Boundary Optimal Control of the Westervelt and the Kuznetsov equations☆

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Abstract

This paper is concerned with optimal Neumann boundary control for the Westervelt and the Kuznetsov equation, which are equations of nonlinear acoustics. Specifically, functionals of tracking type with applications in noninvasive ultrasonic medical treatments are considered. Existence of optimal controls is established and first order necessary optimality conditions are derived. Stability of the minimizer with respect to perturbations in the data as well as convergence of the controls when the regularization parameter tends to zero is shown.

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1. Introduction

High intensity focused ultrasound plays a role in several medical and industrial applications such as lithotripsy, thermotherapy, ultrasound cleaning or welding, and sonochemistry (see, e.g. [1, 2] and the references therein). Typically, one wishes to induce by boundary excitation an acoustic field such that the acoustic pressure in specified regions is sufficiently high in order to destroy unwanted inclusions (stones, tumors, deposits) or to create enough heat to initiate desired reactions. Everywhere else, the field should be at a strength well below a prescribed safety threshold. For the acoustic pressure necessary in these applications, a linear model of wave propagation such as the acoustic wave equation is no longer valid, and nonlinear effects have to be considered. This leads to boundary control problems for the Westervelt equation (formulated in terms of the acoustic pressure fluctuation u)

$$D_t^2 u - c^2\Delta u - b\Delta(D_t u) = \frac{\beta_a}{\rho c^2} D_t^2 u^2 \quad \text{in } (0,T)\times\Omega \tag{1}$$

or the Kuznetsov equation (formulated in terms of the acoustic velocity potential ψ)

$$D_t^2\psi - c^2\Delta\psi - b\Delta(D_t\psi) = D_t\left(\frac{\beta_a - 1}{c^2}(D_t\psi)^2 + |\nabla\psi|^2\right) \quad \text{in } (0,T)\times\Omega$$

modeling nonlinear acoustic wave propagation in a smooth bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \in \{1,2,3\}$. Here, $c > 0$ is the speed of sound, $b > 0$ the diffusivity of sound, $\rho > 0$ the mass density, and $\beta_a > 1$ the parameter of nonlinearity. Using the relation

$$\rho D_t\psi = u\,,$$

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we can formulate the Kuznetsov equation in terms of the acoustic pressure as well:

$$D_t^2 u - c^2 \Delta u - b\Delta(D_t u) = \frac{\beta_a - 1}{\rho c^2} D_t^2 u^2 + \frac{1}{\rho} D_t^2 |\nabla(\int_0^t u\, d\tau)|^2 \quad \text{in } (0,T)\times\Omega\,, \tag{2}$$

where we have set $\psi|_{t=0} = 0$.

For a derivation of the above models we refer to, e.g., [3, 4, 2, 5, 6]. Whereas the Kuznetsov equation is the more generally valid model, the Westervelt equation is technically somewhat simpler to treat from a mathematical point of view and therefore will be discussed first in this paper. However, note that we extend all results to the Kuznetsov case as well.

Usually, the acoustic waves are excited by a magnetomechanical or by a piezoelectric principle. We here concentrate on the latter case, where a two-dimensional array (often called mosaic) composed of a large number of separately controllable small piezoelectric transducers is used, see e.g., [1, 7]. The normal derivative of the acoustic pressure at the interface Γ is prescribed by the normal acceleration of the transducers. This allows to model the controlled ultrasound excitation by Neumann boundary conditions g in

$$\begin{cases} \dfrac{\partial u}{\partial n} = g & \text{on } (0,T)\times\Gamma \\ D_t u + c\dfrac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\hat{\Gamma} \\ u(0,\cdot) = u_0, D_t u(0,\cdot) = u_1 & \text{in } \Omega\,. \end{cases} \tag{3}$$

The absorbing boundary conditions on the rest $\hat{\Gamma} := \partial\Omega\backslash\Gamma$ of the boundary are used to avoid reflections on the artificial boundary of the computational domain, that is, to mimic an unbounded domain or rather an unknown outer boundary, which is actually the boundary of the patient to be treated by high intensity ultrasound.

To the authors' best knowledge, this is the first work on the practically relevant problem of optimal boundary control for a fully nonlinear acoustic wave equation. The central aim of the present paper is therefore to answer the question of existence and derive and rigorously justify first order necessary optimality conditions which can be used for the numerical computation of the optimal control. The main difficulty here is the proof of the differentiability of the reduced gradient, which relies on a careful well-posedness analysis of the nonlinear partial differential equations which describe the state, the adjoint state and the reduced gradient. Related works on optimal control for nonlinear wave equations include [8, 9] (distributed control), [10] (boundary control of semilinear equations) and [11] on coupled parabolic-hyperbolic and hyperbolic-hyperbolic systems.

The paper is organized as follows. In Section 2 we establish well-posedness of the problem of Neumann boundary optimal control for an appropriate class of cost functionals containing especially practically relevant tracking type functionals. The following Section 3 is devoted to the derivation and justification of necessary first order optimality conditions. Section 4 briefly discusses stability of the minimizer with respect to perturbations in the data as well as convergence of the controls when the regularization parameter tends to zero.

2. Existence of an optimal control

In this paper we establish well-posedness of an optimal control problem for the Westervelt (1) and the Kuznetsov (2) equations using a Neumann boundary control $g \in L^2((0,T)\times\Gamma)$ on $\Gamma \subseteq \partial\Omega$. So we want to minimize a regularized tracking type functional of the form

$$J_\alpha^{u^d}(g,u) = \frac{1}{2}J^1(u,u^d) + \frac{\alpha}{2}J^2(g)$$

with $\alpha > 0$ and u^d representing a desired pressure distribution, e.g.,

$$J_\alpha^{u^d}(g,u) = \frac{1}{2}\|u - u^d\|_{\mathcal{U}}^2 + \frac{\alpha}{2}\|g\|_{\mathcal{G}}^2 \tag{4}$$

with Banach space norms $\|\cdot\|^2_{\mathcal{U}}$, $\|\cdot\|^2_{\mathcal{G}}$ or

$$J_\alpha^{u^d}(g,u) = \frac{1}{2}\int_\Omega |u(T)-u^d|^2\,dx + \frac{\alpha}{2}\int_0^T\int_\Gamma |g|^2\,d\Gamma\,dt \tag{5}$$

over the class of admissible controls

$$G^{ad} = \{g \in G \,|\, \|g\|_G \le K \text{ and } g(0,\cdot) = \frac{\partial u_0}{\partial n} \text{ on } \Gamma\} \tag{6}$$

where

$$\begin{aligned} G &= \{g \in L^2((0,T)\times\Gamma) \,|\, \|g\|_G < \infty\}\,, \\ \|g\|_G^2 &= \|g\|^2_{H^1(0,T;H^{1/2}(\Gamma))} + \|D_t^2 g\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \end{aligned}$$

and $u = u(g)$ is the solution of the Westervelt or the Kuznetsov equation with g as Neumann boundary data.

Remark 1. The definition of the set G^{ad} is forced by well-posedness analysis of the state equations, i.e. the bounds on g are mandatory in order to guarantee existence of a solution. The equality constraint in (6) is a compatibility condition for the initial and boundary data in (3). In the following we will assume that $\frac{\partial u_0}{\partial n}$ is not too large as compared to K so that

$$\exists \hat{g} \in G: \quad \|\hat{g}\|_G < K \text{ and } \hat{g}(0,\cdot) = \frac{\partial u_0}{\partial n} \text{ on } \Gamma \tag{9}$$

which implies not only that G^{ad} is nonempty but also that the Slater condition is satisfied, which will be required later on to formulate first order optimality conditions.

With the abbreviation $k = \frac{\beta_a}{\rho c^2}$ for the Westervelt and $k = \frac{\beta_a - 1}{\rho c^2}$, $\gamma = \frac{2}{\rho}$ for the Kuznetsov equation, the weak forms of (1), (2) can be written as:

$$\left\{\begin{aligned} &\text{Find } u \in W \text{ such that} \\ &\int_0^T\int_\Omega (1-2ku)D_t^2 u w\,dx\,dt + \int_0^T\int_\Omega (c^2\nabla u + b\nabla D_t u)\nabla w\,dx\,dt + \int_0^T\int_{\hat\Gamma} (cD_t u + \frac{b}{c}D_t^2 u)w\,d\Gamma\,dt \\ &\qquad = 2k\int_0^T\int_\Omega (D_t u)^2 w\,dx\,dt + \int_0^T\int_\Gamma (c^2 g + bD_t g)w\,d\Gamma\,dt\,, \\ &u(0,\cdot) = u_0, \quad D_t u(0,\cdot) = u_1\,, \\ &\text{holds for all test functions } w \in L^2(0,T;H^1(\Omega)), \end{aligned}\right. \tag{10}$$

and

$$\left\{\begin{aligned} &\text{Find } u \in W \text{ such that} \\ &\int_0^T\int_\Omega (1-2ku)D_t^2 u w\,dx\,dt + \int_0^T\int_\Omega (c^2\nabla u + b\nabla D_t u)\nabla w\,dx\,dt + \int_0^T\int_{\hat\Gamma} (cD_t u + \frac{b}{c}D_t^2 u)w\,d\Gamma\,dt \\ &\qquad = 2k\int_0^T\int_\Omega (D_t u)^2 w\,dx\,dt \quad + \gamma\int_0^T\int_\Omega \left(\nabla(\int_0^t u\,d\tau)\nabla D_t u + |\nabla u|^2\right) w\,dx\,dt + \int_0^T\int_\Gamma (c^2 g + bD_t g)w\,d\Gamma\,dt \\ &u(0,\cdot) = u_0, \quad D_t u(0,\cdot) = u_1\,, \\ &\text{holds for all test functions } w \in L^2(0,T;H^1(\Omega)), \end{aligned}\right. \tag{11}$$

respectively. For these weak forms to be well defined, we fix

$$W = \{u \in L^\infty((0,T)\times\Omega) \,|\, u, D_t u \in H^1((0,T)\times\Omega)\}$$

in the Westervelt case and

$$W = \{u \in L^\infty((0,T)\times\Omega) \,|\, u, D_t u \in H^1((0,T)\times\Omega)\} \cap L^2(0,T;W^{1,12/5}(\Omega))$$

in the Kuznetsov case. Moreover, note that the Neumann boundary condition $\frac{\partial u}{\partial n} = g$ is equivalent to $c^2\frac{\partial u}{\partial n} + bD_t\frac{\partial u}{\partial n} = c^2 g + D_t g$ by uniqueness of the solution to the initial value problem for the ordinary differential equation $c^2 y + by' = f$ on $[0, T]$ together with the compatibility condition $g(0,\cdot) = \frac{\partial u_0}{\partial n}$ on Γ.

An analysis of the well-posedness of (1), (2) with (3) gives us existence and uniqueness of solutions to these initial-boundary value problems (see [12]). More precisely, we define

$$\begin{aligned}\mathcal{W} := \Big\{u \in L^\infty((0,T)\times\Omega) \,|\, &\|u\|_{L^\infty((0,T)\times\Omega)} \le m,\\ &c^2\|\Delta u\|_{L^2(0,T;L^2(\Omega))} \le \sqrt[4]{|\Omega|}\bar{a},\\ &\|D_t^2 u\|_{L^2(0,T;L^4(\Omega))} \le \bar{a},\\ &\|D_t u\|_{L^\infty(0,T;L^4(\Omega))} \le \bar{a}\sqrt{T}\Big\}\end{aligned}$$

and

$$\|u\|_{\mathcal{W}} := \max\{\|\Delta u\|_{L^\infty(0,T;L^2(\Omega))}, \|D_t^2 u\|_{L^\infty(0,T;L^2(\Omega))}, \|\nabla D_t u\|_{L^\infty(0,T;L^2(\Omega))}, \|\nabla D_t^2 u\|_{L^2(0,T;L^2(\Omega))}\}.$$

and show

Proposition 1. *Let $g \in G^{ad}$ with $\|g\|_G$, $\|u_0\|_{H^2(\Omega)}$, $\|u_1\|_{H^2(\Omega)}$ sufficiently small. Then there exist $T > 0$, $\bar{a} > 0$, $m < \frac{1}{2k}$ such that there exists a solution $u \in \mathcal{W}$ of (1) (or of (2)), which is uniquely determined by (10) (or by (11)) and satisfies*

$$\|u\|_{\mathcal{W}} \le C(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^2(\Omega)} + \|g\|_G)\,. \tag{12}$$

Remark 2. The existence proof is based on the Banach contraction mapping principle as well as appropriate Sobolev embedding results to preserve strict positivity of the "coefficient" $(1 - 2ku)$ of the second time derivative in order to avoid degeneracy. Note that the use of monotonicity (in place of contraction) arguments is excluded by the fact that the nonlinearities are of quadratic type and therefore the correspondig operators would not be monotone. Moreover, we would also expect compactness arguments to fail because of the lack of smoothing properties of the wave equation as well as the relatively high order of differentiation on the nonlinearities. As a matter of fact, in view of the Sobolev smoothness that we need for obtaining the L^∞ bound on u to control $(1-2ku)$ and for handling the remaining nonlinearities, the regularity assumptions we work with seem to be indispensable.

We mention in passing that we expect a global in time existence result analogous to Corollary 1 in [12] to hold for both the Westervelt and the Kuznetsov case, however its proof would be too technical to be presented in this paper.

Proof. Consider, like in [12], the fixed point operator $\mathcal{T}$ mapping $v \in \mathcal{W}$ to the solution u of

$$\begin{cases}(1-2kv)D_t^2 u - c^2\Delta u - b\Delta(D_t u) = 2kD_t u D_t v & \text{in } (0,T)\times\Omega\\ \dfrac{\partial u}{\partial n} = g & \text{on } (0,T)\times\Gamma\\ D_t u + c\dfrac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\hat{\Gamma}\\ u(0,\cdot) = u_0, \quad D_t u(0,\cdot) = u_1 & \text{in } \Omega\end{cases} \tag{13}$$

in the Westervelt case and of

$$\begin{cases}(1-2kv)D_t^2 u - c^2\Delta u - b\Delta(D_t u) = 2kD_t u D_t v + \dfrac{2}{\rho}\nabla v\nabla u + \dfrac{2}{\rho}\nabla(\displaystyle\int_0^t v\,d\tau)\nabla D_t u & \text{in } (0,T)\times\Omega\\ \dfrac{\partial u}{\partial n} = g \quad \text{on } (0,T)\times\Gamma\\ D_t u + c\dfrac{\partial u}{\partial n} = 0 \quad \text{on } (0,T)\times\hat{\Gamma}\\ u(0,\cdot) = u_0, \quad D_t u(0,\cdot) = u_1 \quad \text{in } \Omega\end{cases} \tag{14}$$

in the Kuznetsov case, respectively. Testing (13) and (14) with Δu and $-\Delta D_t u$, we get

$$\begin{aligned}
&\max\left\{c^2\|\Delta u\|^2_{L^2(0,T;L^2(\Omega))}, \frac{b}{2}\|\Delta u\|^2_{L^\infty(0,T;L^2(\Omega))}\right\}\\
&\le (1+2km)\|D_t^2 u\|_{L^2(0,T;L^2(\Omega))}\cdot\|\Delta u\|_{L^2(0,T;L^2(\Omega))}\\
&+2k\|D_t v\|_{L^2(0,T;L^4(\Omega))}\cdot\|D_t u\|_{L^\infty(0,T;L^4(\Omega))}\cdot\|\Delta u\|_{L^2(0,T;L^2(\Omega))}\\
&+\frac{b}{2}\|\Delta u_0\|^2_{L^2(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
&\max\left\{\frac{c^2}{2}\|\Delta u\|^2_{L^\infty(0,T;L^2(\Omega))}, b\|\Delta D_t u\|^2_{L^2(0,T;L^2(\Omega))}\right\}\\
&\le (1+2km)\|D_t^2 u\|_{L^2(0,T;L^2(\Omega))}\cdot\|\Delta D_t u\|_{L^2(0,T;L^2(\Omega))}\\
&+2k\|D_t v\|_{L^2(0,T;L^4(\Omega))}\cdot\|D_t u\|_{L^\infty(0,T;L^4(\Omega))}\cdot\|\Delta D_t u\|_{L^2(0,T;L^2(\Omega))}\\
&+\frac{c^2}{2}\|\Delta u_0\|^2_{L^2(\Omega)}\\
&+\gamma\|\nabla v\|_{L^2(0,T;L^4(\Omega))}\cdot\|\nabla u\|_{L^\infty(0,T;L^4(\Omega))}\cdot\|\Delta D_t u\|_{L^2(0,T;L^2(\Omega))}\\
&+\gamma\sqrt{T}\|\nabla v\|_{L^2(0,T;L^4(\Omega))}\cdot\|\nabla D_t u\|_{L^2(0,T;L^4(\Omega))}\cdot\|\Delta D_t u\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

respectively. Differentiating (13), (14) with respect to t as well as testing the resulting partial differential equation with $D_t^2 u$, yields

$$\begin{aligned}
&\max\left\{\frac{1-2km}{2}\|D_t\tilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))}, \frac{c^2}{2}\|\nabla\tilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))}, b\|\nabla(D_t\tilde{u})\|^2_{L^2(0,T;L^2(\Omega))} \quad c\|D_t\tilde{u}\|^2_{L^2(0,T;L^2(\hat{\Gamma}))}, \frac{b}{2c}\|D_t\tilde{u}\|^2_{L^\infty(0,T;L^2(\hat{\Gamma}))}\right\}\\
&\le 2k\|D_t^2 v\|_{L^2(0,T;L^2(\Omega))}\cdot\|\tilde{u}\|_{L^\infty(0,T;L^4(\Omega))}\cdot\|D_t\tilde{u}\|_{L^2(0,T;L^4(\Omega))}\\
&+3k\|D_t v\|_{L^\infty(0,T;L^2(\Omega))}\cdot\|D_t\tilde{u}\|^2_{L^2(0,T;L^4(\Omega))}\\
&+\frac{1+2km}{2}\|D_t^2 u(0)\|_{L^2(\Omega)}+\frac{c^2}{2}\|\nabla u_1\|_{L^2(\Omega)}\\
&+c^2\|D_t g\|_{L^2(0,T;H^{-1/2}(\Gamma))}\|D_t\tilde{u}\|_{L^2(0,T;H^{1/2}(\Gamma))}\\
&+b\|D_t^2 g\|_{L^2(0,T;H^{-1/2}(\Gamma))}\|D_t\tilde{u}\|_{L^2(0,T;H^{1/2}(\Gamma))}
\end{aligned}$$

and

$$\begin{aligned}
&\max\left\{\frac{1-2km}{2}\|D_t\tilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))}, \frac{c^2}{2}\|\nabla\tilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))}, b\|\nabla(D_t\tilde{u})\|^2_{L^2(0,T;L^2(\Omega))}, c\|D_t\tilde{u}\|^2_{L^2(0,T;L^2(\hat{\Gamma}))}, \frac{b}{2c}\|D_t\tilde{u}\|^2_{L^\infty(0,T;L^2(\hat{\Gamma}))}\right\}\\
&\le 2k\|D_t^2 v\|_{L^2(0,T;L^2(\Omega))}\cdot\|\tilde{u}\|_{L^\infty(0,T;L^4(\Omega))}\cdot\|D_t\tilde{u}\|_{L^2(0,T;L^4(\Omega))}\\
&+3k\|D_t v\|_{L^\infty(0,T;L^2(\Omega))}\cdot\|D_t\tilde{u}\|^2_{L^2(0,T;L^4(\Omega))}\\
&+\frac{1+2km}{2}\|D_t^2 u(0)\|_{L^2(\Omega)}+\frac{c^2}{2}\|\nabla u_1\|_{L^2(\Omega)}\\
&+2\gamma\|\nabla v\|_{L^2(0,T;L^4(\Omega))}\cdot\|\nabla\tilde{u}\|_{L^\infty(0,T;L^2(\Omega))}\cdot\|D_t\tilde{u}\|_{L^2(0,T;L^4(\Omega))}\\
&+\gamma\|D_t v\|_{L^\infty(0,T;L^4(\Omega))}\Big(\|\Delta u\|_{L^2(0,T;L^2(\Omega))}\|D_t\tilde{u}\|_{L^2(0,T;L^4(\Omega))}+\|\nabla u\|_{L^2(0,T;L^4(\Omega))}\cdot\|\nabla D_t\tilde{u}\|_{L^2(0,T;L^2(\Omega))}\Big)\\
&+\gamma\sqrt{T}\|\nabla v\|_{L^2(0,T;L^4(\Omega))}\cdot\|\nabla D_t\tilde{u}\|_{L^2(0,T;L^2(\Omega))}\cdot\|D_t\tilde{u}\|_{L^2(0,T;L^4(\Omega))}\\
&+c^2\|D_t g\|_{L^2(0,T;H^{-1/2}(\Gamma))}\|D_t\tilde{u}\|_{L^2(0,T;H^{1/2}(\Gamma))}\\
&+b\|D_t^2 g\|_{L^2(0,T;H^{-1/2}(\Gamma))}\|D_t\tilde{u}\|_{L^2(0,T;H^{1/2}(\Gamma))}
\end{aligned}$$

respectively for $\tilde{u} = D_t u$. Using the following Sobolev embeddings, trace theorem, and elliptic estimates

$$\begin{aligned} \|f\|_{L^4(\Omega)} &\leq \hat{K}_1 \|f\|_{H^1(\Omega)}\,, \quad f \in H^1(\Omega) \\ \|f\|_{L^\infty(\Omega)} &\leq \hat{K}_2 \|f\|_{H^2(\Omega)}\,, \quad f \in H^2(\Omega) \\ \|f\|_{H^{1/2}(\Gamma)} &\leq \hat{K}_3 \|f\|_{H^1(\Omega)}\,, \quad f \in H^1(\Omega) \\ \|f\|_{H^2(\Omega)} &\leq \hat{K}_4 \left(\|\Delta f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\frac{\partial f}{\partial n}\|_{H^{1/2}(\partial\Omega)} \right), \quad f \in H^2(\Omega)\,, \end{aligned}$$

we can therefore conclude that $\mathcal{T}$ is a self-mapping on $\mathcal{W}$ provided $\|g\|_G$, $\|u_0\|_{H^2(\Omega)}$, $\|u_1\|_{H^2(\Omega)}$ and T are sufficiently small.

In the contractivity estimates of $\mathcal{T}$ on $\mathcal{W}$, we only get additional nonnegative terms on the left-hand side due to the absorbing boundary conditions as compared to [12], so that the estimates stated there remain valid.

Uniqueness follows similarly to the contractivity proofs for the fixed point operators in the proofs of Theorems 3, 5 in [12], so we only show it for the Westervelt case here and refer to the proof of Theorem 5 in [12] for the Kuznetsov case: Let $u^{(1)}, u^{(2)}$ be solutions of (10), then $\hat{u} = u^{(1)} - u^{(2)}$ satisfies

$$\begin{cases} \text{Find } \hat{u} \in W \text{ such that} \\ \int_0^T \int_\Omega D_t^2\Big((1 - k(u^{(1)} + u^{(2)}))\hat{u}\Big) w\, dx\, dt + \int_0^T \int_\Omega (c^2 \nabla \hat{u} + b \nabla D_t \hat{u}) \nabla w\, dx\, dt + \int_0^T \int_{\hat{\Gamma}} (c D_t \hat{u} + \frac{b}{c} D_t^2 \hat{u}) w\, d\Gamma\, dt = 0\,, \\ \hat{u}(0,\cdot) = 0, \quad D_t \hat{u}(0,\cdot) = 0\,, \\ \text{holds for all test functions } w \in L^2(0,T;H^1(\Omega)). \end{cases} \tag{19}$$

With $w = \chi_{(0,t)} D_t \hat{u} \in L^2(0,T;H^1(\Omega))$ and

$$D_t^2\Big((1 - k(u^{(1)} + u^{(2)}))\hat{u}\Big) D_t \hat{u} = \frac{1}{2} D_t \Big((1 - k(u^{(1)} + u^{(2)}))(D_t \hat{u})^2\Big) - k\Big(D_t^2(u^{(1)} + u^{(2)})\hat{u} + \frac{3}{2} D_t(u^{(1)} + u^{(2)}) D_t \hat{u}\Big) D_t \hat{u}$$

we get

$$\begin{aligned} &\frac{1}{2}\int_0^t D_t \int_\Omega \Big((1 - k(u^{(1)} + u^{(2)}))(D_t\hat{u})^2\Big)\, dx\, dt \\ &\quad + \frac{c^2}{2}\int_0^t D_t \int_\Omega |\nabla \hat{u}|^2\, dx\, dt + b \int_0^t \int_\Omega |\nabla D_t \hat{u}|^2\, dx\, dt + c \int_0^t \int_{\hat{\Gamma}} (D_t \hat{u})^2\, d\Gamma\, dt + \frac{b}{c}\int_0^t D_t \int_{\hat{\Gamma}} (D_t\hat{u})^2\, d\Gamma\, dt \\ &\qquad = k \int_0^t \int_\Omega \Big(D_t^2(u^{(1)} + u^{(2)})\hat{u} D_t \hat{u} + \frac{3}{2} D_t(u^{(1)} + u^{(2)})(D_t\hat{u})^2\Big)\, dx\, dt \end{aligned}$$

hence, by taking the supremum over $t \in [0,T]$ and making use of the fact that $u^{(1)}, u^{(2)} \in \mathcal{W}$, we get

$$\begin{aligned} &\max\left\{ \frac{1 - 2km}{2}\|D_t\hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))}, \frac{c^2}{2}\|\nabla\hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))}, b\|\nabla D_t\hat{u}\|^2_{L^2(0,T;L^2(\Omega))}, c\|D_t\hat{u}\|^2_{L^2(0,T;L^2(\hat{\Gamma}))}, \frac{b}{2c}\|D_t\hat{u}\|^2_{L^\infty(0,T;L^2(\hat{\Gamma}))} \right\} \\ &\quad \leq 2k\bar{a}\|\hat{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\|D_t\hat{u}\|_{L^2(0,T;L^4(\Omega))} + 3k\sqrt{T}\sqrt[4]{|\Omega|}\bar{a}\|D_t\hat{u}\|^2_{L^2(0,T;L^4(\Omega))} \leq 5k\sqrt{T}\sqrt[4]{|\Omega|}\bar{a}\|D_t\hat{u}\|^2_{L^2(0,T;L^4(\Omega))} \end{aligned} \tag{20}$$

and therewith by $\|D_t\hat{u}\|_{L^2(0,T;L^2(\Omega))} \leq \sqrt{T}\|D_t\hat{u}\|_{L^\infty(0,T;L^2(\Omega))}$ and $\max\{c_A A, c_B B\} \geq \frac{1}{2}\min\{c_A, c_B\}(A + B)$

$$\frac{1}{2}\min\left\{\frac{1 - 2km}{2T}, b\right\}\|D_t\hat{u}\|^2_{L^2(0,T;H^1(\Omega))} \leq 5k\sqrt{T}\sqrt[4]{|\Omega|}\bar{a}\hat{K}_1^2\|D_t\hat{u}\|^2_{L^2(0,T;H^1(\Omega))}\,,$$

which implies $\hat{u} = 0$ for T sufficiently small so that

$$10k\sqrt{T}\sqrt[4]{|\Omega|}\bar{a}\hat{K}_1^2 < \min\left\{\frac{1 - 2km}{2T}, b\right\}\,.$$

For the Kuznetsov case the main necessary technical modifications as compared to [12] arise due to the fact that we consider Neumann instead of Dirichlet boundary conditions here, so that we have

$$\begin{aligned}
\|f\|_{L^4(\Omega)} &\leq \hat{K}_1\|f\|_{H^1(\Omega)} && \text{in place of} \quad \|f\|_{L^4(\Omega)} \leq K_1\|\nabla f\|_{L^2(\Omega)} \\
\|f\|_{L^\infty(\Omega)} &\leq \hat{K}_2\left(\|\Delta f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\frac{\partial f}{\partial n}\|_{H^{1/2}(\Gamma)}\right) && \text{in place of} \quad \|f\|_{L^\infty(\Omega)} \leq K_2\|\Delta f\|_{L^2(\Omega)} \\
\|\nabla f\|_{L^4(\Omega)} &\leq \hat{K}_5\left(\|\Delta f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\frac{\partial f}{\partial n}\|_{H^{1/2}(\Gamma)}\right) && \text{in place of} \quad \|\nabla f\|_{L^4(\Omega)} \leq K_5\|\Delta f\|_{L^2(\Omega)}\,.
\end{aligned} \tag{21}$$

The additional $\|f\|_{L^2(\Omega)}$ terms in (21) with $\hat{u}(t)$ inserted in place of f can be dominated by means of the obtained estimates for $\|D_t^2\hat{u}\|_{L^\infty(0,T;L^2(\Omega)}$, using the fact that $\|\hat{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq 4T^2\|D_t^2\hat{u}\|_{L^\infty(0,T;L^2(\Omega))}$. □

Therewith the control-to-state mapping

$$S : G^{ad} \to \mathcal{W}\,, \quad g \mapsto S(g) = u = u(g) \text{ solving (10)/(11)} \tag{22}$$

is well defined and we can write the boundary optimal control problem either as

$$\text{Find } (g^*, u^*) \in M \text{ s.t. } J_\alpha^{u^d}(g^*, u^*) = \inf_{(g,u)\in M} J_\alpha^{u^d}(g, u)$$

with

$$M = \{(g, u) \in G^{ad} \times W \,|\, u \text{ solves (10)/(11)}\}$$

or in its reduced form with

$$j(g) = J_\alpha^{u^d}(g, S(g))$$

as

$$\text{Find } g^* \in G^{ad} \text{ s.t. } j(g^*) = \inf_{g\in G^{ad}} j(g)$$

In order to be able to prove existence of a minimizer, we first of all show weak sequential closedness of the mapping S:

Lemma 2. *Let $\mathcal{U}$ and $\mathcal{G}$ be Banach spaces such that*

$$G^{ad} \textit{ is closed with respect to the weak}^* \textit{ topology on } \mathcal{G}\,. \tag{23}$$

Then S is weakly sequentially closed as a mapping from $\mathcal{G}$ to $\mathcal{U}$ in the sense that for any sequence $\{g_m\}_{m\in\mathbb{N}} \subseteq G^{ad}$

$$\left(g_m \overset{*}{\rightharpoonup} g^* \text{ in } \mathcal{G} \wedge S(g_m) \overset{*}{\rightharpoonup} u^* \text{ in } \mathcal{U}\right) \Rightarrow \left(g^* \in G^{ad} \wedge S(g^*) = u^*\right).$$

Proof. The implication $g_m \overset{*}{\rightharpoonup} g^* \Rightarrow g^* \in G^{ad}$ follows directly from the assumption (23). To show $S(g^*) = u^*$, we use the fact that $\{g_m\}_{m\in\mathbb{N}}$ and $\{u_m\}_{m\in\mathbb{N}}$ are uniformly bounded in G and U (denoting the Banach space induced by the norm $\|\cdot\|_U$ from (12)) by the constant K from (6) and — according to (12) — some $\bar{C}$, respectively. According to the Banach Alaoglu Theorem, $\mathcal{B}_K(0)^G \times \mathcal{B}_{\bar{C}}(0)^U$ is weak* compact in $G \times U$ since the latter is the dual of a normed vector space, see, e.g., Theorem 8.18.3 in [13]. Hence, there exists a subsequence, denoted by $\{g_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$, and

$g^* \in G^{ad}$, $u^* \in \mathcal{B}_{\tilde{C}}(0)^U$, such that

$$\begin{aligned}
u_n &\to u^* \text{ in } L^\infty(0,T;L^2(\Omega))\\
D_t u_n &\overset{*}{\rightharpoonup} D_t u^* \text{ in } L^\infty(0,T;L^2(\Omega))\\
D_t^2 u_n &\rightharpoonup D_t^2 u^* \text{ in } L^2((0,T)\times\Omega))\\
D_t^2 u_n &\text{ uniformly bounded in } L^2(0,T;L^4(\Omega))\\
\nabla u_n &\rightharpoonup \nabla u^* \text{ in } L^2((0,T)\times\Omega)\\
\nabla D_t u_n &\rightharpoonup \nabla D_t u^* \text{ in } L^2((0,T)\times\Omega)\\
(D_t u_n)^2 &\rightharpoonup (D_t u^*)^2 \text{ in } L^2((0,T)\times\Omega)\\
\nabla(\int_0^t u_n\, d\tau) &\to \nabla(\int_0^t u^*\, d\tau) \text{ in } L^4((0,T)\times\Omega)\\
\nabla u_n &\to \nabla u^* \text{ in } L^4((0,T)\times\Omega)\\
g_n &\rightharpoonup g^* \text{ in } L^2(0,T;H^{-1/2}(\Gamma))\\
D_t g_n &\rightharpoonup D_t g^* \text{ in } L^2(0,T;H^{-1/2}(\Gamma))\,,
\end{aligned}$$

where we have used continuity and compactness of the embedding $H^1((0,T)\times\Omega)\to L^4((0,T)\times\Omega)$ together with the estimate

$$\begin{aligned}
\|D_t u_n\|^2_{H^1((0,T)\times\Omega)} &= \|D_t^2 u_n\|^2_{L^2((0,T)\times\Omega)} + \|\nabla D_t u_n\|^2_{L^2((0,T)\times\Omega)} + \|D_t u_n\|^2_{L^2((0,T)\times\Omega)}\\
&\le 2T\|D_t^2 u_n\|^2_{L^\infty(0,T;L^2(\Omega))} + T\|\nabla D_t u_n\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_1\|_{L^2(\Omega)} \le \tilde{C}\,,
\end{aligned}$$

$$\|\nabla u_n\|^2_{H^1((0,T)\times\Omega)} \le C\left(\|\nabla D_t u_n\|^2_{L^2((0,T)\times\Omega)} + \|\Delta u_n\|^2_{L^2((0,T)\times\Omega)} + \|\nabla u_n\|^2_{L^2((0,T)\times\Omega)}\right) \le \tilde{C}\,,$$

and similarly for $\|\nabla(\int_0^t u_n\, d\tau)\|^2_{H^1((0,T)\times\Omega)}$, as well as compactness of the embedding $H^2((0,T)\times\Omega)\to L^\infty((0,T)\times\Omega)$ with

$$\|u_n\|^2_{H^2((0,T)\times\Omega)} \le C\left(\|D_t^2 u_n\|^2_{L^2((0,T)\times\Omega)} + \|\nabla D_t u_n\|^2_{L^2((0,T)\times\Omega)} + \|\Delta u_n\|^2_{L^2((0,T)\times\Omega)} + \|u_n\|^2_{L^2((0,T)\times\Omega)}\right) \le \tilde{C}\,,$$

by (12). Therewith, in the weak forms of the state equations (10), (11) with u replaced by u_n and in the initial conditions $u_n(0,\cdot) = u_0$, $\quad D_t u_n(0,\cdot) = u_1$, we can pass to the limit to obtain $u^* = S(g^*)$. Note that for the convergence of the first term on the left-hand side we can estimate

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \left((1-2ku_n)D_t^2 u_n - (1-2ku^*)D_t^2 u^*\right) w\, dx\, dt\right|\\
&\qquad\le \left|\int_0^T\int_\Omega 2k(u^*-u_n)D_t^2 u_n w\, dx\, dt\right| + \left|\int_0^T\int_\Omega (1-2ku^*)(D_t^2 u_n - D_t^2 u^*) w\, dx\, dt\right|\\
&\qquad\le 2k\|u^*-u_n\|_{L^\infty(0,T;L^2(\Omega))}\|D_t^2 u_n\|_{L^2(0,T;L^4(\Omega))}\|w\|_{L^2(0,T;L^4(\Omega))}\\
&\qquad\quad + \left|\langle D_t^2 u_n - D_t^2 u^*, (1-2ku^*)w\rangle_{L^2((0,T)\times\Omega)}\right| \to 0.
\end{aligned}$$

Convergence of the additional terms in the Kuznetsov case can be seen as follows:

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \left(\nabla(\int_0^t u_n\, d\tau)\nabla D_t u_n - \nabla(\int_0^t u^*\, d\tau)\nabla D_t u^*\right) w\, dx\, dt\right|\\
&\qquad\le \underbrace{\|\nabla D_t u_n\|_{L^2((0,T)\times\Omega))}}_{\le\sqrt{T}\tilde{C}} \underbrace{\|\nabla(\int_0^t (u_n - u^*)\, d\tau)\|_{L^4(0,T)\times\Omega)}}_{\to 0} \|w\|_{L^2(0,T;L^4(\Omega))}\\
&\qquad\quad + \left|\langle \nabla D_t(u_n - u^*), \nabla(\int_0^t u^*\, d\tau) w\rangle_{L^2((0,T)\times\Omega)}\right| \to 0\,, \qquad (24)
\end{aligned}$$

$$\left|\int_0^T\int_\Omega \left(|\nabla u_n|^2 < -|\nabla u^*|^2\right) w\,dx\,dt\right| \leq \underbrace{\|\nabla(u_n+u^*)\|_{L^\infty(0,T;L^2(\Omega))}}_{\leq 2T\bar{C}+2\|\nabla u_0\|_{L^2(\Omega)}} \underbrace{\|\nabla(u_n-u^*)\|_{L^2(0,T;L^4(\Omega))}}_{\to 0} \|w\|_{L^2(0,T;L^4(\Omega))} \to 0. \tag{25}$$

□

Now we state and prove the main result of this section.

Theorem 3. *Let $J_\alpha^{u^d}$ be bounded from below and lower semicontinuous with respect to the weak* topology on $G\times U$. Then there exists an optimal control $g^* \in G^{ad}$ which minimizes the cost functional $J_\alpha^{u^d}(g, S(g))$ over $g \in G^{ad}$.*

Note that in case of (4) with Banach space norms $\|.\|_{\mathcal{G}}$, $\|.\|_{\mathcal{U}}$, continuity of the embeddings $U \hookrightarrow \mathcal{U}$, $G \hookrightarrow \mathcal{G}$ is sufficient for the assumptions of Theorem 3. It is readily checked that (5) satisfies the assumptions of Theorem 3 as well.

Proof. Let $\{g_n\}_{n\in\mathbb{N}} \in G^{ad}$ be a minimizing sequence such that

$$\lim_{n\to\infty} J_\alpha^{u^d}(g_n, u_n) = \inf_{g\in G^{ad}} J_\alpha^{u^d}(g, S(g)),$$

where $u_n = S(g_n)$.

By (6), (12), there exists a constant $\bar{C}$ such that for all n

$$\|u_n\|_U \leq \bar{C}\,.$$

Hence, by weak* compactness of $G^{ad}\times\mathcal{B}_{\bar{C}}$ in $G\times U$, there exist weak* $G\times U$ convergent subsequences $\{g_m\}_{m\in\mathbb{N}}$, $\{u_m\}_{m\in\mathbb{N}}$ with limits g^*, u^* in G^{ad} and $\mathcal{B}_{\bar{C}}$, respectively. By Lemma 2 we have $S(g^*) = u^*$. Now, having lower-semicontinuity of the cost functional, we conclude that g^* is an optimal control. □

3. Optimality system

In this section, we derive the first order necessary optimality conditions for the optimal boundary control problem

$$\min J_\alpha^{u^d}(g,u) \text{ s.t. } (10)/(11) \text{ and } g \in G^{ad}$$

Using the control-to-state map S defined by (22), we again consider the reduced control problem

$$\min j(g) = J_\alpha^{u^d}(g, S(g)) \text{ s.t. } g \in G^{ad}\,, \tag{26}$$

which has linear equality constraints

$$\int_\Gamma (g^*(x,0) - \frac{\partial u_0}{\partial n})\phi\, d\Gamma = 0 \quad \forall \phi \in H^{-1/2}(\Gamma)$$

and convex inequality constraints

$$\|g\|_G^2 - K^2 \leq 0$$

so that the Slater condition can be used as a constraint qualification, which in our context reads as (9). Therewith, by taking variations (h,ψ) of (g,ϕ) in $G\times H^{-1/2}(\Gamma)$ we can formally state the first order optimality or Karush-Kuhn-Tucker conditions (see, e.g. [14]): There exists $\lambda^* \in \mathbb{R}$ and $\psi^* \in H^{-1/2}(\Gamma)$ such that

$$\begin{cases} j'(g;h) + \displaystyle\int_\Gamma h(x,0)\psi^*\, d\Gamma + 2\lambda^*\langle g^*, h\rangle_G = 0 \quad \forall h \in G \\ \displaystyle\int_\Gamma (g^*(x,0) - \frac{\partial u_0}{\partial n})\psi\, d\Gamma = 0 \quad \forall \psi \in H^{-1/2}(\Gamma) \\ \|g^*\|_G^2 - K^2 \leq 0\,, \quad \lambda^* \geq 0\,, \quad \lambda^*(\|g^*\|_G^2 - K^2) = 0. \end{cases} \tag{27}$$

In order to rigorously establish the existence of the Lagrange multipliers and justify the choice of function spaces, we first have to show the differentiability of j with respect to the control g and since $J_\alpha^{u^d}$ depends on the state variable u we need differentiability of u with respect to g.

Remark 3. It is of course possible to treat the Westervelt (respectively the Kuznetsov) equation as an additional constraint, and prove existence of the corresponding Lagrange multiplier by a well-posedness argument for a linearized equation; this would eliminate the need to show differentiability of j with respect to g. On the other hand, we believe that this result (Proposition 5) is of independent interest, e.g., for establishing the convergence of numerical methods or for directly applying additional rates results from [15].

We will make use of the following auxiliary result:

Proposition 4. *For $h \in H^2(0,T;H^{-1/2}(\Gamma))$, there exists a unique solution $\bar h$ of*

$$\begin{cases} D_t^2\bar h - c^2\Delta\bar h - b\Delta D_t\bar h = 0 & \text{in } (0,T)\times\Omega \\ \dfrac{\partial\bar h}{\partial n} = h & \text{on } (0,T)\times\Gamma \\ D_t\bar h + c\dfrac{\partial\bar h}{\partial n} = 0 & \text{on } (0,T)\times\hat\Gamma \\ \bar h(0,\cdot) = 0,\ \ D_t\bar h(0,\cdot) = 0 & \text{in } \Omega \end{cases} \tag{28}$$

and

$$\max\left\{\|D_t^2\bar h\|_{L^2(0,T;L^2(\Omega))}, \|\nabla D_t\bar h\|_{L^\infty(0,T;L^2(\Omega))}, \|\Delta\bar h\|_{L^\infty(0,T;L^2(\Omega))}, \|\Delta D_t\bar h\|_{L^2(0,T;L^2(\Omega))}\right\} \le C\|c^2D_th + bD_t^2h\|_{L^2(0,T;H^{-1/2}(\Gamma))} \tag{29}$$

Proof. First of all, we will derive energy estimates for $\bar h$. Multiplying the equation by $\Delta\bar h$ and integrating over Ω we get

$$c^2\|\Delta\bar h\|^2_{L^2(\Omega)} + \frac{b}{2}D_t\|\Delta\bar h\|^2_{L^2(\Omega)} = \int_\Omega D_t^2\bar h\Delta\bar h\,dx\,dt$$

which after integration over $(0,t)$ and taking $\sup_{t\in[0,T]}$ together with the Cauchy-Schwarz inequality reduces to

$$\max\left\{c^2\|\Delta\bar h\|^2_{L^2(0,T;L^2(\Omega))}, \frac{b}{2}\|\Delta\bar h\|^2_{L^\infty(0,T;L^2(\Omega))}\right\} \le \|D_t^2\bar h\|_{L^2(0,T;L^2(\Omega))}\cdot\|\Delta\bar h\|_{L^2(0,T;L^2(\Omega))}.$$

Using an inequality of the type $\alpha\beta \le \frac{1}{4\varepsilon}\alpha^2 + \varepsilon\beta^2$, with $0 < \varepsilon < c^2$, we can conclude

$$\max\{\|\Delta\bar h\|_{L^2(0,T;L^2(\Omega))}, \|\Delta\bar h\|_{L^\infty(0,T;L^2(\Omega))}\} \le C\|D_t^2\bar h\|_{L^2(0,T;L^2(\Omega))}$$

where C denotes the generic constant (i.e., with changing values), which will also be used in the following estimates. In the same fashion, if we replace $\Delta\bar h$ by $\Delta D_t\bar h$ in the previous analysis, we obtain

$$\max\{\|\Delta\bar h\|_{L^\infty(0,T;L^2(\Omega))}, \|\Delta D_t\bar h\|_{L^2(0,T;L^2(\Omega))}\} \le C\|D_t^2\bar h\|_{L^2(0,T;L^2(\Omega))}$$

so we need an estimate on the second order time derivative of $\bar h$ in an appropriate norm. To obtain it, we proceed as follows: Differentiate (28) with respect to time, multiply it by $D_t\tilde h$, where $\tilde h = D_t\bar h$, and integrate over Ω to get

$$\int_\Omega D_t^2\tilde h D_t\tilde h\,dx - c^2\int_\Omega \Delta\tilde h D_t\tilde h\,dx - b\int_\Omega \Delta D_t\tilde h D_t\tilde h\,dx = 0\,.$$

Integration by parts, taking into account the boundary conditions, yields

$$\frac12 D_t\|D_t\tilde h\|^2_{L^2(\Omega)} + \frac{c^2}{2}D_t\|\nabla\tilde h\|^2_{L^2(\Omega)} + b\|\nabla D_t\tilde h\|^2_{L^2(\Omega)} + \int_{\hat\Gamma}\left(cD_t\tilde h + \frac{b}{c}D_t^2\tilde h\right)D_t\tilde h\,d\Gamma = \int_\Gamma\left(c^2D_th + bD_t^2h\right)D_t\tilde h\,d\Gamma,$$

hence

$$\max\left\{\frac12\|D_t^2\bar h\|^2_{L^\infty(0,T;L^2(\Omega))}, \frac{c^2}{2}\|\nabla D_t\bar h\|^2_{L^\infty(0,T;L^2(\Omega))}, b\|\nabla D_t^2\bar h\|^2_{L^2(0,T;L^2(\Omega))}, c\|D_t^2\bar h\|^2_{L^2(0,T;L^2(\hat\Gamma))}, \frac{b}{2c}\|D_t^2\bar h\|^2_{L^\infty(0,T;L^2(\hat\Gamma))}\right\}$$
$$\le K\|c^2D_th + bD_t^2h\|_{L^2(0,T;H^{-1/2}(\Gamma))}\cdot\|D_t^2\bar h\|_{L^2(0,T;H^1(\Omega))}$$

by the trace theorem ($\|f\|_{H^{1/2}(\Gamma)} \leq K\|f\|_{H^1(\Omega)}$). Now, the same argument as before ($\alpha\beta \leq \frac{1}{4\varepsilon}\alpha^2 + \varepsilon\beta^2$ with this time $0 < \varepsilon < \min\{\frac{1}{2}, b\}$) enables us to deduce

$$\max\left\{\|D_t^2\bar{h}\|_{L^\infty(0,T;L^2(\Omega))}, \|\nabla D_t\bar{h}\|_{L^\infty(0,T;L^2(\Omega))}, \|\nabla D_t^2\bar{h}\|_{L^2(0,T;L^2(\Omega))}, \|D_t^2\bar{h}\|_{L^2(0,T;L^2(\hat{\Gamma}))}, \|D_t^2\bar{h}\|_{L^\infty(0,T;L^2(\hat{\Gamma}))}\right\} \leq C\|c^2 D_t h + bD_t^2 h\|_{L^2(0,T;H^{-1/2}(\Gamma))}$$

and altogether we get (29). Now the existence and uniqueness proof can be carried out analogously to, e.g., Theorem 3.1 and Proposition 3.7 in [16]. □

Still weaker forms of the state equations (10), (11) can be achieved by integration by parts also with respect to time:

$$\begin{cases} \text{Find } u \in \check{V} \text{ such that} \\ \int_0^T\int_\Omega -D_t((1-ku)u)D_t w\,dx\,dt + \int_0^T\int_\Omega (c^2\nabla u + b\nabla D_t u)\nabla w\,dx\,dt + \int_0^T\int_{\hat{\Gamma}} (cD_t u + \frac{b}{c}D_t^2 u)w\,d\Gamma\,dt \\ \qquad = \int_0^T\int_\Gamma (c^2 g + bD_t g)w\,d\Gamma\,dt + \int_\Omega u_1(1-2ku_0)w(0,\cdot)\,dx\,, \\ u(0,\cdot) = u_0\,, \\ \text{holds for all test functions } w \in V \text{ with } w(T,\cdot) = 0, \end{cases} \tag{30}$$

and

$$\begin{cases} \text{Find } u \in \check{V} \text{ such that} \\ \int_0^T\int_\Omega -D_t((1-ku)u)D_t w\,dx\,dt + \int_0^T\int_\Omega (c^2\nabla u + b\nabla D_t u)\nabla w\,dx\,dt \\ \quad + \int_0^T\int_{\hat{\Gamma}} (cD_t u + \frac{b}{c}D_t^2 u)w\,d\Gamma\,dt + \gamma\int_0^T\int_\Omega \nabla(\int_0^t u\,d\tau)\nabla u D_t w\,dx\,dt \\ \qquad = \int_0^T\int_\Gamma (c^2 g + bD_t g)w\,d\Gamma\,dt + \int_\Omega u_1(1-2ku_0)w(0,\cdot)\,dx\,, \\ u(0,\cdot) = u_0, \\ \text{holds for all test functions } w \in V \text{ with } w(T,\cdot) = 0, \end{cases} \tag{31}$$

respectively, where

$$\begin{aligned} V &= \{v \in L^2(0,T;H^1(\Omega))\,|\,D_t v \in L^2(0,T;L^2(\Omega))\}\,, \\ \check{V} &= \{v \in L^2((0,T)\times\Omega)\,|\,D_t v \in L^2(0,T;H^1(\Omega))\}\,. \end{aligned}$$

Proposition 5. *Let T be sufficiently small so that the assumptions of Proposition 3 are satisfied.*

Then the mapping S according to (22) is directionally differentiable with respect to the $H^s(0,T;H^{-1/2}(\Gamma))$ topology in preimage space and the weak $\check{V}$ topology in image space in the sense that for all $g \in G^{ad}$

$$\frac{S(g+\varepsilon h) - S(g)}{\varepsilon} \rightharpoonup z \text{ in } \check{V} \quad \forall h \in H^s(0,T;H^{-1/2}(\Gamma)) \text{ with } g + \varepsilon h \in G^{ad}\,,$$

where

$$s\begin{cases} \geq 1 & \text{in case of the Westervelt equation} \\ \geq 2 & \text{in case of the Kuznetsov equation} \end{cases},$$

and z solves

$$\begin{cases} D_t^2 z - c^2\Delta z - b\Delta(D_t z) = 2kD_t^2(zu)\left[+\gamma D_t^2\left(\nabla(\int_0^t u\,d\tau)\cdot\nabla(\int_0^t z\,d\tau)\right)\right] & \text{in } (0,T)\times\Omega \\ \frac{\partial z}{\partial n} = h & \text{on } (0,T)\times\Gamma \\ D_t z + c\frac{\partial z}{\partial n} = 0 & \text{on } (0,T)\times\hat{\Gamma} \\ z(0,\cdot) = D_t z(0,\cdot) = 0 & \text{in } \Omega \end{cases} \tag{32}$$

in the weaker sense, i.e.,

$$
\begin{cases}
\textit{Find } z \in \check{V} \textit{ such that} \\
\int_0^T\int_\Omega -D_t\Big((1-2ku)z\Big)D_t w\,dx\,dt + \int_0^T\int_\Omega (c^2\nabla z + b\nabla D_t z)\nabla w\,dx\,dt + \int_0^T\int_{\hat\Gamma}(cD_t z + \frac{b}{c}D_t^2 z)w\,d\Gamma\,dt \\
\qquad \left[+\gamma\int_0^T\int_\Omega\left(\nabla(\int_0^t u\,d\tau)\nabla z + \nabla(\int_0^t z\,d\tau)\nabla u\right)D_t w\,dx\,dt\right] \\
\qquad = \int_0^T\int_\Gamma (c^2 h + bD_t h)w\,d\Gamma\,dt\,, \\
z(0,\cdot) = 0, \\
\textit{holds for all test functions } w \in V \textit{ with } w(T,\cdot) = 0,
\end{cases}
\tag{33}
$$

where the terms in brackets are to be omitted (i.e., $\gamma := 0$*) in the Westervelt case.*

Proof. Again the estimates used in this proof are similar to those derived for showing contractivity of the self-mapping used in the existence proof for the solution of the state equations, so we show the details here only for the Westervelt case. For the full details, we again refer to [12].

Let $u_\varepsilon = S(g+\varepsilon h)$ and $u = S(g)$. Then the quotient $\frac{u_\varepsilon - u}{\varepsilon} =: v_\varepsilon$ satisfies the following initial-boundary value problem:

$$
\begin{cases}
D_t^2 v_\varepsilon - c^2\Delta v_\varepsilon - b\Delta(D_t v_\varepsilon) = kD_t^2(v_\varepsilon(u_\varepsilon + u)) & \text{in } (0,T)\times\Omega \\
\dfrac{\partial v_\varepsilon}{\partial n} = h & \text{on } (0,T)\times\Gamma \\
D_t v_\varepsilon + c\dfrac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } (0,T)\times\hat\Gamma \\
v_\varepsilon(0,\cdot) = D_t v_\varepsilon(0,\cdot) = 0 & \text{in } \Omega\,,
\end{cases}
\tag{34}
$$

whose weak form is just (19) with $u^{(1)}$, $u^{(2)}$, $\hat u$ and the zero right-hand side replaced by u_ε, u, v_ε and $\int_0^T\int_\Gamma(c^2h + bD_th)w\,d\Gamma\,dt$, respectively. Therewith, analogously to (20) we obtain

$$
\begin{aligned}
\max\left\{\frac{1-2km}{2}\|D_t v_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))}, \frac{c^2}{2}\|\nabla v_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))}, b\|\nabla D_t v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))}, c\|D_t v_\varepsilon\|^2_{L^2(0,T;L^2(\hat\Gamma))}, \frac{b}{2c}\|D_t v_\varepsilon\|^2_{L^\infty(0,T;L^2(\hat\Gamma))}\right\} \\
\le 5k\sqrt{T}\sqrt[4]{|\Omega|}\bar a\|D_t v_\varepsilon\|^2_{L^2(0,T;L^4(\Omega))} + \|c^2h + bD_th\|_{L^2(0,T;H^{-1/2}(\Gamma))}\|D_t v_\varepsilon\|_{L^2(0,T;H^{1/2}(\Gamma))}
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{1}{2}\min\left\{\frac{1-2km}{2T}, b\right\}\|D_t v_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} &\le 5k\sqrt{T}\sqrt[4]{|\Omega|}\bar a\hat K_1^2\|D_t v_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} \\
&\quad + \hat K_3\|c^2h + bD_th\|_{L^2(0,T;H^{-1/2}(\Gamma))}\|D_t v_\varepsilon\|_{L^2(0,T;H^1(\Omega))}
\end{aligned}
$$

which implies

$$
\|D_t v_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le \frac{\hat K_3\|c^2h + bD_th\|_{L^2(0,T;H^{-1/2}(\Gamma))}}{\frac{1}{2}\min\left\{\frac{1-2km}{2T}, b\right\} - 5k\sqrt{T}\sqrt[4]{|\Omega|}\bar a\hat K_1^2}\,,
$$

i.e., uniform boundedness of $(v_\varepsilon)_{\varepsilon>0}$ in $\check V$, which in turn implies weak $\check V$ convergence along a subsequence $\{v_{\varepsilon_n}\}_{n\in\mathbb N}$ with $\varepsilon_n \to 0$. For any weak $\check V$ convergent subsequence $\{v_{\varepsilon_n}\}_{n\in\mathbb N}$ with $\varepsilon_n \to 0$ and weak limit $\bar z$ now passing to the limit $n\to\infty$ in the weaker form of (34)

$$
\begin{cases}
\text{Find } v_\varepsilon \in \check V \text{ such that} \\
\int_0^T\int_\Omega -D_t\Big((1-k(u+u_\varepsilon))v_\varepsilon\Big)D_t w\,dx\,dt + \int_0^T\int_\Omega (c^2\nabla v_\varepsilon + b\nabla D_t v_\varepsilon)\nabla w\,dx\,dt = \int_0^T\int_\Gamma (c^2h + bD_th)w\,d\Gamma\,dt\,, \\
v_\varepsilon(0,\cdot) = 0 \\
\text{holds for all test functions } w \in V \text{ with } w(T,\cdot) = 0,
\end{cases}
\tag{35}
$$

we can conclude that $\bar{z}$ solves (33) with $\gamma = 0$. To see this for the first term—for all other terms it is straightforward—consider the estimate

$$\left|\int_0^T\int_\Omega D_t\Big((1-k(u+u_{\varepsilon_n}))v_{\varepsilon_n} - (1-2ku)\bar{z}\Big)D_t w\,dx\,dt\right| = \left|\int_0^T\int_\Omega D_t\Big(-k\varepsilon v_{\varepsilon_n}^2 + (1-2ku)(v_{\varepsilon_n}-\bar{z})\Big)D_t w\,dx\,dt\right|$$
$$= \left|\int_0^T\int_\Omega \Big(-2k\varepsilon v_{\varepsilon_n} D_t v_{\varepsilon_n} - 2kD_t u(v_{\varepsilon_n}-\bar{z}) + (1-2ku)D_t(v_{\varepsilon_n}-\bar{z})\Big)D_t w\,dx\,dt\right|$$
$$\leq 2k\varepsilon\|v_{\varepsilon_n}\|_{L^\infty(0,T;L^4(\Omega))}\|D_t v_{\varepsilon_n}\|_{L^2(0,T;L^4(\Omega))}\|D_t w\|_{L^2((0,T)\times\Omega)}$$
$$+ 2k\left|\int_0^T\int_\Omega \underbrace{(v_{\varepsilon_n}-\bar{z})}_{\rightharpoonup 0 \text{ in } \check{V}} \underbrace{D_t u D_t w}_{\in L^2(0,T;L^4(\Omega)^*)\subset \check{V}^*} dx\,dt\right| + \left|\int_0^T\int_\Omega \underbrace{D_t(v_{\varepsilon_n}-\bar{z})}_{\rightharpoonup 0 \text{ in } L^2(0,T;H^1(\Omega))} \underbrace{(1-2ku)D_t w}_{\in L^2((0,T)\times\Omega)\subset L^2(0,T;H^1(\Omega))^*} dx\,dt\right|.$$

By derivation of an energy estimate analogous to the one for v_ε above, it is readily checked that the solution of (32) is unique, which by a subsequence-subsequence argument implies weak convergence as stated in the theorem.

In the Kuznetsov case (34) gets an additional term

$$\left[+\gamma D_t^2\left(\nabla(\int_0^t v_\varepsilon\,d\tau)\nabla(\int_0^t (u_\varepsilon+u)\,d\tau)\right)dx\,dt\right]$$

on the right- hand side. We proceed analogously to the contractivity proof in Theorem 5 of [12] with the replacements $u_1 \leftrightarrow u_\varepsilon$, $v_1 \leftrightarrow u_\varepsilon$, $u_2 \leftrightarrow u$, $v_2 \leftrightarrow u$, $\hat{u} \leftrightarrow v_\varepsilon$, $\hat{v} \leftrightarrow v_\varepsilon$. The inhomogeneous Neumann boundary conditions are taken into account by considering $v_\varepsilon^h := v_\varepsilon - \bar{h}$, where $\bar{h}$ solves (28), so that v_ε^h satisfies homogeneous Neumann boundary conditions, and making use of Proposition 4 to estimate the terms on the right-hand side of the state equations appearing due to the subtraction of $\bar{h}$. (Note that a direct use of the Neumann boundary conditions of v_ε as arising from spatial integration by parts is not possible due to the lack of a trace theorem for the trace operator from $L^2(\Omega)$ into $H^{-1/2}(\partial\Omega)$.) Therewith we can proceed analogously to the (quite technical) contractivity proof of Theorem 5 in [12] (see especially the left-hand sides of (68), (70) there) to arrive at

$$\begin{aligned}|||v_\varepsilon||| := \max\{&\|D_t^2 v_\varepsilon\|_{L^2(0,T;L^2(\Omega))}, \|\nabla D_t v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))},\\ &\|\Delta v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}, \|\Delta D_t v_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\}\\ &\leq \|h\|_{H^2(0,T;H^{-1/2}(\Gamma))}\end{aligned} \tag{36}$$

by Proposition 4 provided T is sufficiently small. Note that the absorbing boundary conditions again pose no difficulties, and that the difference (21) to the Dirichlet case in [12] can be treated as in the proof of Proposition 1. Using the uniform boundedness (36), convergence of the additional Kuznetsov term on the left hand side of (35) can be seen analogously to (24), (25). □

The first equation in (27) can be simplified by introducing $\bar{p} \in V$ as the variational solution of the adjoint equation

$$\begin{aligned}\int_0^T\int_\Omega -D_t\Big((1-2ku)v\Big)D_t p\,dx\,dt + \int_0^T\int_\Omega (c^2\nabla v + b\nabla D_t v)\nabla p\,dx\,dt + \int_0^T\int_{\hat{\Gamma}} (cD_t v + \frac{b}{c}D_t^2 v)p\,d\Gamma\,dt\\ \left[+\gamma\int_0^T\int_\Omega \left(\nabla(\int_0^t u\,d\tau)\nabla v + \nabla(\int_0^t v\,d\tau)\nabla u\right)D_t p\,dx\,dt\right] = D_u J_\alpha^{u^d}(g,u;v),\end{aligned} \tag{37}$$

for all test functions $v \in \check{V}$ with $v(0,\cdot) = 0$, which satisfies the end condition $p(T,\cdot) = 0$. The well-posedness of this equation can be shown with Proposition 5. Using (32) and (37), with $u = S(g)$, we have that

$$\begin{aligned}
j'(g;h) &= D_g J_\alpha^{u^d}(g, S(g); h) + D_u J_\alpha^{u^d}(g, S(g); S'(g;h)) \\
&= D_g J_\alpha^{u^d}(g, S(g); h) + \int_0^T \int_\Omega -D_t\Big((1-2kS(g))S'(g;h)\Big) D_t p \, dx \, dt \\
&+ \int_0^T \int_\Omega (c^2 \nabla S'(g;h) + b\nabla D_t S'(g;h)) \nabla \bar{p} \, dx \, dt + \int_0^T \int_{\hat\Gamma} (c D_t S'(g;h) + \frac{b}{c} D_t^2 S'(g;h)) p \, d\Gamma \, dt \\
&\left[+\gamma \int_0^T \int_\Omega \Big(\nabla (\int_0^t S(g)\, d\tau) \nabla S'(g;h) + \nabla(\int_0^t S'(g;h)\, d\tau) \nabla S(g) \Big) D_t p \, dx \, dt \right] \\
&= D_g J_\alpha^{u^d}(g, S(g); h) + \int_0^T \int_\Gamma (c^2 h + b D_t h) p \, d\Gamma \, dt
\end{aligned}$$

by (33) for $z = S'(g;h)$.

Using standard results from optimization theory (see, e.g., the review article [14] and the references therein) we thus get the following result:

Theorem 6. *Let $g^* \in G^{ad}$ be a solution to the minimization problem (26) and let (9) hold.*
Then there exist $\lambda^* \in \mathbb{R}$, $\phi^* \in H^{-1/2}(\Gamma)$, $u \in \mathcal{W}$, $\bar{p} \in V$ such that

$$\begin{cases}
D_g J_\alpha^{u^d}(g^*, u; h) + \int_0^T \int_\Gamma (c^2 h + b D_t h) \bar{p} \, d\Gamma \, dt + \int_\Gamma h(x,0) \psi^* \, d\Gamma + 2\lambda^* \langle g^*, h \rangle_G = 0 \quad \forall h \in G \\
\int_\Gamma (g^*(x,0) - \frac{\partial u_0}{\partial n}) \psi \, d\Gamma = 0 \quad \forall \psi \in H^{-1/2}(\Gamma) \\
\|g^*\|_G^2 - K^2 \le 0 \,, \quad \lambda^* \ge 0 \,, \quad \lambda^*(\|g^*\|_G^2 - K^2) = 0
\end{cases}$$

as well as the state equation (30) or (31) and the adjoint equation (37) hold.

Proof. The result follows from Proposition 3.2 in [14] in the special setting of (3.7) in [14], which with

$$G^{ad} = \{ g \in G \,|\, F_1(g) = 0, F_2(g) \in (-\infty, 0] \} \,,$$

$$\begin{array}{rcl} F_1 : G & \to & H^{1/2}(\partial\Omega) \\ g & \mapsto & g(x,0) - \frac{\partial u_0}{\partial n} \end{array} , \quad \begin{array}{rcl} F_2 : G & \to & \mathbb{R} \\ g & \mapsto & \|g\|_G^2 - K^2 \end{array}$$

becomes

$$\left. \begin{array}{rcl} F_1'(g) : G & \to & H^{1/2}(\partial\Omega) \\ h & \mapsto & h(x,0) \end{array} \right\} \text{ is onto} \tag{38}$$

and

$$\exists h \in G \; : \quad F_1'(g)h = 0 \,, \; F_2(g) + F_2'(g)h \in (-\infty, 0) \tag{39}$$

Condition (38) is trivially satisfied, since for any $h_0 \in H^{1/2}(\partial\Omega)$ we can easily find a function $h \in G$ such that $h(t=0) = h_0$, e.g. the function constant in time with value h_0. As far as condition (39) is concerned, it is readily checked that with $F_2(g) + F_2'(g)h = \|g+h\|_G^2 - K^2 - \|h\|_G^2$, $\hat{g} := g + h$, it is implied by (9). □

4. Stability and Convergence as $\alpha \to 0$

Finally, we consider stability of a minimizer with respect to perturbations in u^d as well as convergence as $\alpha \to 0$. For simplicity of exposition we make use of the results in Hilbert spaces from [15] and mention in passing that a generalization to Banach spaces can be done according to, e.g., [17], see also [14] for a more general setting.

An application of Theorem 2.1 in [15] together with Lemma 2 immediately yields

Corollary 7. *Let $\mathcal{U}, \mathcal{G}$ be Hilbert spaces with $U \hookrightarrow \mathcal{U}$, $G \hookrightarrow \mathcal{G}$, let (23) hold, and let $J_\alpha^{u^d}$ be defined by (4).*

If $u_k^d \to u^d$ in $\mathcal{U}$ then the corresponding minimizers g_k of $J_\alpha^{u_k^d}(\cdot, S(\cdot))$ over G^{ad} (according to Theorem 3) have a $\mathcal{G}$ convergent subsequence and the limit of each $\mathcal{G}$ convergent subsequence is a minimizer of $J_\alpha^{u^d}(\cdot, S(\cdot))$ over G^{ad}.

Likewise, we can apply Theorem 2.3 in [15] to conclude

Corollary 8. *Let $\mathcal{U}, \mathcal{G}$ be Hilbert spaces, with $U \hookrightarrow \mathcal{U}$, $G \hookrightarrow \mathcal{G}$, let (23) hold, let, for $u_\alpha^d \in \mathcal{U}$, $J_\alpha^{u_\alpha^d}$ be given by (4), and g_α be defined as an approximate minimizer of $J_\alpha^{u_\alpha^d}$ in the sense that*

$$J_\alpha^{u_\alpha^d}(g_\alpha) \leq J_\alpha^{u_\alpha^d}(g) + \eta(\alpha) \quad \text{for all } g \in G^{ad} \,.$$

Moreover assume that u^d is attainable, i.e., there exists a $g^\dagger \in G^{ad}$ such that $S(g^\dagger) = u^d$. Then for any sequence $(\alpha_k)_{k\in\mathbb{N}}$ with

$$\alpha_k \to 0\,, \quad \|u_{\alpha_k}^d - u^d\|_{\mathcal{U}} = o(\sqrt{\alpha_k})\,, \quad \eta(\alpha_k) = o(\alpha_k)$$

the sequence $(g_{\alpha_k})_{k\in\mathbb{N}}$ has a $\mathcal{G}$ convergent subsequence and the limit $\bar{g}$ of every $\mathcal{G}$ convergent subsequence satisfies $S(\bar{g}) = u^d$.

5. Conclusions and Remarks

In this paper, we study boundary optimal control problems with regularized tracking type functionals in nonlinear acoustics, modeled by the Westervelt or by the Kuznetsov equation. We establish existence of an optimal control, derive and justify first order optimality conditions, and shortly discuss stability as well as convergence with vanishing regularization.

Future research will be devoted to deriving second order optimality conditions and establishing local uniqueness. Also of interest is a closely related shape optimization problem in nonlinear acoustics, where the optimal control to a desired pressure distribution is to be done by shape design of an acoustic lens. This leads to a coupled acoustic–acoustic or acoustic–elastic field problem, for which the well-posedness of the forward problem already poses interesting challenges.

Finally, it would be worthwhile to derive efficient schemes for the numerical solution of the optimal control problem (cf. Remark 3).

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