}^{N_h}$. Similarly, we have for all $u_h \in U_h$ that

$$\|u_h\|_{\mathcal{M}} = \sup_{\|v\|_\infty = 1} \sum_{j=1}^{N_h} u_j \langle \delta_{x_j}, v \rangle = \sum_{j=1}^{N_h} |u_j| = |\vec{u}_h|_1 \quad \text{for all } u_h \in U_h.$$

Hence endowed with these norms, U_h is the topological dual of Y_h with respect to the duality pairing

$$\langle u_h, y_h \rangle = \sum_{j=1}^{N_h} u_j y_j = \vec{u}_h^T \vec{y}_h.$$

For every σ we define the space of discrete controls and states by

$$\mathcal{U}_\sigma = \{u_\sigma \in L^2(I, U_h) : u_\sigma|_{I_k} \in U_h,\ 1 \leqslant k \leqslant N_\tau\}$$

and

$$\mathcal{Y}_\sigma = \{y_\sigma \in L^2(I, Y_h) : y_\sigma|_{I_k} \in Y_h,\ 1 \leqslant k \leqslant N_\tau\},$$

where $I_k = (t_{k-1}, t_k]$. The elements $u_\sigma \in \mathcal{U}_\sigma$ and $y_\sigma \in \mathcal{Y}_\sigma$ can be represented in the form

$$u_\sigma = \sum_{k=1}^{N_\tau} u_{k,h} \chi_k \quad \text{and} \quad y_\sigma = \sum_{k=1}^{N_\tau} y_{k,h} \chi_k,$$

where χ_k is the indicator function of I_k, $u_{k,h} \in U_h$ and $y_{k,h} \in Y_h$. Moreover, by definition of U_h and Y_h, we can write

$$u_\sigma = \sum_{k=1}^{N_\tau}\sum_{j=1}^{N_h} u_{kj}\chi_k\delta_{x_j} \text{ and } y_\sigma = \sum_{k=1}^{N_\tau}\sum_{j=1}^{N_h} y_{kj}\chi_k e_j.$$

Thus $\mathcal{U}_\sigma$ and $\mathcal{Y}_\sigma$ are finite dimensional spaces of dimension $N_\tau \times N_h$, and bases are given by $\{\chi_k\delta_{x_j}\}_{k,j}$ and $\{\chi_k e_j\}_{k,j}$. Identifying again u_σ with the vector $\vec{u}_\sigma$ of expansion coefficients u_{kj}, we have for all $u_\sigma \in \mathcal{U}_\sigma$ that

$$\begin{aligned}\|u_\sigma\|^2_{L^2(\mathcal{M})} &= \int_0^T \Big\| \sum_{k=1}^{N_\tau}\sum_{j=1}^{N_h} u_{kj}\chi_k\delta_{x_j}\Big\|^2_{\mathcal{M}} dt = \sum_{k=1}^{N_\tau}\int_{I_k} \Big\|\sum_{j=1}^{N_h} u_{kj}\delta_{x_j}\Big\|^2_{\mathcal{M}} dt \\ &= \sum_{k=1}^{N_\tau}\tau_k\left(\sum_{j=1}^{N_h}|u_{kj}|\right)^2 = \sum_{k=1}^{N_\tau}\tau_k|\vec{u}_k|_1^2\end{aligned}$$

for $\vec{u}_k = (u_{k1}, \dots, u_{kN_h})^T$, and similarly for all $y_\sigma \in \mathcal{Y}_\sigma$ that

$$\|y_\sigma\|^2_{L^2(C_0)} = \sum_{k=1}^{N_\tau}\tau_k\left(\max_{1\leqslant j\leqslant N_h}|y_{kj}|\right)^2 = \sum_{k=1}^{N_\tau}\tau_k|\vec{y}_k|_\infty^2.$$

It is thus straightforward to verify that endowed with these norms, $\mathcal{U}_\sigma$ is the topological dual of $\mathcal{Y}_\sigma$ with respect to the duality pairing

$$\langle u_\sigma, y_\sigma\rangle = \sum_{k=1}^{N_\tau}\tau_k\sum_{j=1}^{N_h}u_{kj}y_{kj} = \sum_{k=1}^{N_\tau}\tau_k(\vec{u}_k^T\vec{y}_k). \tag{4.3}$$

Next we define the linear operators $\Lambda_h : \mathcal{M}(\Omega) \to U_h \subset \mathcal{M}(\Omega)$ and $\Pi_h : C_0(\Omega) \to Y_h \subset C_0(\Omega)$ by

$$\Lambda_h u = \sum_{j=1}^{N_h}\langle u, e_j\rangle\delta_{x_j} \quad \text{and} \quad \Pi_h y = \sum_{j=1}^{N_h} y(x_j)e_j.$$

The operator Π_h is the nodal interpolation operator for Y_h. Concerning the operator Λ_h we have the following result.

Theorem 4.1 ([Casas, Clason, et al. 2012, Theorem 3.1]). *The following properties hold.*

(i) For every $u \in \mathcal{M}(\Omega)$ and every $y \in C_0(\Omega)$ and $y_h \in Y_h$ we have

$$\begin{aligned}\langle u, y_h\rangle &= \langle\Lambda_h u, y_h\rangle,\\ \langle u, \Pi_h y\rangle &= \langle\Lambda_h u, y\rangle.\end{aligned}$$

(ii) For every $u \in \mathcal{M}(\Omega)$ *we have*

$$\|\Lambda_h u\|_{\mathcal{M}} \leqslant \|u\|_{\mathcal{M}},$$
$$\Lambda_h u \overset{*}{\rightharpoonup} u \textit{ in } \mathcal{M}(\Omega) \textit{ and } \|\Lambda_h u\|_{\mathcal{M}} \to \|u\|_{\mathcal{M}} \textit{ as } h \to 0.$$

(iii) There exists a constant $C > 0$ *such that for every* $u \in \mathcal{M}(\Omega)$ *we have*

$$\|u - \Lambda_h u\|_{W^{-1,p}(\Omega)} \leqslant Ch^{1-n/p'}\|u\|_{\mathcal{M}}, \quad 1 < p < \frac{n}{n-1},$$
$$\|u - \Lambda_h u\|_{(W_0^{1,\infty}(\Omega))^*} \leqslant Ch\|u\|_{\mathcal{M}},$$

with $1/p' + 1/p = 1$.

Similarly to Λ_h and Π_h we define the linear operators

$$\Phi_\sigma : L^2(I, \mathcal{M}(\Omega)) \to \mathcal{U}_\sigma \subset L^2(I, \mathcal{M}(\Omega))$$

and

$$\Psi_\sigma : L^2(I, C_0(\Omega)) \to \mathcal{Y}_\sigma \subset L^2(I, C_0(\Omega))$$

by

$$\Phi_\sigma u = \sum_{k=1}^{N_\tau} \frac{1}{\tau_k} \int_{I_k} \Lambda_h(u(t))\, dt\, \chi_k = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} \frac{1}{\tau_k} \int_{I_k} \langle u(t), e_j \rangle\, dt\, \chi_k \delta_{x_j},$$

$$\Psi_\sigma y = \sum_{k=1}^{N_\tau} \frac{1}{\tau_k} \int_{I_k} \Pi_h(y(t))\, dt\, \chi_k = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} \frac{1}{\tau_k} \int_{I_k} y(x_j, t)\, dt\, \chi_k e_j.$$

Analogously to Theorem 4.1 we obtain the following result concerning Φ_σ and Ψ_σ.

Theorem 4.2. *The following properties hold.*

(i) For every $u_\sigma \in \mathcal{U}_\sigma$ *and every* $y_\sigma \in \mathcal{Y}_\sigma$ *we have*

$$\Phi_\sigma u_\sigma = u_\sigma \textit{ and } \Psi_\sigma y_\sigma = y_\sigma. \tag{4.4}$$

(ii) For every $u \in L^2(I, \mathcal{M}(\Omega))$ *and every* $y \in L^2(I, C_0(\Omega))$ *and* $y_\sigma \in \mathcal{Y}_\sigma$ *we have*

$$\langle u, y_\sigma \rangle = \langle \Phi_\sigma u, y_\sigma \rangle, \tag{4.5}$$
$$\langle u, \Psi_\sigma y \rangle = \langle \Phi_\sigma u, y \rangle. \tag{4.6}$$

(iii) For every $u \in L^2(I, \mathcal{M}(\Omega))$ *and* $y \in L^2(I, C_0(\Omega))$ *we have*

$$\|\Phi_\sigma u\|_{L^2(\mathcal{M})} \leqslant \|u\|_{L^2(\mathcal{M})}, \tag{4.7}$$
$$\|\Psi_\sigma y\|_{L^2(C_0)} \leqslant \|y\|_{L^2(C_0)}. \tag{4.8}$$

(iv) *For every* $u \in L^2(I, \mathcal{M}(\Omega))$ *and* $y \in L^2(I, C_0(\Omega))$ *we have*

$$\Phi_\sigma u \xrightharpoonup{*} u \text{ in } L^2(I, \mathcal{M}(\Omega)) \text{ and } \|\Phi_\sigma u\|_{L^2(\mathcal{M})} \to \|u\|_{L^2(\mathcal{M})}, \tag{4.9}$$

$$\Psi_\sigma y \to y \text{ in } L^2(I, C_0(\Omega)). \tag{4.10}$$

Proof. The formulas of (4.4) follow from the linearity of the operators and the identities $\Phi_\sigma(\chi_l \delta_{x_i}) = \chi_l \delta_{x_i}$ and $\Psi_\sigma(\chi_l e_i) = \chi_l e_i$ for all $1 \leqslant l \leqslant N_\tau$ and $1 \leqslant i \leqslant N_h$.

Identity (4.5) is a consequence of (4.4) and (4.6). Let us prove the latter. First we observe that

$$\Phi_\sigma u = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} u_{kj} \chi_k \delta_{x_j}, \quad \text{with } u_{kj} = \frac{1}{\tau_k} \int_{I_k} \langle u(t), e_j \rangle \, dt, \tag{4.11}$$

$$\Psi_\sigma y = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} y_{kj} \chi_k e_j, \quad \text{with } y_{kj} = \frac{1}{\tau_k} \int_{I_k} y(x_j, t) \, dt. \tag{4.12}$$

From (4.11) and (4.12) we have

$$\begin{aligned} \langle \Phi_\sigma u, y \rangle &= \int_0^T \langle (\Phi_\sigma u)(t), y(t) \rangle \, dt = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} u_{kj} \int_0^T \langle \chi_k \delta_{x_j}, y(t) \rangle \, dt \\ &= \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} u_{kj} \int_{I_k} y(x_j, t) \, dt = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} \tau_k u_{kj} y_{kj}. \end{aligned}$$

Analogously we get

$$\begin{aligned} \langle u, \Psi_\sigma y \rangle &= \int_0^T \langle u(t), (\Psi_\sigma y)(t) \rangle \, dt = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} y_{kj} \int_0^T \langle u(t), \chi_k e_j \rangle \, dt \\ &= \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} y_{kj} \int_{I_k} \langle u(t), e_j \rangle \, dt = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} \tau_k u_{kj} y_{kj}, \end{aligned}$$

as desired.

We turn to (4.7). First we recall that the norm of $\Phi_\sigma u$ is given by

$$\|\Phi_\sigma u\|_{L^2(\mathcal{M})} = \left(\sum_{k=1}^{N_\tau} \tau_k \left(\sum_{j=1}^{N_h} |u_{kj}| \right)^2 \right)^{1/2}.$$

Next we define $y_\sigma \in \mathcal{Y}_\sigma$ by

$$y_{kj} = \left(\sum_{i=1}^{N_h} |u_{ki}| \right) \operatorname{sign}(u_{kj}),$$

where we set $\operatorname{sign}(0) = 0$. For y_σ we compute the expressions

$$\langle u, y_\sigma\rangle = \int_0^T \langle u(t), y_\sigma(t)\rangle\,dt = \sum_{k=1}^{N_\tau}\int_{I_k}\sum_{j=1}^{N_h} y_{kj}\langle u(t), e_j\rangle\,dt$$
$$= \sum_{k=1}^{N_\tau}\tau_k\sum_{j=1}^{N_h} y_{kj}u_{kj} = \sum_{k=1}^{N_\tau}\tau_k\left(\sum_{j=1}^{N_h}|u_{kj}|\right)^2 = \|\Phi_\sigma u\|^2_{L^2(\mathcal{M})} \tag{4.13}$$

and

$$\|y_\sigma\|_{L^2(C_0)} = \left(\int_0^T \|y_\sigma(t)\|^2_\infty\,dt\right)^{1/2} = \left(\sum_{k=1}^{N_\tau}\int_{I_k}\Big\|\sum_{j=1}^{N_h} y_{kj}e_j\Big\|^2_\infty\,dt\right)^{1/2}$$
$$= \left(\sum_{k=1}^{N_\tau}\tau_k\left(\sum_{j=1}^{N_h}|u_{kj}|\right)^2\right)^{1/2} = \|\Phi_\sigma u\|_{L^2(\mathcal{M})}. \tag{4.14}$$

From (4.13) and (4.14) we deduce

$$\|\Phi_\sigma u\|^2_{L^2(\mathcal{M})} = \langle u, y_\sigma\rangle \leqslant \|u\|_{L^2(\mathcal{M})}\|y_\sigma\|_{L^2(C_0)} = \|u\|_{L^2(\mathcal{M})}\|\Phi_\sigma u\|_{L^2(\mathcal{M})},$$

which implies (4.7).

To establish (4.8) we choose $y \in L^2(I, C_0(\Omega))$ and estimate

$$\|\Psi_\sigma y\|_{L^2(C_0)} = \left(\sum_{k=1}^{N_\tau}\int_{I_k}\|(\Psi_\sigma y)(t)\|^2_\infty\,dt\right)^{1/2}$$
$$= \left(\sum_{k=1}^{N_\tau}\frac{1}{\tau_k}\Big\|\sum_{j=1}^{N_h}\left(\int_{I_k} y(x_j,t)\,dt\right)e_j\Big\|^2_\infty\right)^{1/2}$$
$$\leqslant \left(\sum_{k=1}^{N_\tau}\frac{1}{\tau_k}\left(\int_{I_k}\|y(t)\|_\infty\,dt\right)^2\right)^{1/2} \leqslant \left(\sum_{k=1}^{N_\tau}\int_{I_k}\|y(t)\|^2_\infty\,dt\right)^{1/2}$$
$$= \|y\|_{L^2(C_0)}.$$

Before proving (4.9), we will consider (4.10). It is well known that (4.10) holds for functions in $C^\infty(\bar\Omega_T)$ vanishing on Σ_T. From the density of these functions in $L^2(I, C_0(\Omega))$ and from inequality (4.8) we deduce (4.10).

Finally, we prove (4.9). From (4.7) we know that $\{\Phi_\sigma u\}_\sigma$ is bounded in the space $L^2(I, \mathcal{M}(\Omega))$. Then, there exists a subsequence, denoted in the same way, and an element $\tilde u \in L^2(I, \mathcal{M}(\Omega))$ such that $\Phi_\sigma u \stackrel{*}{\rightharpoonup} \tilde u$ in $L^2(I, \mathcal{M}(\Omega))$. Then, for every $y \in L^2(I, C_0(\Omega))$ it holds that

$$\lim_{\sigma\to 0}\int_0^T \langle(\Phi_\sigma u)(t), y(t)\rangle\,dt = \int_0^T \langle\tilde u(t), y(t)\rangle\,dt.$$

Using (4.6) and (4.10) we find

$$\lim_{\sigma\to 0}\int_0^T \langle(\Phi_\sigma u)(t), y(t)\rangle\,dt = \lim_{\sigma\to 0}\int_0^T \langle u(t), (\Psi_\sigma y)(t)\rangle\,dt = \int_0^T \langle u(t), y(t)\rangle\,dt.$$

Combining these two equalities we have that

$$\int_0^T \langle \tilde{u}(t), y(t)\rangle\,dt = \int_0^T \langle u(t), y(t)\rangle\,dt \quad \forall y \in L^2(I, C_0(\Omega)),$$

therefore $u = \tilde{u}$ and the whole sequence $\{\Phi_\sigma u\}_\sigma$ converges weakly-$\star$ to u.

By the convergence $\Phi_\sigma u \stackrel{*}{\rightharpoonup} u$ and (4.7) we obtain

$$\|u\|_{L^2(\mathcal{M})} \leqslant \liminf_{\sigma\to 0}\|\Phi_\sigma u\|_{L^2(\mathcal{M})} \leqslant \limsup_{\sigma\to 0}\|\Phi_\sigma u\|_{L^2(\mathcal{M})} \leqslant \|u\|_{L^2(\mathcal{M})},$$

which concludes the proof of (4.9). □

We finish this section by proving the following approximation result.

Theorem 4.3. *Let y and y^σ be the solutions to (1.1) corresponding to u and $\Phi_\sigma u$, respectively. Then there exists a constant $C > 0$ independent of u and σ such that*

$$\|y - y^\sigma\|_{L^2(\Omega_T)} \leqslant Ch^{2-\frac{n}{2}}\|u\|_{L^2(\mathcal{M})} \quad \forall u \in L^2(I, \mathcal{M}(\Omega)). \tag{4.15}$$

Proof. Let $f \in L^2(\Omega_T)$ be arbitrary and take $z \in \mathcal{Z}$ satisfying

$$\begin{cases} -\partial_t z - \Delta z &= f \quad \text{in } \Omega_T, \\ z &= 0 \quad \text{on } \Sigma_T, \\ z(x,T) &= 0 \quad \text{in } \Omega. \end{cases} \tag{4.16}$$

Due to the convexity of Ω, there exists a constant $\tilde{C}$ independent of f such that $\|z\|_{H^{2,1}(\Omega_T)} \leqslant \tilde{C}\|f\|_{L^2(\Omega_T)}$. By (2.1) and (4.6) we get

$$\begin{aligned} \int_{\Omega_T}(y - y^\sigma)f\,dx\,dt &= \int_0^T \langle u(t) - (\Phi_\sigma u)(t), z(t)\rangle\,dt \\ &= \int_0^T \langle u(t), z(t) - (\Psi_\sigma z)(t)\rangle\,dt \\ &\leqslant \|u\|_{L^2(\mathcal{M})}\|z - \Psi_\sigma z\|_{L^2(C_0)}. \end{aligned} \tag{4.17}$$

Now, we will prove that

$$\|z - \Psi_\sigma z\|_{L^2(C_0)} \leqslant Ch^{2-\frac{n}{2}}\|z\|_{H^{2,1}(\Omega_T)}. \tag{4.18}$$

From the error estimates of the interpolation in Sobolev spaces [Ciarlet 1978, Chapter 3] we get

$$\|z-\Pi_h z\|_{L^2(C_0)} = \left(\int_0^T \|z(t)-\Pi_h z(t)\|_\infty^2 \, dt\right)^{1/2} \leqslant Ch^{2-\frac{n}{2}} \left(\int_0^T \|z(t)\|_{H^2(\Omega)}^2 \, dt\right)^{1/2} \leqslant Ch^{2-\frac{n}{2}} \|z\|_{H^{2,1}(\Omega_T)}. \tag{4.19}$$

Here and below C denotes a constant independent of σ. By an inverse inequality (see [Ciarlet 1978, Theorem 17.2]) and using (4.2) for the last inequality in the following estimate we obtain

$$\begin{aligned}
\|\Pi_h z-\Psi_\sigma z\|_{L^2(C_0)} &= \left(\sum_{k=1}^{N_\tau}\int_{I_k}\left\|\Pi_h z(t)-\frac{1}{\tau_k}\int_{I_k}\Pi_h z(s)\,ds\right\|_\infty^2 dt\right)^{1/2}\\
&\leqslant \left(\sum_{k=1}^{N_\tau}\frac{1}{\tau_k}\int_{I_k}\int_{I_k}\|\Pi_h z(t)-\Pi_h z(s)\|_\infty^2\,ds\,dt\right)^{1/2}\\
&\leqslant \frac{C}{h^{n/2}\sqrt{\tau}}\left(\sum_{k=1}^{N_\tau}\int_{I_k}\int_{I_k}\|\Pi_h z(t)-\Pi_h z(s)\|_{L^2(\Omega)}^2\,ds\,dt\right)^{1/2}\\
&\leqslant \frac{C}{h^{n/2}\sqrt{\tau}}\left(\sum_{k=1}^{N_\tau}\int_{I_k}\int_{I_k}\|\Pi_h z(t)-z(t)\|_{L^2(\Omega)}^2\,ds\,dt\right)^{1/2}\\
&\quad+\frac{C}{h^{n/2}\sqrt{\tau}}\left(\sum_{k=1}^{N_\tau}\int_{I_k}\int_{I_k}\|\Pi_h z(s)-z(s)\|_{L^2(\Omega)}^2\,ds\,dt\right)^{1/2}\\
&\quad+\frac{C}{h^{n/2}\sqrt{\tau}}\left(\sum_{k=1}^{N_\tau}\int_{I_k}\int_{I_k}\|z(t)-z(s)\|_{L^2(\Omega)}^2\,ds\,dt\right)^{1/2}\\
&\leqslant \frac{Ch^2}{h^{n/2}}\|z\|_{H^{2,1}(\Omega_T)}+\frac{C}{h^{n/2}\sqrt{\tau}}\left(\sum_{k=1}^{N_\tau}\int_{I_k}\int_{I_k}\left\|\int_{I_k}\partial_t z(\theta)\,d\theta\right\|_{L^2(\Omega)}^2\,ds\,dt\right)^{1/2}\\
&\leqslant C\frac{h^2+\tau}{h^{n/2}}\|z\|_{H^{2,1}(\Omega_T)} \leqslant Ch^{2-\frac{n}{2}}\|z\|_{H^{2,1}(\Omega_T)}.
\end{aligned} \tag{4.20}$$

Inequality (4.18) follows from (4.19) and (4.20). Finally, (4.17) and (4.18) leads to

$$\int_{\Omega_T}(y-y^\sigma)f\,dx\,dt \leqslant Ch^{2-\frac{n}{2}}\|z\|_{H^{2,1}(\Omega_T)} \leqslant Ch^{2-\frac{n}{2}}\|f\|_{L^2(\Omega_T)} \quad \forall f\in L^2(\Omega_T),$$

which implies (4.15). □

4.2 DISCRETE STATE EQUATION

In this section we approximate the state equation and provide error estimates. We recall that I_k was defined as $(t_{k-1}, t_k]$ and consequently $y_{k,h} = y_\sigma(t_k) = y_\sigma|_{I_k}$, $1 \leqslant k \leqslant N_\tau$. To approximate the state equation in time we use a dG(0) discontinuous Galerkin method, which can be formulated as an implicit Euler time stepping scheme. Given a control $u \in L^2(I, \mathcal{M}(\Omega))$, for $k = 1, \ldots, N_\tau$ and $z_h \in Y_h$ we set

$$\left\{ \begin{array}{l} \left(\dfrac{y_{k,h} - y_{k-1,h}}{\tau_k}, z_h\right) + a(y_{k,h}, z_h) = \dfrac{1}{\tau_k}\displaystyle\int_{I_k} \langle u(t), z_h\rangle \, dt, \\ y_{0,h} = y_{0h}, \end{array} \right. \tag{4.21}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$, a is the bilinear form associated to the operator $-\Delta$, i.e.,

$$a(y, z) = \int_\Omega \nabla y \nabla z \, dx,$$

and y_{0h} is an element of Y_h satisfying for some $C_0 > 0$

$$\|y_0 - y_{0h}\|_{H^{-1}(\Omega)} \leqslant C_0 h \|y_0\|_{L^2(\Omega)}. \tag{4.22}$$

For instance we can choose for y_{0h} the projection $P_h y_0$ of y_0 on Y_h given by the variational equation

$$(P_h y_0, z_h) = (y_0, z_h) \;\; \forall z_h \in Y_h.$$

For any such choice of y_{0h}, the estimate (4.22) implies that there exists a constant $C_1 > 0$ independent of h such that

$$\|y_{0h}\|_{L^2(\Omega)} \leqslant C_1 \|y_0\|_{L^2(\Omega)}. \tag{4.23}$$

Indeed, by using an inverse inequality and the well known estimates for the projection operator $P_h : L^2(\Omega) \to Y_h$, we obtain

$$\begin{aligned} \|y_{0h}\|_{L^2(\Omega)} &\leqslant \|y_{0h} - P_h y_0\|_{L^2(\Omega)} + \|P_h y_0\|_{L^2(\Omega)} \leqslant \frac{C}{h}\|y_{0h} - P_h y_0\|_{H^{-1}(\Omega)} + \|y_0\|_{L^2(\Omega)} \\ &\leqslant \frac{C}{h}\left(\|y_{0h} - y_0\|_{H^{-1}(\Omega)} + \|y_0 - P_h y_0\|_{H^{-1}(\Omega)}\right) + \|y_0\|_{L^2(\Omega)} \\ &\leqslant (C+1)\|y_0\|_{L^2(\Omega)}. \end{aligned}$$

Obviously (4.21) defines a unique solution y_σ. Let us observe that from (4.5) we have the following important consequence.

Lemma 4.4. *Let y_σ and $\tilde{y}_\sigma$ denote the solutions to (4.21) associated to the controls u and $\Phi_\sigma u$, respectively. Then the identity $y_\sigma = \tilde{y}_\sigma$ holds.*

The rest of the section is devoted to the proof of the stability of the scheme (4.21) and to the derivation of error estimates for $\|y - y_\sigma\|_{L^2(\Omega_T)}$, where y and y_σ are the solutions to (1.1) and (4.21) associated to a given control $u \in L^2(I, \mathcal{M}(\Omega))$. To this end, we introduce some operators that will be used in the proof of the theorems. For every h we consider the Ritz projection $R_h : H^1_0(\Omega) \to Y_h$ given by

$$a(y_h, R_h z) = a(y_h, z) \quad \forall y_h \in Y_h.$$

From the theory of finite elements we know that for all $z \in H^2(\Omega) \cap H^1_0(\Omega)$,

$$\left\{ \begin{array}{l} \|z - R_h z\|_{L^2(\Omega)} + h\|z - R_h z\|_{H^1(\Omega)} \leqslant Ch^2 \|z\|_{H^2(\Omega)}, \\ \|z - R_h z\|_{L^\infty(\Omega)} \leqslant Ch^{2-\frac{n}{2}} \|z\|_{H^2(\Omega)}. \end{array} \right. \tag{4.24}$$

Now, for every $\sigma = (\tau, h)$ we define $\mathcal{R}_\sigma : L^2(I, H^1_0(\Omega)) \to \mathcal{Y}_\sigma$ by

$$\mathcal{R}_\sigma z = \sum_{k=1}^{N_\tau} \frac{1}{\tau_k} \int_{I_k} R_h z(t)\, dt \chi_k = \sum_{k=1}^{N_\tau} z_{k,h} \chi_k.$$

The operator $\mathcal{R}_\sigma$ enjoys for all $z \in L^2(I, H^1_0(\Omega))$ and $y_\sigma \in \mathcal{Y}_\sigma$ the property

$$\int_0^T a(y_\sigma(t), z(t) - \mathcal{R}_\sigma z(t))\, dt = \sum_{k=1}^{N_\tau} \int_{I_k} a(y_{k,h}, z(t) - z_{k,h})\, dt = 0. \tag{4.25}$$

Indeed, for every $k = 1, \ldots, N_\tau$ we have

$$\begin{aligned} \int_{I_k} a(y_{k,h}, z(t))\, dt &= \int_{I_k} a(y_{k,h}, R_h z(t))\, dt \\ &= \tau_k a(y_{k,h}, \frac{1}{\tau_k} \int_{I_k} R_h z(t)\, dt) \\ &= \int_{I_k} a(y_{h,k}, z_{h,k})\, dt. \end{aligned}$$

Theorem 4.5. *Given a control $u \in L^2(I, \mathcal{M}(\Omega))$, let y_σ be the solution to (4.21) corresponding to u. Then, there exist constants $C_i > 0$, $i = 1, 2$, independent of u and σ such that*

$$\sum_{k=1}^{N_\tau} \|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)} + \tau \max_{1 \leqslant k \leqslant N_\tau} \|\nabla y_{k,h}\|^2_{L^2(\Omega)} \leqslant C_1 \left(\|y_0\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\mathcal{M})} \right), \tag{4.26}$$

$$\|y_\sigma\|_{L^2(\Omega_T)} \leqslant C_2 \left(\|y_0\|_{L^2(\Omega)} + \|u\|_{L^2(\mathcal{M})} \right). \tag{4.27}$$

Proof. Let us set $z_h = y_{k,h} - y_{k-1,h}$ in (4.21). Then we obtain for $1 \leqslant k \leqslant N_\tau$ that

$$\frac{1}{\tau_k}\|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)} + a(y_{k,h}, y_{k,h} - y_{k-1,h}) = \frac{1}{\tau_k}\int_{I_k} \langle u(t), y_{k,h} - y_{k-1,h}\rangle\, dt.$$

From here we get with the aid of an inverse estimate [Ciarlet 1978, Theorem 17.2]

$$\begin{aligned}
&\frac{1}{\tau}\|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)} + \frac{1}{2}[a(y_{k,h}, y_{k,h}) - a(y_{k-1,h}, y_{k-1,h})] \\
&\qquad\qquad \leqslant \frac{1}{\tau_k}\|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)} + \frac{1}{2}[a(y_{k,h}, y_{k,h}) - a(y_{k-1,h}, y_{k-1,h}) \\
&\qquad\qquad\quad + a(y_{k,h} - y_{k-1,h}, y_{k,h} - y_{k-1,h})] \\
&\qquad\qquad = \frac{1}{\tau_k}\|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)} + a(y_{k,h}, y_{k,h} - y_{k-1,h}) \\
&\qquad\qquad = \frac{1}{\tau_k}\int_{I_k} \langle u(t), y_{k,h} - y_{k-1,h}\rangle\, dt \\
&\qquad\qquad \leqslant \frac{1}{\sqrt{\tau_k}}\|u\|_{L^2(I_k,\mathcal{M})}\|y_{k,h} - y_{k-1,h}\|_\infty \\
&\qquad\qquad \leqslant \frac{Ch^{-n/2}}{\sqrt{\tau_k}}\|u\|_{L^2(I_k,\mathcal{M})}\|y_{k,h} - y_{k-1,h}\|_{L^2(\Omega)} \\
&\qquad\qquad \leqslant \frac{C^2h^{-n}\tau}{2\tau_k}\|u\|^2_{L^2(I_k,\mathcal{M})} + \frac{1}{2\tau}\|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)} \\
&\qquad\qquad \leqslant \frac{C^2\rho_T C_{\Omega,T}}{2\tau}\|u\|^2_{L^2(I_k,\mathcal{M})} + \frac{1}{2\tau}\|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)}.
\end{aligned}$$

In the last inequality we have used (4.2). Summing from $k = 1$ to m and using (4.2), it follows that

$$\frac{1}{\tau}\sum_{k=1}^{m}\|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)} + a(y_{m,h}, y_{m,h}) - a(y_{0h}, y_{0h}) \leqslant \frac{C^2\rho_T C_{\Omega,T}}{\tau}\|u\|^2_{L^2(\mathcal{M})}.$$

Hence

$$\sum_{k=1}^{m}\|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega)} + \tau\|\nabla y_{m,h}\|^2_{L^2(\Omega)} \leqslant C(\|y_0\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\mathcal{M})}). \tag{4.28}$$

Here we have used an inverse inequality, (4.2), and (4.23) to get

$$\tau\|y_{0h}\|^2_{H^1(\Omega)} \leqslant \frac{C\tau}{h^2}\|y_{0h}\|^2_{L^2(\Omega)} \leqslant C\|y_0\|^2_{L^2(\Omega)}.$$

Finally, since $1 \leqslant m \leqslant N_\tau$ is arbitrary, (4.26) follows from (4.28).

Now we prove (4.27). Given $f \in L^2(\Omega_T)$, we take $z \in \mathcal{Z}$ satisfying (4.16). Integrating by parts we get

$$\begin{aligned}\int_{\Omega_T} y_\sigma f\,dxdt &= \sum_{k=1}^{N_\tau}\int_{I_k}\int_\Omega y_{k,h}(x)f(x,t)\,dxdt\\&= \sum_{k=1}^{N_\tau}\int_{I_k}\{-\partial_t(y_{k,h},z(t))+a(y_{k,h},z(t))\}\,dt\\&= \sum_{k=1}^{N_\tau}\left\{(y_{k,h},z(t_{k-1})-z(t_k))+\int_{I_k}a(y_{k,h},z(t))\,dt\right\}\\&= \sum_{k=1}^{N_\tau}\left\{(y_{k,h}-y_{k-1,h},z(t_{k-1}))+\int_{I_k}a(y_{k,h},z(t))\,dt\right\}+(y_{0h},z(0)).\end{aligned}$$

Taking $z_\sigma = \mathcal{R}_\sigma z$, we get from the above identity and (4.25) that

$$\begin{aligned}\int_{\Omega_T} y_\sigma f\,dxdt &= \sum_{k=1}^{N_\tau}\{(y_{k,h}-y_{k-1,h},z_{k,h})+\tau_k a(y_{k,h},z_{k,h})\}+(y_{0h},z(0))\\&\quad+\sum_{k=1}^{N_\tau}\left\{(y_{k,h}-y_{k-1,h},z(t_{k-1})-z_{k,h})+\int_{I_k}a(y_{k,h},z(t)-z_{k,h})\,dt\right\}\\&= \int_0^T\langle u(t),z_\sigma(t)\rangle\,dt+(y_{0h},z(0))+\sum_{k=1}^{N_\tau}(y_{k,h}-y_{k-1,h},z(t_{k-1})-z_{k,h}).\end{aligned} \tag{4.29}$$

Let us estimate each of these terms. From the definition of z_σ and (4.23) we obtain

$$\begin{aligned}\int_0^T\langle u(t),z_\sigma(t)\rangle\,dt+(y_{0h},z(0)) &\leqslant \|u\|_{L^2(\mathcal{M})}\|z_\sigma\|_{L^2(C_0)}+\|y_{0h}\|_{L^2(\Omega)}\|z(0)\|_{L^2(\Omega)}\\&\leqslant C\|z\|_{H^{2,1}(\Omega_T)}(\|u\|_{L^2(\mathcal{M})}+\|y_0\|_{L^2(\Omega)}),\end{aligned} \tag{4.30}$$

where we have used that there exists a constant $C>0$ independent of σ such that

$$\|\mathcal{R}_\sigma v\|_{L^2(C_0)} \leqslant C\|v\|_{H^{2,1}(\Omega_T)} \quad \forall v\in H^{2,1}(\Omega_T). \tag{4.31}$$

Indeed,

$$\begin{aligned}\|\mathcal{R}_\sigma v\|_{L^2(C_0)} &= \left(\sum_{k=1}^{N_\tau}\int_{I_k}\|\mathcal{R}_\sigma v(t)\|_\infty^2\,dt\right)^{1/2} = \left(\sum_{k=1}^{N_\tau}\int_{I_k}\left\|\frac{1}{\tau_k}\int_{I_k}R_h v(s)\,ds\right\|_\infty^2 dt\right)^{1/2}\\&\leqslant \left(\sum_{k=1}^{N_\tau}\int_{I_k}\|R_h v(s)\|_\infty^2\,ds\right)^{1/2}.\end{aligned}$$

Using (4.24) we deduce that

$$\|R_h w\|_\infty \leqslant \|R_h w-w\|_\infty+\|w\|_\infty \leqslant Ch^\kappa\|w\|_{H^2(\Omega)}+\|w\|_\infty \leqslant C\|w\|_{H^2(\Omega)}$$

for every $w \in H^2(\Omega) \cap H^1_0(\Omega)$, with $\kappa = 1$ if $n \leqslant 2$ and $\kappa = 1/2$ if $n = 3$. Then, (4.31) follows from the above inequalities.

Concerning the last term of (4.29), we will prove

$$\sum_{k=1}^{N_\tau} (y_{k,h} - y_{k-1,h}, z(t_{k-1}) - z_{k,h}) \leqslant Ch^\kappa \|z\|_{H^{2,1}(\Omega_T)}(\|u\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)}), \tag{4.32}$$

where κ is defined as above. First we observe that (4.26) implies

$$\begin{aligned}\sum_{k=1}^{N_\tau} &|(y_{k,h} - y_{k-1,h}, z(t_{k-1}) - z_{k,h})| \\ &\leqslant \left(\sum_{k=1}^{N_\tau} \|y_{k,h} - y_{k-1,h}\|^2_{L^2(\Omega_h)}\right)^{1/2} \left(\sum_{k=1}^{N_\tau} \|z(t_{k-1}) - z_{k,h}\|^2_{L^2(\Omega_h)}\right)^{1/2} \\ &\leqslant C(\|u\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)}) \left(\sum_{k=1}^{N_\tau} \|z(t_{k-1}) - z_{k,h}\|^2_{L^2(\Omega_h)}\right)^{1/2}.\end{aligned} \tag{4.33}$$

From the definition of z_σ and (4.24) we deduce

$$\begin{aligned}\|z(t_{k-1}) - z_{k,h}\|_{L^2(\Omega_h)} &= \left(\int_{\Omega_h} \left|\frac{1}{\tau_k}\int_{I_k} \{z(t_{k-1}) - R_h z(s)\}\, ds\right|^2 dx\right)^{1/2} \\ &\leqslant \left(\frac{1}{\tau_k}\int_{\Omega_h}\int_{I_k} |z(t_{k-1}) - R_h z(s)|^2\, ds dx\right)^{1/2} \\ &\leqslant \left(\frac{1}{\tau_k}\int_{\Omega_h}\int_{I_k} |z(t_{k-1}) - z(s)|^2\, ds dx\right)^{1/2} \\ &\quad + \left(\frac{1}{\tau_k}\int_{I_k} \|z(s) - R_h z(s)\|^2_{L^2(\Omega_h)}\, ds\right)^{1/2} \\ &\leqslant \left(\int_{\Omega_h}\int_{I_k}\int_{I_k} |\partial_t z(\theta)|^2\, d\theta\, ds\, dx\right)^{1/2} \\ &\quad + Ch^2 \left(\frac{1}{\tau_k}\int_{I_k} \|z(s)\|^2_{H^2(\Omega)}\, ds\right)^{1/2} \\ &\leqslant \sqrt{\tau}\|\partial_t z\|_{L^2(I_k, L^2(\Omega))} + \frac{Ch^2\sqrt{\rho_T}}{\sqrt{\tau}}\|z\|_{L^2(I_k, H^2(\Omega))} \\ &\leqslant Ch^\kappa(\|\partial_t z\|_{L^2(I_k, L^2(\Omega))} + \|z\|_{L^2(I_k, H^2(\Omega))}).\end{aligned}$$

Inserting this estimate in (4.33) we infer (4.32). Finally, (4.29), (4.30) and (4.32) imply that

$$\int_{\Omega_T} y_\sigma f\, dx dt \leqslant C\|f\|_{L^2(\Omega_T)}(\|u\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)}) \quad \forall f \in L^2(\Omega_T),$$

which is equivalent to (4.27) $\square$

In the next theorem we show error estimates for the discretization of the state equation.

Theorem 4.6. *Given $u \in L^2(I, \mathcal{M}(\Omega))$, let y and y_σ be the solutions to (1.1) and (4.21). Then, there exists a constant C independent of $u \in L^2(I, \mathcal{M}(\Omega))$, $y_0 \in L^2(\Omega)$, and σ such that*

$$\|y - y_\sigma\|_{L^2(\Omega_T)} \leqslant Ch^\kappa(\|u\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)}), \tag{4.34}$$

where $\kappa = 1$ if $n \leqslant 2$ and $\kappa = 1/2$ if $n = 3$.

Proof. As in the proof of Theorem 4.5, we take an arbitrary element $f \in L^2(\Omega_T)$, $z \in \mathcal{Z}$ solution to (4.16), and $z_\sigma = \mathcal{R}_\sigma z$. Then, from (2.1) we obtain

$$\begin{aligned}\int_{\Omega_T} (y - y_\sigma) f \, dx \, dt = \int_0^T \langle u(t), z(t)\rangle \, dt + \int_\Omega y_0(x) z(x, 0) \, dx \\ - \sum_{k=1}^{N_\tau} \int_{I_k} \{-(y_{k,h}, \partial_t z(t)) + a(y_{k,h}, z(t))\} \, dt.\end{aligned} \tag{4.35}$$

Integrating by parts we get

$$\begin{aligned}\sum_{k=1}^{N_\tau} \int_{I_k} -(y_{k,h}, \partial_t z(t)) \, dt &= \sum_{k=1}^{N_\tau} (y_{k,h}, z(t_{k-1}) - z(t_k)) \\ &= \sum_{k=1}^{N_\tau} (y_{k,h} - y_{k-1,h}, z(t_{k-1})) + (y_{0h}, z(0)).\end{aligned}$$

From this identity, (4.21), and (4.25) we deduce

$$\begin{aligned}&\sum_{k=1}^{N_\tau} \int_{I_k} \{-(y_{k,h}, \partial_t z(t)) + a(y_{k,h}, z(t))\} \, dt \\ &\quad = \sum_{k=1}^{N_\tau} \int_{I_k} \left\{(y_{k,h} - y_{k-1,h}, z(t_{k-1})) + \int_{I_k} a(y_{k,h}, z(t))\right\} dt + (y_{0h}, z(0)) \\ &\quad = \sum_{k=1}^{N_\tau} \int_{I_k} \left\{(y_{k,h} - y_{k-1,h}, z_{k,h}) + \int_{I_k} a(y_{k,h}, z_{k,h})\right\} dt \\ &\qquad + \sum_{k=1}^{N_\tau} \int_{I_k} (y_{k,h} - y_{k-1,h}, z(t_{k-1}) - z_{k,h}) + (y_{0h}, z(0)) \\ &\quad = \int_0^T \langle u(t), z_\sigma(t)\rangle \, dt + \int_\Omega y_{0h}(x) z(x, 0) \, dx \\ &\qquad + \sum_{k=1}^{N_\tau} \int_{I_k} (y_{k,h} - y_{k-1,h}, z(t_{k-1}) - z_{k,h}).\end{aligned}$$

Inserting this identity in (4.35) we infer

$$\int_{\Omega_T}(y-y_\sigma)f\,dx\,dt=\int_0^T\langle u(t),z(t)-\mathcal{R}_\sigma z(t)\rangle\,dt+\int_\Omega(y_0(x)-y_{0h}(x))z(x,0)\,dx \\ -\sum_{k=1}^{N_\tau}\int_{I_k}(y_{k,h}-y_{k-1,h},z(t_{k-1})-z_{k,h}). \tag{4.36}$$

Let us estimate each of these three terms. For the first term we observe that

$$\|z-\mathcal{R}_\sigma z\|_{L^2(C_0)}\leqslant Ch^\kappa\|z\|_{H^{2,1}(\Omega_T)}.$$

The proof of this inequality is the same than the one of (4.18); it is enough to replace Π_h by R_h and to use (4.24). Using this inequality we obtain the first estimate as follows:

$$\begin{aligned}\left|\int_0^T\langle u(t),z(t)-\mathcal{R}_\sigma z(t)\rangle\,dt\right|&\leqslant\|u\|_{L^2(\mathcal{M})}\|z-\mathcal{R}_\sigma z\|_{L^2(C_0)}\\&\leqslant ch^\kappa\|u\|_{L^2(\mathcal{M})}\|z\|_{H^{2,1}(\Omega_T)}.\end{aligned}\tag{4.37}$$

For the second term we proceed with the aid of (4.23):

$$\begin{aligned}\left|\int_\Omega(y_0(x)-y_{0h}(x))z(x,0)\,dx\right|&\leqslant\|y_0-y_{0h}\|_{H^{-1}(\Omega)}\|z(0)\|_{H_0^1(\Omega)}\\&\leqslant Ch\|y_0\|_{L^2(\Omega)}\|z\|_{H^{2,1}(\Omega_T)}.\end{aligned}\tag{4.38}$$

Finally, the third term of (4.36) was estimated in (4.32). Thus, using (4.37), (4.38), and (4.32) in (4.36) the inequality

$$\begin{aligned}\int_{\Omega_T}(y-y_\sigma)f\,dx\,dt&\leqslant Ch^\kappa(\|u\|_{L^2(\mathcal{M})}+\|y_0\|_{L^2(\Omega)})\|z\|_{H^{2,1}(\Omega_T)}\\&\leqslant Ch^\kappa(\|u\|_{L^2(\mathcal{M})}+\|y_0\|_{L^2(\Omega)})\|f\|_{L^2(\Omega_T)}\end{aligned}$$

is obtained, which leads to (4.34) □

4.3 DISCRETE OPTIMAL CONTROL PROBLEM

The approximation of the optimal control problem (P) is defined as

$$\min_{u\in L^2(I,\mathcal{M}(\Omega))}J_\sigma(u)=\frac{1}{2}\|y_\sigma-y_d\|^2_{L^2(\Omega_{hT})}+\alpha\|u\|_{L^2(\mathcal{M})}, \tag{P_σ}$$

where y_σ is the discrete state associated to u, i.e., the solution to (4.21). Let us observe that analogously to J, the functional J_σ is convex. However, it is not strictly convex due to the non-injectivity of the control-to-discrete-state mapping and the non-strict convexity of the norm of $L^2(I,\mathcal{M}(\Omega))$. Although the existence of a solution can be shown in the same way as for the problem (P), we therefore cannot deduce its uniqueness. On the other hand, if $\tilde{u}_\sigma$ is a solution to (P_σ) and if we take $\bar{u}_\sigma = \Phi_\sigma \tilde{u}_\sigma$, then Lemma 4.4 and the inequality (4.7) imply that $J_\sigma(\bar{u}_\sigma) \leqslant J_\sigma(\tilde{u}_\sigma)$, hence $\bar{u}_\sigma$ is also a solution to (P_σ). Since for $u_\sigma \in \mathcal{U}_\sigma$, the mapping $u_\sigma \mapsto y_\sigma(u_\sigma)$, with $y_\sigma(u_\sigma)$ the solution to (4.21) for $u = u_\sigma$, is linear, injective and $\dim \mathcal{U}_\sigma = \dim \mathcal{Y}_\sigma$, this mapping is bijective. Therefore, the cost functional J_σ is strictly convex on $\mathcal{U}_\sigma$, hence (P_σ) has a unique solution in $\mathcal{U}_\sigma$, which will be denoted by $\bar{u}_\sigma$ hereafter. We summarize this discussion in the following theorem.

Theorem 4.7. *Problem* (P_σ) *admits at least one solution. Among all solutions, there exists a unique solution $\bar{u}_\sigma$ belonging to $\mathcal{U}_\sigma$. Moreover, any other solution $\tilde{u} \in L^2(I,\mathcal{M}(\Omega))$ to* (P_σ) *satisfies $\Phi_\sigma \tilde{u} = \bar{u}_\sigma$.*

Remark 4.8. The fact that problem (P_σ) has exactly one solution in $\mathcal{U}_\sigma$ is of practical interest.Indeed, recall that $\bar{u}_\sigma$, as an element of $\mathcal{U}_\sigma$, can be uniquely represented as

$$\bar{u}_\sigma = \sum_{k=1}^{N_\tau} \sum_{j=1}^{N_h} \bar{u}_{kj} \chi_k \delta_{x_j}.$$

The numerical computation of $\bar{u}_\sigma$ therefore is equivalent to the computation of the coefficients $\{\bar{u}_{kj} : 1 \leqslant k \leqslant N_\tau,\ 1 \leqslant j \leqslant N_h\}$; see section 6.

We finish this section by analyzing the convergence of the solution in $\mathcal{U}_\sigma$ to (P_σ) to the solution to (P).

Theorem 4.9. *For every σ, let $\bar{u}_\sigma$ be the unique solution to problem (P_σ) belonging to $\mathcal{U}_\sigma$ and let $\bar{u}$ be the solution to problem (P). Then the following convergence properties hold for $\sigma \to 0^+$:*

$$\bar{u}_\sigma \overset{*}{\rightharpoonup} \bar{u} \ \textit{in}\ L^2(I,\mathcal{M}(\Omega)), \tag{4.39}$$

$$\|\bar{u}_\sigma\|_{L^2(\mathcal{M})} \to \|\bar{u}\|_{L^2(\mathcal{M})}, \tag{4.40}$$

$$\|\bar{y} - \bar{y}_\sigma\|_{L^2(\Omega_T)} \to 0, \tag{4.41}$$

$$J_\sigma(\bar{u}_\sigma) \to J(\bar{u}), \tag{4.42}$$

where $\bar{y}$ and $\bar{y}_\sigma$ are the continuous and discrete states associated to $\bar{u}$ and $\bar{u}_\sigma$, respectively.

Proof. First of all, let us show that

$$u_\sigma \overset{*}{\rightharpoonup} u \text{ in } L^2(I,\mathcal{M}(\Omega)) \quad \text{implies} \quad \|y_\sigma - y\|_{L^2(\Omega_T)} \to 0, \tag{4.43}$$

where y_σ and y are the discrete and continuous states associated to the controls u_σ and u, respectively. Indeed, let us write $y - y_\sigma = (y - y^\sigma) + (y^\sigma - y_\sigma)$, where y^σ is the continuous state associated to u_σ. Then by Theorems 2.4 and 4.6 we deduce (4.43).

Turning to the verification of (4.39), we observe that

$$\alpha\|\bar{u}_\sigma\|_{L^2(\mathcal{M})} \leqslant J_\sigma(\bar{u}_\sigma) \leqslant J_\sigma(0) = \frac{1}{2}\|\hat{y}_{\sigma 0} - y_d\|^2_{L^2(\Omega_{hT})} \leqslant \frac{1}{2}\|\hat{y}_{\sigma 0} - y_d\|^2_{L^2(\Omega_T)}$$

with $\hat{y}_{\sigma 0}$ denoting the uncontrolled discrete state, which implies the boundedness of $\{\bar{u}_\sigma\}_\sigma$ in $L^2(I, \mathcal{M}(\Omega))$. By taking a subsequence, we have that $\bar{u}_\sigma \overset{*}{\rightharpoonup} u$ in $L^2(I, \mathcal{M}(\Omega))$. Then using (4.1), (4.43), lower semicontinuity of the norm $\|\cdot\|_{L^2(\mathcal{M})}$ and (4.9) we obtain

$$J(u) \leqslant \liminf_{\sigma \to 0} J_\sigma(\bar{u}_\sigma) \leqslant \limsup_{\sigma \to 0} J_\sigma(\bar{u}_\sigma) \leqslant \limsup_{\sigma \to 0} J_\sigma(\Psi_\sigma \bar{u}) = J(\bar{u}).$$

Hence $u = \bar{u}$ by the uniqueness of the solution to (P), and the whole sequence $\{\bar{u}_\sigma\}_\sigma$ converges weakly-$\star$ to $\bar{u}$. In addition, the above inequality implies (4.42). Using again (4.43), we deduce (4.41). Finally, (4.40) follows immediately from (4.41) and (4.42). □

5 ERROR ESTIMATES

We now turn to the proof of error estimates for the optimal costs and for the optimal states. We still require Ω to be convex and assume in addition

$$y_d \in L^2(I, L^r(\Omega)) \text{ with } r = \begin{cases} 2 & \text{if } n = 1, \\ 4 & \text{if } n = 2, \\ \frac{8}{3} & \text{if } n = 3. \end{cases} \tag{5.1}$$

Recall that $\bar{y}$ and $\bar{y}_\sigma$ denote the continuous and discrete states associated to the optimal controls $\bar{u}$ and $\bar{u}_\sigma$, respectively.

Theorem 5.1. *There exists a constant $C > 0$ independent of σ such that*

$$|J(\bar{u}) - J_\sigma(\bar{u}_\sigma)| \leqslant Ch^\kappa, \tag{5.2}$$

where $\kappa = 1$ if $n \leqslant 2$ and $\kappa = 1/2$ if $n = 3$.

Proof. Taking r as in (5.1) and using Hölder's inequality and (4.1), we deduce that for all $\phi \in L^2(I, L^r(\Omega))$ and $n = 2$ or 3,

$$\|\phi\|_{L^2(I, L^2(\Omega \setminus \Omega_h))} \leqslant \|\phi\|_{L^2(I, L^r(\Omega \setminus \Omega_h))} |\Omega \setminus \Omega_h|^{\frac{r-2}{2r}} \leqslant C \|\phi\|_{L^2(I, L^r(\Omega \setminus \Omega_h))} h^{\frac{\kappa}{2}} \tag{5.3}$$

holds. Observe that $\Omega = \Omega_h$ for $n = 1$; consequently (5.3) holds with $C = 0$.

Let y and y_σ be the continuous and discrete states associated to a given control u. As a consequence of (4.34) and (5.3), with $\phi = y - y_d$, we obtain

$$\begin{aligned} \left| \|y - y_d\|^2_{L^2(\Omega_T)} - \|y_\sigma - y_d\|^2_{L^2(\Omega_{hT})} \right| &\leqslant \|y - y_d\|^2_{L^2(I, L^2(\Omega \setminus \Omega_h))} \\ &\quad + \left(\|y - y_d\|_{L^2(\Omega_{hT})} + \|y_\sigma - y_d\|_{L^2(\Omega_{hT})} \right) \|y - y_\sigma\|_{L^2(\Omega_{hT})} \\ &\leqslant C \left(\|y - y_d\|^2_{L^2(I, L^r(\Omega \setminus \Omega_h))} + \|u\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)} \right) h^\kappa. \end{aligned} \tag{5.4}$$

Now, by the optimality of $\bar{u}$ and $\bar{u}_\sigma$ we have

$$J(\bar{u}) - J_\sigma(\bar{u}) \leqslant J(\bar{u}) - J_\sigma(\bar{u}_\sigma) \leqslant J(\bar{u}_\sigma) - J_\sigma(\bar{u}_\sigma),$$

and hence

$$|J(\bar{u}) - J_\sigma(\bar{u}_\sigma)| \leqslant \max\{|J(\bar{u}) - J_\sigma(\bar{u})|, |J(\bar{u}_\sigma) - J_\sigma(\bar{u}_\sigma)|\}. \tag{5.5}$$

From (4.40) we deduce that $\{\bar{u}_\sigma\}_\sigma$ is bounded in $L^2(I, \mathcal{M}(\Omega))$. Therefore, (2.2) implies that the continuous associated states $\{y_{\bar{u}_\sigma}\}_\sigma$ are bounded in $L^2(I, W_0^{1,p}(\Omega))$ for every $1 \leqslant p < \frac{n}{n-1}$, and therefore in $L^2(I, L^r(\Omega))$ as well. We now apply (5.4) with $u = \bar{u}_\sigma$ and $u = \bar{u}$, respectively. Together with (5.5) this establishes (5.2). □

In the following theorem we establish a rate of convergence for the states.

Theorem 5.2. *There exists a constant $C > 0$ independent of h such that*

$$\|\bar{y} - \bar{y}_\sigma\|_{L^2(\Omega_T)} \leqslant Ch^{\frac{\kappa}{2}}, \tag{5.6}$$

with κ as defined in Theorem 4.1.

Proof. Let $S : L^2(I, \mathcal{M}(\Omega)) \to L^2(\Omega_T)$ and $S_\sigma : L^2(I, \mathcal{M}(\Omega)) \to L^2(\Omega_T)$ be the solution operators associated to the equations (1.1) and (4.21), respectively. From (4.34) it follows that

$$\|Su - S_\sigma u\|_{L^2(\Omega_T)} \leqslant Ch^\kappa(\|u\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)}). \tag{5.7}$$

By the optimality of $\bar{u}$ we have for all $u \in L^2(I, \mathcal{M}(\Omega))$ that

$$(S\bar{u} - y_d, Su - S\bar{u}) + \alpha[\|u\|_{L^2(\mathcal{M})} - \|\bar{u}\|_{L^2(\mathcal{M})}] \geqslant 0,$$

where $(\cdot,\cdot)$ now denotes the scalar product in $L^2(\Omega_T)$. In particular, taking $u = \bar{u}_\sigma$, we get

$$(S\bar{u} - y_d, S\bar{u}_\sigma - S\bar{u}) + \alpha[\|\bar{u}_\sigma\|_{L^2(\mathcal{M})} - \|\bar{u}\|_{L^2(\mathcal{M})}] \geqslant 0. \tag{5.8}$$

Analogously, the optimality of $\bar{u}_\sigma$ implies that

$$(S_\sigma\bar{u}_\sigma - y_d, S_\sigma\bar{u} - S_\sigma\bar{u}_\sigma) + \alpha[\|\bar{u}\|_{L^2(\mathcal{M})} - \|\bar{u}_\sigma\|_{L^2(\mathcal{M})}] \geqslant 0. \tag{5.9}$$

We point out that by definition of Y_h, we have $S_\sigma u = 0$ in $I \times (\Omega \setminus \Omega_h)$. Then, the scalar product above in $L^2(\Omega_T)$ coincides with that in $L^2(\Omega_{hT})$. Now, we rearrange terms in (5.9) as follows:

$$\begin{aligned}(S\bar{u}_\sigma - y_d, S\bar{u} - S\bar{u}_\sigma) + (S_\sigma\bar{u}_\sigma - S\bar{u}_\sigma, S_\sigma\bar{u} - S_\sigma\bar{u}_\sigma)\\ + (y_d, S\bar{u} - S_\sigma\bar{u} + S_\sigma\bar{u}_\sigma - S\bar{u}_\sigma) + (S\bar{u}_\sigma, S_\sigma\bar{u} - S\bar{u} + S\bar{u}_\sigma - S_\sigma\bar{u}_\sigma)\\ + \alpha[\|\bar{u}\|_{L^2(\mathcal{M})} - \|\bar{u}_\sigma\|_{L^2(\mathcal{M})}] \geqslant 0.\end{aligned} \tag{5.10}$$

Adding (5.8) and (5.10) we obtain

$$\begin{aligned}\|S\bar{u}-S_\sigma\bar{u}_\sigma\|^2_{L^2(\Omega_T)} &= (S\bar{u}-S_\sigma\bar{u}_\sigma, S\bar{u}-S_\sigma\bar{u}_\sigma) \\ &\leqslant (S_\sigma\bar{u}_\sigma - S\bar{u}_\sigma, S_\sigma\bar{u}-S_\sigma\bar{u}_\sigma) \\ &\quad + (y_d - S\bar{u}_\sigma, S\bar{u}-S_\sigma\bar{u}+S_\sigma\bar{u}_\sigma - S\bar{u}_\sigma).\end{aligned} \tag{5.11}$$

Let us estimate the right hand terms. For the first one we apply the Cauchy–Schwarz inequality and use (5.7) to deduce

$$(S_\sigma\bar{u}_\sigma - S\bar{u}_\sigma, S_\sigma\bar{u}-S_\sigma\bar{u}_\sigma) \leqslant \|S_\sigma\bar{u}_\sigma - S\bar{u}_\sigma\|_{L^2(\Omega_T)}\|S_\sigma\bar{u}-S_\sigma\bar{u}_\sigma\|_{L^2(\Omega_T)} \leqslant Ch^\kappa, \tag{5.12}$$

where we have used that $\{\bar{u}_\sigma\}_\sigma$, $\{S_\sigma\bar{u}\}_\sigma$ and $\{S_\sigma\bar{u}_\sigma\}_\sigma$ are bounded due to (4.40) and (4.27). For the second term we use once again (5.7) to obtain

$$\begin{aligned}&(y_d - S\bar{u}_\sigma, S\bar{u}-S_\sigma\bar{u}+S_\sigma\bar{u}_\sigma - S\bar{u}_\sigma) \\ &\quad + \|y_d - S\bar{u}_\sigma\|_{L^2(\Omega_T)}\|(S-S_\sigma)(\bar{u}-\bar{u}_\sigma)\|_{L^2(\Omega_T)} \\ &\qquad + C(\|\bar{u}-\bar{u}_\sigma\|_{L^2(\mathcal{M})} + \|y_0\|_{L^2(\Omega)})h^\kappa \leqslant Ch^\kappa,\end{aligned} \tag{5.13}$$

where we have also used that $y_d \in L^2(I, L^r(\Omega))$ and (2.2). Finally, (5.11), (5.12) and (5.13) prove (5.6). ☐

Remark 5.3. Let us observe that (5.2) and (5.6) imply that

$$\left|\|\bar{u}\|_{L^2(\mathcal{M})} - \|\bar{u}_\sigma\|_{L^2(\mathcal{M})}\right| \leqslant Ch^{\frac{\kappa}{2}}$$

for some constant $C > 0$ independent of σ.

6 NUMERICAL SOLUTION

We now address the computation of minimizers $\bar{u}_\sigma$ of problem (P_σ). First of all, we note that if we define $y_{d,\sigma}$ as the $L^2(\Omega_{hT})$ projection of y_d on $\mathcal{Y}_\sigma$, then

$$J_\sigma(u) = \frac{1}{2}\|y_\sigma - y_{d,\sigma}\|^2_{L^2(\Omega_{hT})} + \alpha\|u\|_{L^2(\mathcal{M})} + \frac{1}{2}\|y_d - y_{d,\sigma}\|^2_{L^2(\Omega_{hT})}.$$

Therefore, the problems (P_σ) and

$$\min_{u\in L^2(I,\mathcal{M}(\Omega))} \tilde{J}_\sigma(u) = \frac{1}{2}\|y_\sigma - y_{d,\sigma}\|^2_{L^2(\Omega_{hT})} + \alpha\|u\|_{L^2(\mathcal{M})} \tag{Q_σ}$$

are equivalent. In this section we present a numerical algorithm to solve (Q_σ) as an alternative formulation to (P_σ).

Due to the spatio-temporal coupling of the norm in $L^2(I, \mathcal{M}(\Omega))$, its subdifferential is difficult to characterize. However, using Fenchel duality combined with an equivalent reformulation that decouples the spatio-temporal structure, we can obtain optimality conditions that can be solved using a semismooth Newton method.

For the reader's convenience, we recall the Fenchel duality theory, e.g., from [Ekeland and Témam 1999, Chapter 4]. Let V and Y be Banach spaces with topological duals V^* and Y^*, respectively, and let $\Lambda : V \to Y$ be a continuous linear operator. Setting $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, let $\mathcal{F} : V \to \bar{\mathbb{R}}$, $\mathcal{G} : Y \to \bar{\mathbb{R}}$ be convex lower semi-continuous functionals which are not identically equal ∞ and for which there exists a $v_0 \in V$ such that $\mathcal{F}(v_0) < \infty$, $\mathcal{G}(\Lambda v_0) < \infty$, and $\mathcal{G}$ is continuous at Λv_0. Let $\mathcal{F}^* : V^* \to \bar{\mathbb{R}}$ denote the Fenchel conjugate of $\mathcal{F}$ defined by

$$\mathcal{F}^*(q) = \sup_{v \in V} \langle q, v\rangle_{V^*,V} - \mathcal{F}(v),$$

which we can calculate using the fact that

$$\mathcal{F}^*(q) = \langle q, v\rangle_{V^*,V} - \mathcal{F}(v) \quad \text{if and only if} \quad q \in \partial\mathcal{F}(v). \tag{6.1}$$

Here, $\partial\mathcal{F}$ denotes the subdifferential of the convex function $\mathcal{F}$, which reduces to the Gâteaux-derivative if it exists, and the left hand side arises from differentiating the duality pairing.

The Fenchel duality theorem states that under the assumptions given above,

$$\inf_{v \in V} \mathcal{F}(v) + \mathcal{G}(\Lambda v) = \sup_{q \in Y^*} -\mathcal{F}^*(\Lambda^* q) - \mathcal{G}^*(-q)\,, \tag{6.2}$$

holds, and that the right hand side of (6.2) has at least one solution. Furthermore, the equality in (6.2) is attained at $(\bar{v}, \bar{q})$ if and only if

$$\begin{cases} \Lambda^* \bar{q} \in \partial\mathcal{F}(\bar{v}), \\ -\bar{q} \in \partial\mathcal{G}(\Lambda\bar{v}), \end{cases} \tag{6.3}$$

where the derivative of the duality pairing again enters the left hand side.

We now apply the Fenchel duality theorem to (Q_σ), which we express in terms of the expansion coefficients $\bar{u}_{kj}$. Let $N_\sigma = N_\tau \times N_h$ and identify as above $u_\sigma \in \mathcal{U}_\sigma$ with the vector $\vec{u}_\sigma = (u_{11}, \dots, u_{1N_h}, \dots, u_{N_\tau N_h})^T \in \mathbb{R}^{N_\sigma}$ of coefficients, and similarly $y_{d,\sigma} \in \mathcal{Y}_\sigma$; see section 4.1. To keep the notation simple, we will omit the vector arrows from here on. Denote by $M_h = (\langle e_j, e_k\rangle)_{j,k=1}^{N_h}$ the mass matrix and by $A_h = (a(e_j, e_k))_{j,k=1}^{N_h}$ the stiffness matrix corresponding to Y_h. For the sake of presentation, we fix $y_0 = 0$. Then the discrete state equation (4.21) can be expressed as $L_\sigma y_\sigma = u_\sigma$ with

$$L_\sigma = \begin{pmatrix} \tau_1^{-1} M_h + A_h & 0 & 0 \\ -\tau_1^{-1} M_h & \tau_2^{-1} M_h + A_h & 0 \\ 0 & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{N_\sigma \times N_\sigma}.$$

(Note that the "mass matrix" corresponding to $(\langle \delta_{x_j}, e_k\rangle)_{j,k=1}^{N_h}$ is the identity.)

Introducing for $v_\sigma \in \mathbb{R}^{N_\sigma}$ the vectors $v_k = (v_{k1}, \dots, v_{kN_h})^T \in \mathbb{R}^{N_h}$, $1 \leqslant k \leqslant N_\tau$, the discrete optimal control problem (Q_σ) can be stated in reduced form as

$$\min_{u_\sigma \in \mathbb{R}^{N_\sigma}} \frac{1}{2} \sum_{k=1}^{N_\tau} \tau_k [L_\sigma^{-1} u_\sigma - y_{d,\sigma}]_k^T M_h [L_\sigma^{-1} u_\sigma - y_{d,\sigma}]_k + \alpha \left(\sum_{k=1}^{N_\tau} \tau_k |u_k|_1^2 \right)^{1/2}.$$

We now set $\Lambda : \mathbb{R}^{N_\sigma} \to \mathbb{R}^{N_\sigma}$, $\Lambda v = L_\sigma^{-1} v$,

$$\mathcal{F} : \mathbb{R}^{N_\sigma} \to \mathbb{R}, \qquad \mathcal{F}(v) = \alpha \left(\sum_{k=1}^{N_\tau} \tau_k |v_k|_1^2 \right)^{1/2},$$

$$\mathcal{G} : \mathbb{R}^{N_\sigma} \to \mathbb{R}, \qquad \mathcal{G}(v) = \frac{1}{2} \sum_{k=1}^{N_\tau} \tau_k (v_k - y_{d,k})^T M_h (v_k - y_{d,k}),$$

and calculate the Fenchel conjugates with respect to the topology induced by the duality pairing (4.3). For $\mathcal{G}$, we have by direct calculation that

$$\begin{aligned} \mathcal{G}^*(q) &= \sup_{v \in \mathbb{R}^{N_\sigma}} \sum_{k=1}^{N_\tau} \tau_k q_k^T v_k - \frac{1}{2} \sum_{k=1}^{N_\tau} \tau_k (v_k - y_{d,k})^T M_h (v_k - y_{d,k}) \\ &= \frac{1}{2} \sum_{k=1}^{N_\tau} \tau_k \left((q_k + M_h y_{d,k})^T M_h^{-1} (q_k + M_h y_{d,k}) - y_{d,k}^T M_h y_{d,k} \right) \end{aligned}$$

since the supremum is attained if and only if $q_k = M_h(v_k - y_{d,k})$ for each $1 \leqslant k \leqslant N_\tau$ due to (6.1) and the definition of the duality pairing. For $\mathcal{F}$, we appeal to the fact that in any Banach space the Fenchel conjugate (with respect to the weak-$\star$ topology) of a norm is the indicator function of the unit ball with respect to the dual norm (see, e.g., [Schirotzek 2007, Example 2.2.6]), and to the duality between $\mathcal{U}_\sigma$ and $\mathcal{Y}_\sigma$, to obtain

$$\mathcal{F}^*(q) = \iota_\alpha(q) := \begin{cases} 0 & \text{if } \left(\sum_{k=1}^{N_\tau} \tau_k |q_k|_\infty^2 \right)^{1/2} \leqslant \alpha, \\ \infty & \text{otherwise.} \end{cases}$$

The adjoint $\Lambda^* : \mathbb{R}^{N_\sigma} \to \mathbb{R}^{N_\sigma}$ (with respect to the above duality pairing) is given by L_σ^{-T}. Dropping the constant term in $\mathcal{G}^*$ and substituting $p_\sigma = \Lambda^* q_\sigma$, i.e., $q_\sigma = L_\sigma^T p_\sigma$, we obtain the dual problem

$$\min_{p_\sigma \in \mathbb{R}^{N_\sigma}} \frac{1}{2} \sum_{k=1}^{N_\tau} \tau_k ([L_\sigma^T p_\sigma]_k - M_h y_{d,k})^T M_h^{-1} ([L_\sigma^T p_\sigma]_k - M_h y_{d,k}) + \iota_\alpha(p_\sigma). \tag{6.4}$$

Since $v_0 = 0 = \Lambda v_0$ satisfies the regular point condition, the Fenchel duality theorem is applicable, implying the existence of a solution $\bar{p}_\sigma$ which is unique due to the strict convexity in (6.4).

While the second relation of (6.3),

$$\tau_k(L_\sigma^T\bar{p}_\sigma)_k = \tau_k M_h(L_\sigma^{-1}\bar{u}_\sigma - y_{d,\sigma})_k \quad \text{for all} \quad 1 \leqslant k \leqslant N_\tau, \tag{6.5}$$

can in principle be used to obtain $\bar{u}_\sigma$ from $\bar{p}_\sigma$, the first relation remains impractical for numerical computation. We thus consider the following equivalent reformulation of (6.4), which decouples the spatio-temporal constraint given by the term $\iota_\alpha(p_\sigma)$:

$$\begin{cases} \min\limits_{p_\sigma\in\mathbb{R}^{N_\sigma}, c_\sigma\in\mathbb{R}^{N_\tau}} \dfrac{1}{2}\displaystyle\sum_{k=1}^{N_\tau} \tau_k([L_\sigma^T p_\sigma]_k - M_h y_{d,k})^T M_h^{-1}([L_\sigma^T p_\sigma]_k - M_h y_{d,k}) \\ \text{s.t.} \quad |p_k|_\infty \leqslant c_k \text{ for all } 1 \leqslant k \leqslant N_\tau \quad \text{and} \quad \displaystyle\sum_{k=1}^{N_\tau} \tau_k c_k^2 = \alpha^2, \end{cases}$$

where $c_\sigma = (c_1, \ldots, c_{N_\tau})^T \in \mathbb{R}^{N_\tau}$. Since the constraints satisfy a Slater condition (take $p_\sigma = 0$ and $c_k = T^{-1/2}\alpha$, $1 \leqslant k \leqslant N_\tau$), we obtain (e.g., from [Maurer and Zowe 1979]) existence of Lagrange multipliers $\mu_k^1, \mu_k^2 \in \mathbb{R}^{N_h}$, $1 \leqslant k \leqslant N_\tau$, and $\lambda \in \mathbb{R}$ such that the (unique) solution $(\bar{p}_\sigma, \bar{c}_\sigma)$ satisfies the optimality conditions

$$\begin{cases} \tau_k[L_\sigma M_\sigma^{-1}(L_\sigma^T\bar{p}_\sigma - M_\sigma y_{d,\sigma})]_k = \mu_k^1 + \mu_k^2, \quad 1 \leqslant k \leqslant N_\tau, \\ \displaystyle\sum_{j=1}^{N_h}(-\mu_{kj}^1 + \mu_{kj}^2) + 2\lambda\tau_k\bar{c}_k = 0, \quad 1 \leqslant k \leqslant N_\tau, \\ (\mu_k^1)^T(\bar{p}_k - \bar{c}_k) = 0, \quad (\mu_k^2)^T(\bar{p}_k + \bar{c}_k) = 0, \quad \mu_k^1 \leqslant 0,\ \mu_k^2 \geqslant 0, \quad 1 \leqslant k \leqslant N_\tau, \\ \displaystyle\sum_{k=1}^{N_\tau} \tau_k\bar{c}_k^2 - \alpha^2 = 0, \end{cases} \tag{6.6}$$

where $M_\sigma \in \mathbb{R}^{N_\sigma \times N_\sigma}$ is a block diagonal matrix containing N_τ copies of M_h.

We now rewrite the optimality system in a form amenable to the numerical solution using a semismooth Newton method. First, μ_k^1 and μ_k^2 are scaled by $\tau_k > 0$ to eliminate this factor from the first and second relation (which does not affect the complementarity conditions). Using the componentwise max and min functions, the complementarity conditions for μ_k^1, μ_k^2 and $\bar{p}_k$ can be expressed equivalently for any $\gamma > 0$ as

$$\mu_k^1 + \max(0, -\mu_k^1 + \gamma(\bar{p}_k - \bar{c}_k)) = 0, \qquad \mu_k^2 + \min(0, -\mu_k^2 + \gamma(\bar{p}_k + \bar{c}_k)) = 0.$$

Since $\mu_k^2 = 0$ if $\bar{p}_k > -\bar{c}_k$ and $\mu_k^1 = 0$ if $\bar{p}_k < \bar{c}_k$, we have by componentwise inspection

$$\max(0, -\mu_k^1 + \gamma(\bar{p}_k - \bar{c}_k)) = \max(0, -\mu_k^1 - \mu_k^2 + \gamma(\bar{p}_k - \bar{c}_k)).$$

We argue similarly for the min term. Furthermore, comparing the first relation of (6.6) with (6.5), we deduce that $\bar{u}_k = \mu_k^1 + \mu_k^2$ for all $1 \leqslant k \leqslant N_\tau$. Finally, to avoid having to form M_σ^{-1}, we introduce $\bar{y}_\sigma \in \mathbb{R}^{N_\sigma}$ satisfying

$$L_\sigma^T\bar{p}_\sigma = M_\sigma(\bar{y}_\sigma - y_{d,\sigma}).$$

Inserting these relations into (6.6), we obtain for every $\gamma > 0$ the optimality system

(6.7)

$$\begin{cases} L_\sigma \bar{y}_\sigma - \bar{u}_\sigma = 0, \\ L_\sigma^T \bar{p}_\sigma - M_\sigma(\bar{y}_\sigma - y_{d,\sigma}) = 0, \\ \bar{u}_k + \max(0, -\bar{u}_k + \gamma(\bar{p}_k - \bar{c}_k)) + \min(0, -\bar{u}_k + \gamma(\bar{p}_k + \bar{c}_k)) = 0, \quad 1 \leqslant k \leqslant N_\tau \\ \displaystyle\sum_{j=1}^{N_h} [-\max(0, -\bar{u}_k + \gamma(\bar{p}_k - \bar{c}_k)) + \min(0, -\bar{u}_k + \gamma(\bar{p}_k + \bar{c}_k))]_j + 2\lambda \bar{c}_k = 0, \\ \hfill 1 \leqslant k \leqslant N_\tau, \\ \displaystyle\sum_{k=1}^{N_\tau} \tau_k \bar{c}_k^2 - \alpha^2 = 0. \end{cases}$$

Since the max and min functions are globally Lipschitz mappings in finite dimensions, this defines a semismooth equation which can be solved using a generalized Newton method; see, e.g., [Qi and Sun 1993; Kummer 1992]. Here we recall that the Newton derivative of $\max(0, v)$ with respect to v is given componentwise by

$$[D_N \max(0, v)h]_k = \begin{cases} h_k & \text{if } v_k \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

and similarly that of $\min(0, v)$. In practice, we have to account for the possibly local convergence of the Newton method. To compute a suitable starting point, as an initialization step we successively solve a sequence of approximating problems that are obtained from (6.7) by replacing the max and min terms with

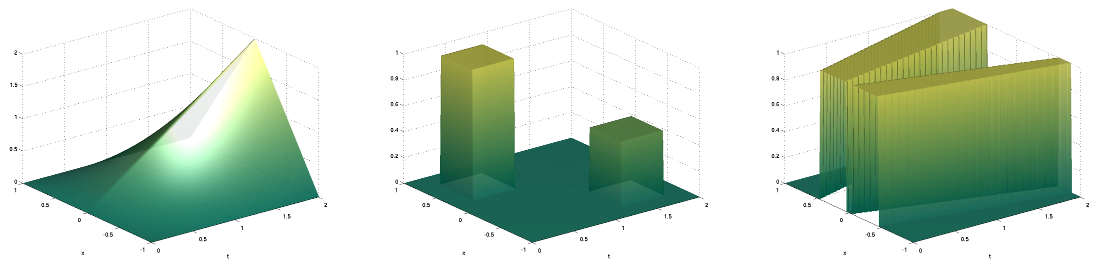
$$\max(0, \gamma(\bar{p}_k - \bar{c}_k)) \quad \text{and} \quad \min(0, \gamma(\bar{p}_k + \bar{c}_k)),$$

respectively, and letting γ tend to infinity. (This can be interpreted as a Moreau–Yosida regularization of the complementarity conditions.) Since now u_k no longer appears in the argument of the max and min functions, it can be eliminated from the optimality system using the third equation (which also allows computing $\bar{u}_k$ given $(\bar{p}_k, \bar{c}_k)$), yielding

(6.8)

$$\begin{cases} L_\sigma^T p_\gamma - M_\sigma(y_\gamma - y_{d,\sigma}) = 0, \\ L_\sigma y_\gamma + \gamma[\max(0, p_\gamma - c_\gamma) + \min(0, p_\gamma + c_\gamma)] = 0, \\ \displaystyle\sum_{j=1}^{N_h} \gamma\, [-\max(0, p_{\gamma,k} - c_{\gamma,k}) + \min(0, p_{\gamma,k} + c_{\gamma,k})]_j + 2\lambda_\gamma c_{\gamma,k} = 0, \quad 1 \leqslant k \leqslant N_\tau \\ \displaystyle\sum_{k=1}^{N_\tau} \tau_k c_{\gamma,k}^2 - \alpha^2 = 0. \end{cases}$$

Starting with $\gamma = 1$ and $p^0 = y^0 = 0$, $c^0 = T^{-1/2}\alpha$ and $\lambda^0 = 1$, we solve (6.8) using a semismooth Newton method, increase γ, and compute a new solution for increased γ with



(a) z_1 (b) z_2 (c) z_3

Figure 1: Targets for numerical experiments

the previous solution as starting point. Once a solution satisfies the constraints (or a stopping value γ^* is reached), we use it as a starting point for the solution of (6.7) with $\gamma = 1$.

Remark 6.1. By virtue of the chosen discretization (specifically, the adjoint consistency of discontinuous Galerkin methods and the discrete topology mirroring the continuous one), the discrete optimality system (6.8) coincides with the discretization of the continuous optimality system obtained by applying Fenchel duality, the relaxation approach and a Moreau–Yosida approximation to problem (P). Since the continuous optimality system may be of independent interest, the derivation is sketched in Appendix A.

7 NUMERICAL EXAMPLES

We illustrate the structure of the optimal controls with some one-dimensional examples. For this purpose we set $\Omega = (-1, 1)$, $T = 2$, $\nu = 10^{-1}$ and consider the state equation

$$\begin{cases} y_t - \nu\Delta y = u, \\ \qquad y(0) = 0, \end{cases}$$

with homogeneous Dirichlet conditions. The spatial domain is discretized using $N_h = 128$ uniformly distributed nodes (which corresponds to $h \approx 0.0156$). Following (4.2), we take $N_\tau = 1024$ time steps (which corresponds to $\tau \approx 0.00195$). The targets are chosen as (see Figure 1)

$$\begin{aligned} z_1 &= t(1 - |x|), \\ z_2 &= \begin{cases} 1 & \text{if } 0.25 \leqslant t \leqslant 0.75 \text{ and } 0.25 \leqslant x \leqslant 0.75, \\ \frac{1}{2} & \text{if } 1.25 \leqslant t \leqslant 1.75 \text{ and } -0.25 \geqslant x \geqslant -0.75, \\ 0 & \text{otherwise}, \end{cases} \end{aligned}$$

$$z_3 = \begin{cases} 1 & \text{if } |x - 0.25 - t/4| < (0.2 + t/20), \\ 1 & \text{if } |x + 0.25 + t/4| < (0.2 - t/20), \\ 0 & \text{otherwise.} \end{cases}$$

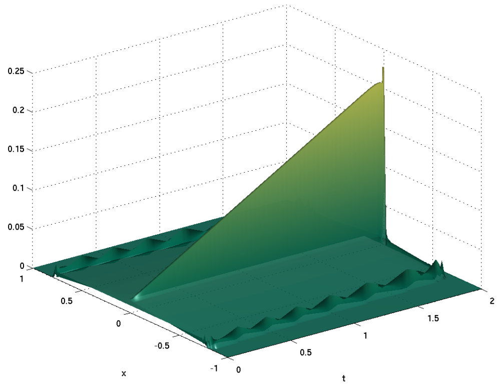
The semismooth Newton method for the solution of the optimality system (6.7) is implemented in MATLAB, where the initialization is calculated as discussed in section 6 with $\gamma_{k+1} = 10\gamma_k$ and $\gamma^* = 10^{12}$. For each target the optimal control is computed for $\alpha = 10^{-3}$ and $\alpha = 10^{-1}$. In every case, the discrete optimality system is solved to an accuracy below 10^{-12}, and the bounds on p_σ and on c_σ are attained within machine precision.

The respective optimal controls u_σ (in the form of linearly interpolated expansion coefficients u_{kj}), optimal states y_σ and bounds c_σ are shown in Figure 2–4. The predicted sparsity structure of the optimal controls can be observed clearly: The spatio-temporal coupling of the control cost predominantly promotes spatial sparsity; see Figure 3b in particular. The structural features of the norm $\|u\|_{L^2(\mathcal{M})}$ are further illustrated by the fact that larger values of α lead to both increased sparsity in space and increased smoothness in time. It is instructive to compare the optimal controls obtained with our $\|u\|_{L^2(\mathcal{M})}$ regularization to those obtained numerically using a (Moreau–Yosida approximation of a) $\mathcal{M}(\Omega_T)$ norm penalty term. Figure 5 shows the latter for all considered targets and values of α. While for $\alpha = 10^{-3}$ both types of control have comparable structure, for $\alpha = 10^{-1}$ the controls in $\mathcal{M}(\Omega_T)$ demonstrate strong temporal sparsity which is absent in the case of controls in $L^2(I, \mathcal{M}(\Omega))$.

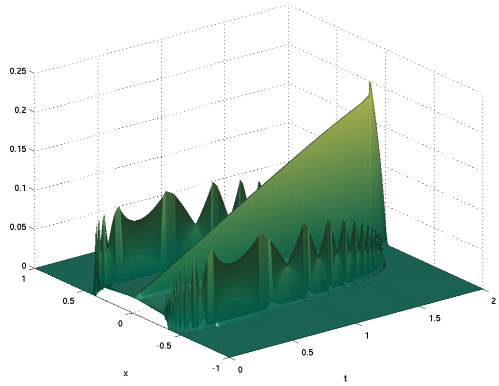
We now investigate the convergence behavior as $h \to 0$. In the absence of a known exact solution, we take as a reference solution the computed optimal discrete control and optimal discrete state on the finest grid with $N_{h^*} = 256$ and $N_{\tau^*} = 4096$, corresponding to $h^* \approx 0.00781$ and $\tau^* \approx 0.000488$. As a representative example, we consider the target z_1 and $\alpha = 0.1$. Figure 6a shows the difference $|J_h - J_{h^*}|$ for a series of successively refined grids with $N_h = 32, 40, \ldots, 128$ and $N_{\tau(h)} = \frac{1}{16}N_h^2$. The observed approximately linear convergence rate agrees with the rate obtained in Theorem 5.1. The corresponding L^2 error $\|y_h - y_{h^*}\|_{L^2}$ of the discrete states also decays with a linear rate, which is faster than predicted by Theorem 5.2. A similar behavior was observed in the elliptic case; see [Casas, Clason, et al. 2012].

8 CONCLUSION

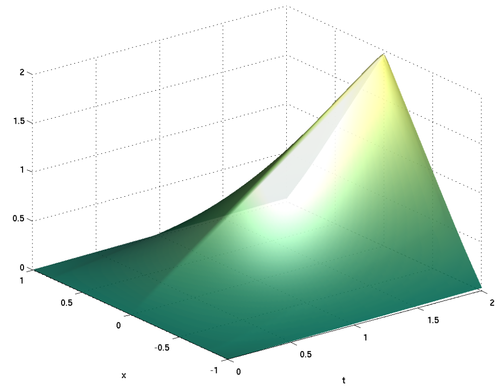
For the appropriate functional-analytic setting of parabolic optimal control problems in measure spaces, there exists a straightforward approximation framework that retains the structural properties of the norm in the measure-valued Banach space and allows deriving numerically accessible optimality conditions as well as convergence rates. In particular, although the state is discretized, the control problem is still formulated and solved in measure space. The numerical results demonstrate that the optimal controls exhibit the expected sparsity pattern.



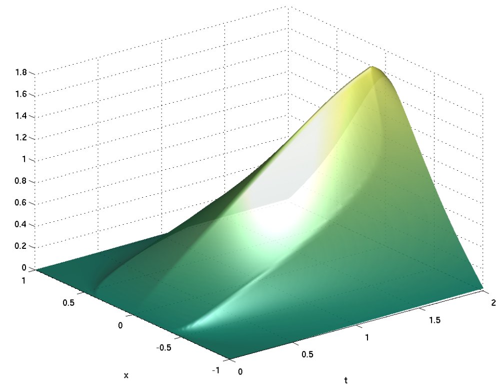
(a) u_σ



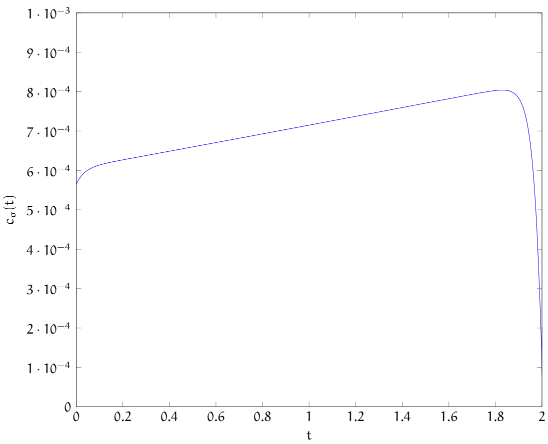
(b) u_σ



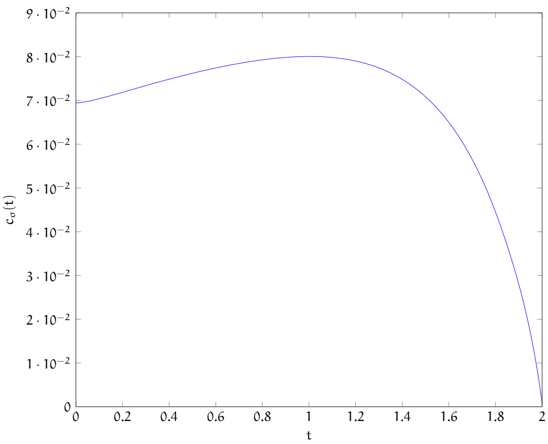
(c) y_σ



(d) y_σ

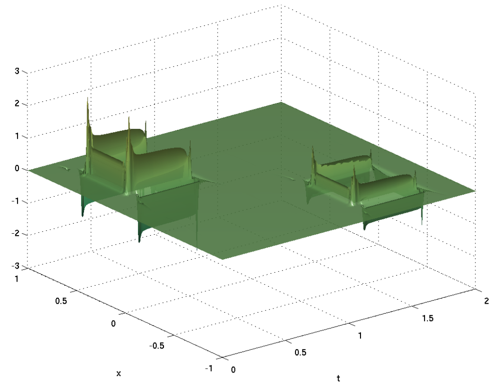


(e) c_σ

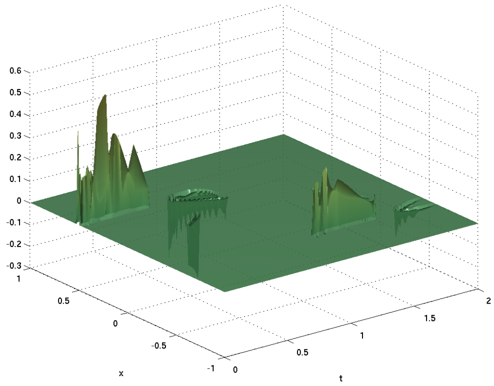


(f) c_σ

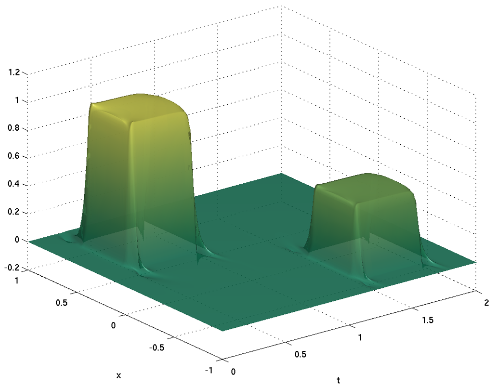
Figure 2: Optimal control u_σ, state y_σ and bound c_σ for target z_1 and $\alpha = 10^{-3}$ (left), $\alpha = 10^{-1}$ (right).



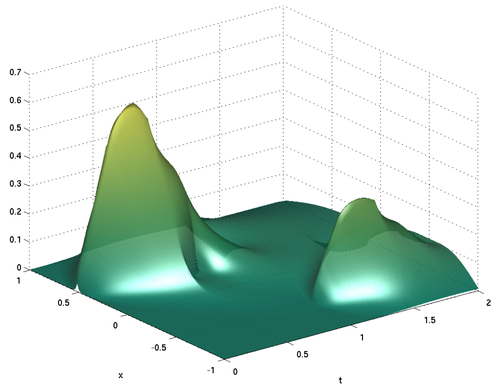
(a) u_σ



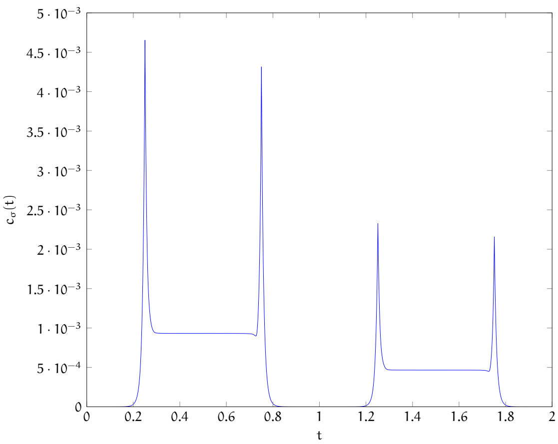
(b) u_σ



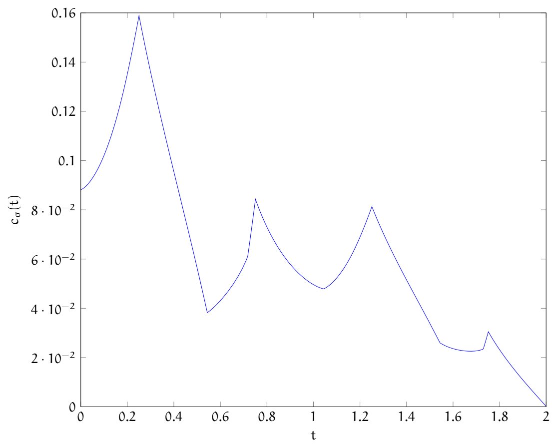
(c) y_σ



(d) y_σ

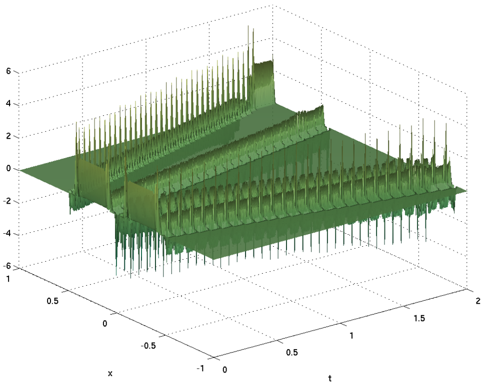


(e) c_σ

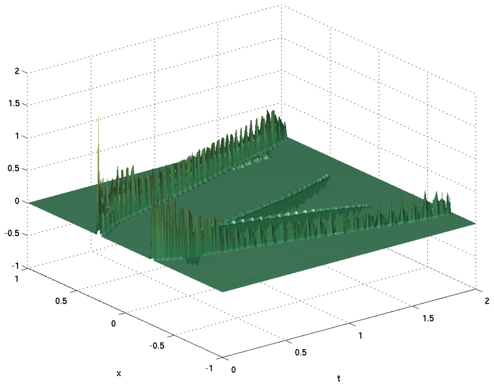


(f) c_σ

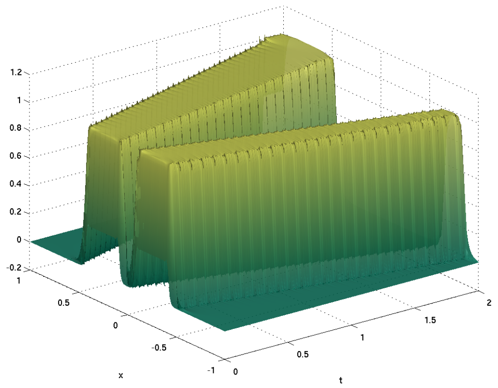
Figure 3: Optimal control u_σ, state y_σ and bound c_σ for target z_2 and $\alpha = 10^{-3}$ (left), $\alpha = 10^{-1}$ (right).



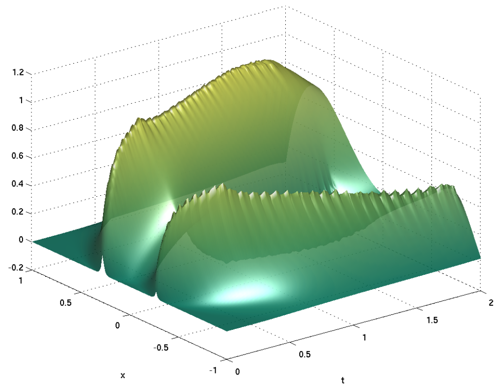
(a) u_σ



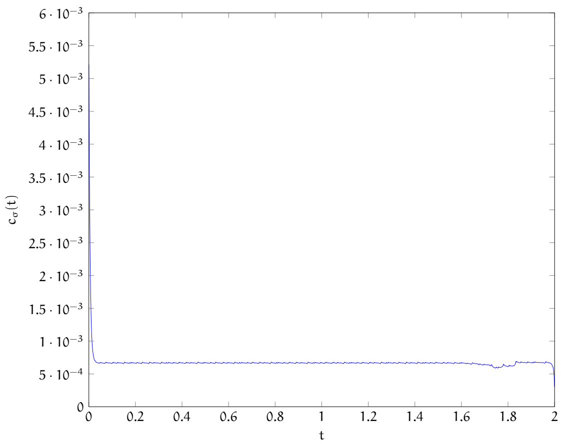
(b) u_σ



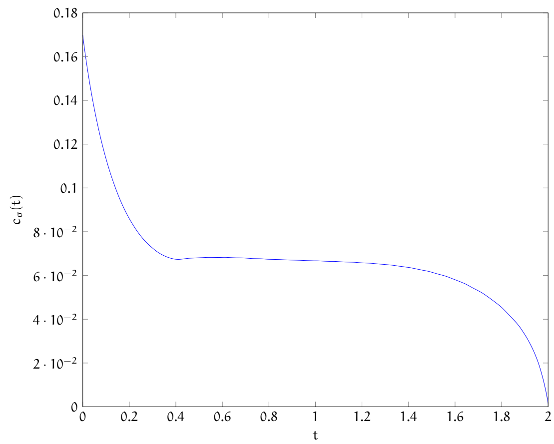
(c) y_σ



(d) y_σ

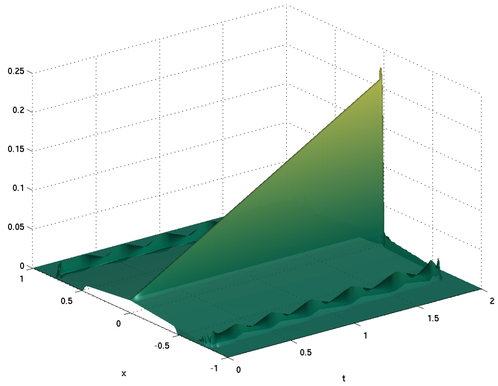


(e) c_σ

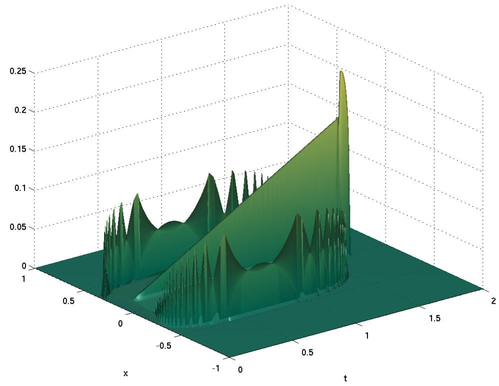


(f) c_σ

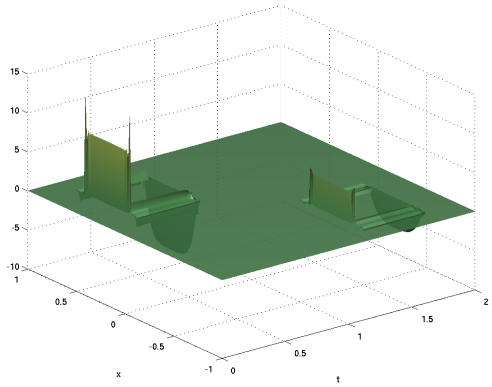
Figure 4: Optimal control u_σ, state y_σ and bound c_σ for target z_3 and $\alpha = 10^{-3}$ (left), $\alpha = 10^{-1}$ (right).



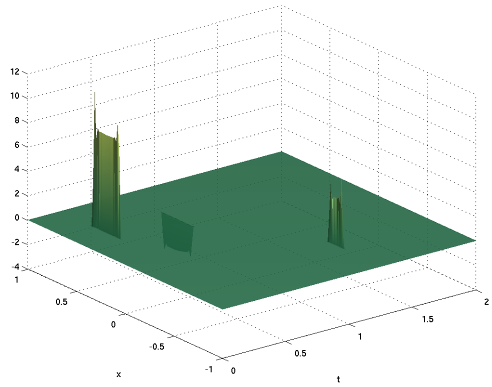
(a) target z_1, $\alpha = 10^{-3}$



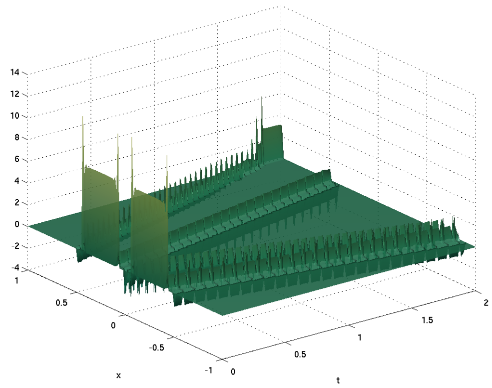
(b) target z_1, $\alpha = 10^{-1}$



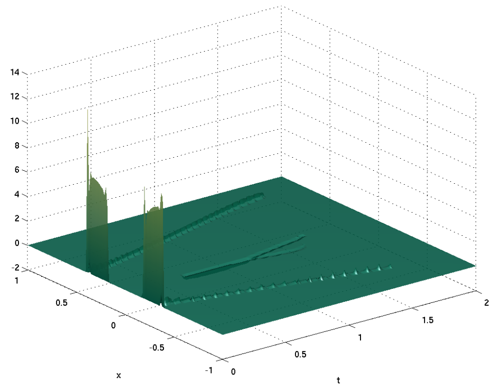
(c) target z_2, $\alpha = 10^{-3}$



(d) target z_2, $\alpha = 10^{-1}$

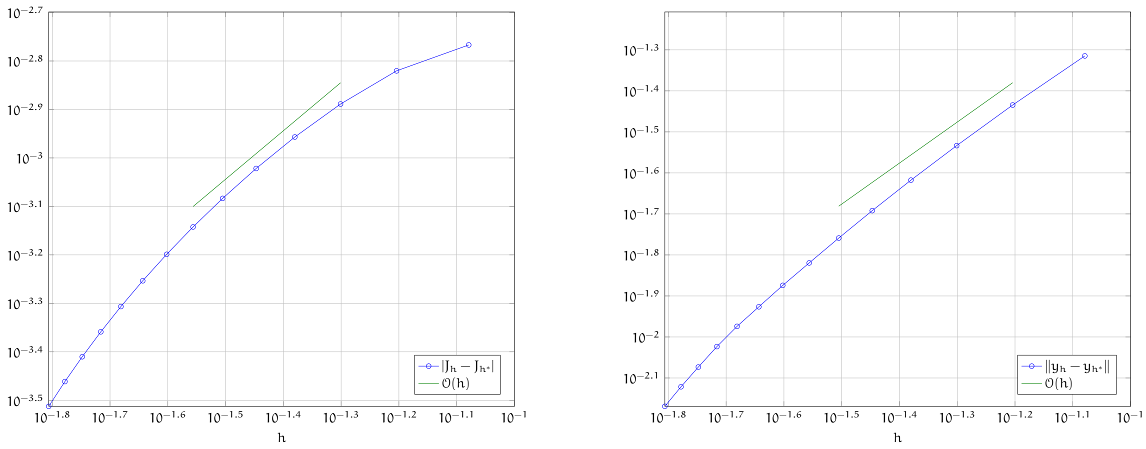


(e) target z_3, $\alpha = 10^{-3}$



(f) target z_3, $\alpha = 10^{-1}$

Figure 5: Optimal controls with $\mathcal{M}(\Omega_T)$ penalty.



(a) Functional value J_h

(b) L^2 norm of state y_h

Figure 6: Illustration of convergence order for target z_1 and $\alpha = 0.1$.

ACKNOWLEDGMENTS

The first author was supported by the Spanish Ministerio de Ciencia e Innovación under projects MTM2011-22711. The remaining authors were supported by the Austrian Science Fund (FWF) under grant SFB F32 (SFB "Mathematical Optimization and Applications in Biomedical Sciences").

APPENDIX A CONTINUOUS OPTIMALITY SYSTEM

In this section we sketch the derivation of the continuous optimality system using Fenchel duality and the relaxation approach. Let $S : L^2(I, \mathcal{M}(\Omega)) \to L^2(\Omega_T)$ denote the solution operator corresponding to the state equation (1.1) with homogeneous initial conditions. It will be convenient to introduce the parabolic differential operator L such that the solution y to (1.1) satisfies $Ly = u$. Then we can express problem (P) in reduced form as

$$\min_{u \in L^2(I,\mathcal{M}(\Omega))} \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega_T)} + \alpha\|u\|_{L^2(I,\mathcal{M}(\Omega))}.$$

To apply Fenchel duality, we set

$$\begin{aligned}
&\mathcal{F} : L^2(I, \mathcal{M}(\Omega)) \to \mathbb{R}, && \mathcal{F}(v) = \alpha\|v\|_{L^2(I,\mathcal{M}(\Omega))},\\
&\mathcal{G} : L^2(\Omega_T) \to \mathbb{R}, && \mathcal{G}(v) = \frac{1}{2}\|v - y_d\|^2_{L^2(\Omega_T)},\\
&\Lambda : L^2(I, \mathcal{M}(\Omega)) \to L^2(\Omega_T), && \Lambda u = Su.
\end{aligned}$$

Similarly to the discrete case, the Fenchel conjugates (with respect to the weak-$\star$ topology) are given by

$$\begin{aligned} &\mathcal{F}^* : L^2(I, C_0(\Omega)) \to \mathbb{R}, && \mathcal{F}^*(q) = \iota_\alpha(q) \\ &\mathcal{G}^* : L^2(\Omega_T) \to \mathbb{R}, && \mathcal{G}^*(q) = \frac{1}{2}\|q + y_d\|_{L^2(\Omega_T)}^2 - \frac{1}{2}\|y_d\|_{L^2(\Omega_T)}^2, \end{aligned}$$

where

$$\iota_\alpha(q) = \begin{cases} 0 & \text{if } \|q\|_{L^2(I,C_0(\Omega))} \leqslant \alpha, \\ \infty & \text{otherwise.} \end{cases}$$

Due to the definition of the solution to (1.1) via duality (see Definition 2.1), we obtain the existence of a weak-$\star$ adjoint operator $\Lambda^* := S^* : L^2(\Omega_T) \to L^2(I, C_0(\Omega))$ defined via the solution to (2.5). Furthermore, there exists a weak-$\star$ adjoint L^* of L such that for given $\psi_0 \in L^2(\Omega_T)$, the solution $z \in L^2(I, C_0(\Omega))$ of (2.5) satisfies $L^* z = \psi_0$. The dual problem is then found to be

$$\min_{q \in L^2(\Omega_T)} \frac{1}{2}\|q - y_d\|_{L^2(\Omega_T)}^2 + \iota_\alpha(S^* q).$$

We again substitute $p = S^* q \in L^2(I, C_0(\Omega))$, i.e., $q = L^* p$, introduce $c \in L^2(I)$ by

$$c(t) := \|p(t)\|_\infty \qquad \text{for a. e. } 0 \leqslant t \leqslant T,$$

and consider

$$\text{(A.1)} \qquad \begin{cases} \displaystyle\min_{p \in L^2(I,C_0(\Omega)), c \in L^2(I)} \frac{1}{2}\|L^* p - y_d\|_{L^2(\Omega_T)}^2 \\ \text{s. t.} \quad \|p(t)\|_\infty \leqslant c(t) \text{ for a. e. } 0 \leqslant t \leqslant T \\ \text{and} \quad \displaystyle\int_0^T c(t)^2 \, dt = \alpha^2. \end{cases}$$

The Moreau–Yosida regularization of (A.1) is given by

$$\begin{cases} \displaystyle\min_{p \in L^2(I,C_0(\Omega)), c \in L^2(I)} \frac{1}{2}\|L^* p - y_d\|_{L^2(\Omega_T)}^2 + \frac{\gamma}{2}\Big[\|\max(0, p - c)\|_{L^2(\Omega_T)}^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \|\min(0, p + c)\|_{L^2(\Omega_T)}^2\Big] \\ \text{s. t.} \quad \displaystyle\int_0^T c(t)^2 \, dt = \alpha^2, \end{cases}$$

where the max and min functions should be understood pointwise in Ω for almost every $0 \leqslant t \leqslant T$. Its solution is denoted by $(p_\gamma, c_\gamma) \in L^2(I, C_0(\Omega)) \times L^2(I)$. Since the cost functional is Fréchet differentiable and a Slater condition is again satisfied for the constraint on c (take $c = T^{-1/2}\alpha$), we obtain existence of a Lagrange multiplier $\lambda_\gamma \in \mathbb{R}$. Introducing once more y_γ satisfying $L^* p_\gamma = y_\gamma - y_d$, this yields the continuous optimality system

$$\begin{cases} L^* p_\gamma - (y_\gamma - y_d) = 0, \\ L y_\gamma + \gamma \max(0, p_\gamma - c_\gamma) + \gamma \min(0, p_\gamma + c_\gamma) = 0, \\ \gamma \displaystyle\int_\Omega - \max(0, p_\gamma - c_\gamma) + \min(0, p_\gamma + c_\gamma)\,dx + 2\lambda_\gamma c_\gamma = 0, \\ \displaystyle\int_0^T c_\gamma^2\,dt - \alpha^2 = 0. \end{cases}$$

By approximating p_γ and y_γ in $\mathcal{Y}_\sigma$, using the fact that for linear finite elements the pointwise maximum and minimum is attained at the nodes, and the adjoint consistency of discontinuous Galerkin methods (i.e., $(L^*)_\sigma = L_\sigma^T$), we recover (6.8).

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