BOULIGAND–LEVENBERG–MARQUARDT ITERATION FOR A NON-SMOOTH ILL-POSED INVERSE PROBLEM

Christian Clason* Vu Huu Nhu†

Abstract In this paper, we consider a modified Levenberg–Marquardt method for solving an ill-posed inverse problem where the forward mapping is not Gâteaux differentiable. By relaxing the standard assumptions for the classical smooth setting, we derive asymptotic stability estimates that are then used to prove the convergence of the proposed method. This method can be applied to an inverse source problem for a non-smooth semilinear elliptic PDE where a Bouligand subdifferential can be used in place of the non-existing Fréchet derivative, and we show that the corresponding *Bouligand–Levenberg–Marquardt iteration* is an iterative regularization scheme. Numerical examples illustrate the advantage over the corresponding Bouligand–Landweber iteration.

1 INTRODUCTION

We consider the inverse problems of the form

$$F(u) = y^\delta, \tag{1.1}$$

where $F : D(F) \subset U \to Y$ is a *non-smooth* (i.e., not necessarily Gâteaux differentiable) nonlinear operator between Hilbert spaces and the available data y^δ are some approximations of the corresponding true data $y^\dagger := F(u^\dagger)$. Furthermore, $D(F)$ denotes the domain of F and $u^\dagger$ is the unknown true solution that needs to be reconstructed.

A typical example of problem (1.1) is the case where $U = Y := L^2(\Omega)$ and F is the solution operator of the non-smooth semilinear elliptic equation

$$-\Delta y + \max(0, y) = u \quad \text{in } \Omega, \quad y \in H_0^1(\Omega) \tag{1.2}$$

with $u \in L^2(\Omega)$ and a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$. In this case, F is not Gâteaux differentiable at $u^\dagger$ if the set of $x \in \Omega$ such that $y^\dagger(x) = 0$ has positive measure; see [3, Prop. 3.4]. Moreover, F is completely continuous (see [3, Lem. 3.2]), and (1.1) is therefore *ill-posed* in the sense that the solution to (1.1) does not depend continuously on the data. A stable solution of (1.1) thus needs regularization techniques. Here we consider *iterative* regularization techniques, which construct a sequence $\{u_n^\delta\}$ of approximations to $u^\dagger$ and ensure stability by early stopping at an iteration index $N(\delta, y^\delta)$ chosen, e.g., according to Morozov's discrepancy principle; see, e.g., [4, 15]. Iterative methods have the advantage over variational methods such as Tikhonov regularization that the selection of the regularization parameter (in this case, the stopping index) is part of the method and does not have to be performed

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*Faculty of Mathematics, University of Duisburg-Essen, Thea-Leymann-Strasse 9, 45127 Essen, Germany (christian.clason@uni-due.de, orcid: 0000-0002-9948-842)

†Faculty of Mathematics, University of Duisburg-Essen, Thea-Leymann-Strasse 9, 45127 Essen, Germany (huu.vu@uni-due.de, orcid: 0000-0003-4279-3937)

by, e.g., checking a sequence of candidates or using additional information on the smoothness of the forward operator that is often not available. An iterative regularization method for (1.2) of Landweber type (which can be interpreted as a generalized gradient descent) was proposed and analyzed in [3]. However, like any first-order scheme, it usually requires a large number of iterations to satisfy the discrepancy principle, especially for small noise. This motivates considering iterative regularization methods of Newton type.

Recall that the Newton method for the smooth version of (1.1) with a continuously Fréchet differentiable operator F reads as

$$F'(u_n^\delta)(u_{n+1}^\delta - u_n^\delta) = y^\delta - F(u_n^\delta),$$

where $F'(u) : U \to Y$ denotes the Gâteaux derivative of F at $u \in D(F)$. However, if (1.1) is ill-posed, this equation is generally ill-posed as well and needs to be regularized. Applying Tikhonov regularization leads to the *Levenberg–Marquardt method*

$$u_{n+1}^\delta = \operatorname{argmin}_{u\in D(F)} \|F'(u_n^\delta)(u - u_n^\delta) - y^\delta - F(u_n^\delta)\|_Y^2 + \alpha_n \|u - u_n^\delta\|_U^2 \tag{1.3}$$

or, equivalently,

$$u_{n+1}^\delta = u_n^\delta + \left(\alpha_n I + F'(u_n^\delta)^* F'(u_n^\delta)\right)^{-1} F'(u_n^\delta)^* \left(y^\delta - F(u_n^\delta)\right), \tag{1.4}$$

where $\alpha_n > 0$ is the Tikhonov parameter. For a linear operator, this method coincides with the non-stationary iterated Tikhonov method studied in, e.g., [1, 7]. As noted above, for noisy data the iteration has to be terminated at a stopping index $N_\delta := N(\delta, y^\delta) < \infty$ in order to be stable. Assuming that $\|y^\delta - y^\dagger\|_Y \le \delta$ and that the Tikhonov parameters α_n are chosen via a Morozov discrepancy principle, [6] showed the *regularization property* $u_{N_\delta}^\delta \to u^\dagger$ as $\delta \to 0$ as well as the logarithmic estimate

$$N_\delta = O\left(1 + |\log(\delta)|\right), \tag{1.5}$$

provided that

$$\|F(u_1) - F(u_2) - F'(u_2)(u_1 - u_2)\|_Y \le c\|u_1 - u_2\|_U \|F(u_1) - F(u_2)\|_Y \tag{1.6}$$

for all $u_1, u_2 \in \overline{B}_U(u^\dagger, \rho)$ and for some constants $c, \rho > 0$. In [9], the regularization property as well as the logarithmic estimate (1.5) of the Levenberg–Marquardt method was shown under the a priori choice

$$\alpha_n = \alpha_0 r^n, \quad n = 0, 1, \ldots \tag{1.7}$$

with $\alpha_0 > 0$ and $r \in (0, 1)$ and under the assumption that for any $u_1, u_2 \in \overline{B}_U(u^\dagger, \rho)$, there exists a bounded linear operator $Q(u_1, u_2) : Y \to Y$ satisfying

$$F'(u_1) = Q(u_1, u_2)F'(u_2) \quad \text{and} \quad \|I - Q(u_1, u_2)\|_{\mathbb{L}(Y)} \le L\|u_1 - u_2\|_U \tag{1.8}$$

for some constant $L > 0$. It is noted that the convergence analysis in [6, 9] requires the *stability* of the method, that is, there holds

$$u_n^\delta \to u_n \quad \text{as } \delta \to 0 \quad \text{for all } n \le N(\delta, y^\delta)$$

with δ small enough, where u_n^δ and u_n are generated by the method corresponding to the noisy ($\delta > 0$) and the noise-free ($\delta = 0$) situations, respectively. The continuity of the derivative F' (or more specifically, of the linear operator in the right-hand side of (1.4)) with respect to u is therefore essential.

The purpose of this work is to present a *modified Levenberg–Marquardt method* for solving (1.1) in the spirit of [24, 3], where we replace the – possibly nonexistent – Fréchet derivative $F'(u)$ in (1.4) by another suitable bounded linear operator G_u. Our main aim is to show the regularization property of the proposed algorithm under the choice (1.7) of Tikhonov parameters and conditions that relax (1.6) and (1.8). We also prove the logarithmic estimate (1.5) of the stopping index. However, unlike the situation in [6, 9], we lack the continuity of the mapping $D(F) \ni u \mapsto G_u \in \mathbb{L}(U, Y)$. To overcome this essential difficulty, we shall combine a technique from [9] with the approach in [3] to prove *asymptotic stability estimates* of iterates u_n^δ; see Section 2.3 and Proposition 2.19 in place of the missing stability of the method. The proposed method is then applied to a non-smooth ill-posed inverse problem where the forward operator is the solution mapping of (1.2). In this case, the operator G_u can be taken from the Bouligand subdifferential of the forward mapping and explicitly characterized by the solution of a suitable linearized PDE, see Proposition 3.1 below. We refer to this special case of the modified Levenberg–Marquardt method as *Bouligand–Levenberg–Marquardt iteration*.

Let us briefly comment on related literature. Newton-type methods, and in particular the Levenberg–Marquardt method, for approximately solving smooth nonlinear ill-posed problems have been extensively investigated in Hilbert spaces; see, e.g. [4, 6, 15, 25, 18, 23, 22, 21] and the references therein. More recently, inverse problems in Banach spaces have attracted increasing attention, and corresponding iterative regularization methods of Newton-type have been developed, e.g., in [10, 13, 11, 12, 20, 19, 16, 14]. Considering (1.3) in Banach spaces (in particular, L^1 or the space of functions of bounded variation) or including additional constraints can lead to non-smooth optimization problems; however, none of the works so far has focused on inverse problems for non-smooth forward operators.

Organization. This paper is organized as follows. After briefly summarizing basic notation, we present the convergence analysis of the modified Levenberg–Marquardt method in Section 2: Section 2.1 is devoted to its well-posedness and the logarithmic estimate of the stopping index N_δ; in Section 2.2, we prove its convergence in the noise-free case; in Section 2.3 we verify its asymptotic stability estimates, which are crucial for the proof of the regularization property of the iterative method in Section 2.4. Section 3 introduces an application of the modified Levenberg–Marquardt method to the non-smooth ill-posed inverse source problem for (1.2). Finally, some numerical examples are provided in Section 4.

Notation. For a Hilbert space X, we denote by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively, the inner product and the norm on X. For a given z belonging to a Banach space Z and $\rho > 0$, by $B_Z(z,\rho)$ and $\overline{B}_Z(z,\rho)$ we denote, respectively, the open and closed balls in Z of radius ρ centered at z. For each measurable function u on Ω and a subset $T \subset \mathbb{R}$, the notation $\{u \in T\}$ stands for the sets of almost every $x \in \Omega$ at which $u(x) \in T$. Similarly, given measurable functions u, v on Ω and subsets $T_1, T_2 \subset \mathbb{R}$, we denote the set of a.e. $x \in \Omega$ such that $u(x) \in T_1$ and $v(x) \in T_2$ by $\{u \in T_1, v \in T_2\}$. For a measurable set S in $\mathbb{R}^d$, we write $|S|$ for the d-dimensional Lebesgue measure of S and denote by $\mathbb{1}_S$ the characteristic function of the set S, i.e., $\mathbb{1}_S(s) = 1$ if $s \in S$ and $\mathbb{1}_S(s) = 0$ if $s \notin S$. The adjoint operator, the null space, and the range of a linear operator G will be denoted by G^*, $\mathcal{N}(G)$, and $\mathcal{R}(G)$, respectively. Finally, we denote by $\mathbb{L}(X)$ and $\mathbb{L}(X,Y)$ the set of all bounded linear operators from Hilbert space X to itself and from X to another Hilbert space Y, respectively.

2 A MODIFIED LEVENBERG–MARQUARDT METHOD

Let U and Y be real Hilbert spaces and F a non-smooth mapping from U to Y with its domain $D(F) \subset U$. We consider the non-smooth ill-posed problem

$$F(u) = y^\delta, \tag{2.1}$$

where the noisy data y^δ satisfy

$$\|y^\delta - y^\dagger\|_Y \le \delta \tag{2.2}$$

with $y^\dagger \in \mathcal{R}(F)$. From now on, let $u^\dagger$ be an arbitrary, but fixed, solution of (2.1) corresponding to the exact data $y^\dagger$. For a given number $\rho > 0$, we denote by $S_\rho(u^\dagger)$ the set of all solutions in $\overline{B}_U(u^\dagger, \rho)$ of (2.1) corresponding the exact data, that is,

$$S_\rho(u^\dagger) := \left\{u^* \in D(F) : F(u^*) = y^\dagger, \|u^* - u^\dagger\|_U \le \rho\right\}.$$

Throughout this work, we make the following assumptions on F.

(A1) There exists a constant $\rho_0 > 0$ such that $\overline{B}_U(u^\dagger, \rho_0) \subset D(F)$. Furthermore, there exists a family of bounded linear operators $\{G_u : u \in \overline{B}_U(u^\dagger, \rho)\} \subset \mathbb{L}(U, Y)$ such that for all $\rho \in (0, \rho_0]$ and $u, \hat{u}$ in $\overline{B}_U(u^\dagger, \rho)$, there holds the *generalized tangential cone condition*

$$\text{(GTCC)} \qquad \|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_Y \le \eta(\rho)\|F(\hat{u}) - F(u)\|_Y$$

for some non-decreasing function $\eta : (0, \rho_0] \to (0, \infty)$ satisfying

$$\eta_0 := \eta(\rho_0) < 1. \tag{2.3}$$

Moreover, for any pair $u_1, u_2 \in \overline{B}_U(u^\dagger, \rho_0)$, there exists a bounded linear operator $Q(u_1, u_2) \in \mathbb{L}(Y)$ such that

$$G_{u_1} = Q(u_1, u_2)G_{u_2} \tag{2.4}$$

and that for all $\rho \in (0, \rho_0]$,

$$\|I - Q(u, \hat{u})\|_{\mathbb{L}(Y)} \le \kappa(\rho) \quad \text{for all } u, \hat{u} \in \overline{B}_U(u^\dagger, \rho) \tag{2.5}$$

and for some non-negative and non-decreasing function κ on $(0, \rho_0]$ with $\kappa_0 := \kappa(\rho_0)$.

(A2) The operator $G_{u^\dagger} : U \to Y$ is compact.

We will furthermore require that $\kappa(\rho)$ and $\eta(\rho)$ can be made sufficiently small by choosing ρ small enough; we will make this more precise during the analysis in this section. In Section 3.2, we will verify that this is possible for the Bouligand–Levenberg–Marquardt method applied to the non-smooth PDE (1.2).

Remark 2.1. If (GTCC) is valid, then for all $\rho \in (0, \rho_0]$ and $u, \hat{u} \in \overline{B}_U(u^\dagger, \rho)$, there hold

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_Y \le \frac{\eta(\rho)}{1 - \eta_0}\|G_u(\hat{u} - u)\|_Y \tag{2.6}$$

and

$$\|F(\hat{u}) - F(u)\|_Y \le \frac{1}{1 - \eta_0}\|G_u(\hat{u} - u)\|_Y, \tag{2.7}$$

where the last inequality leads to the continuity of F at u and hence on $\overline{B}_U(u^\dagger, \rho)$. Moreover, if (2.5) holds, then it holds that

$$\|Q(u_1, u_2)\|_{\mathbb{L}(Y)} \le \kappa_0 + 1 \tag{2.8}$$

for all $u_1, u_2 \in \overline{B}_U(u^\dagger, \rho_0)$. In addition, if Assumptions (A1) and (A2) are satisfied, then all of Assumptions (A1) and (A2) are fulfilled with F and G_u replaced, respectively, by tF and tG_u for some positive number t small enough.

We shall consider the *modified Levenberg–Marquardt method* defined as

$$u_{n+1}^\delta = u_n^\delta + \left(\alpha_n I + G_{u_n^\delta}^* G_{u_n^\delta}\right)^{-1} G_{u_n^\delta}^* \left(y^\delta - F(u_n^\delta)\right), \quad n = 0, 1, \dots, \tag{2.9}$$

where $u_0^\delta := u_0$ and $\{\alpha_n\}$ is given by

$$\alpha_n = \alpha_0 r^n, \quad n = 0, 1, \dots \tag{2.10}$$

for some constants $\alpha_0 > 0$ and $r \in (0, 1)$. We will assume that α_0 is chosen such that

$$\|G_{u^\dagger}\|_{\mathbb{L}(U,Y)} \le \alpha_0^{1/2}. \tag{2.11}$$

Note that according to Remark 2.1, this condition can be enforced for any $\alpha_0 > 0$ by scaling the problem (1.1) as well as G_u accordingly.

The iteration is terminated via the *discrepancy principle*

$$\|y^\delta - F(u_{N_\delta}^\delta)\|_Y \le \tau\delta < \|y^\delta - F(u_n^\delta)\|_Y \quad \text{for all } 0 \le n < N_\delta, \tag{2.12}$$

where $\tau > 1$ is a given number. Here $N_\delta := N(\delta, y^\delta)$ stands for the stopping index of the iterative method.

By $\{u_n\}$, we denote the sequence of iterates defined by (2.9) corresponding to the noise free case ($\delta = 0$), i.e.,

$$u_{n+1} = u_n + \left(\alpha_n I + G_{u_n}^* G_{u_n}\right)^{-1} G_{u_n}^* \left(y^\dagger - F(u_n)\right), \quad n = 0, 1, \dots \tag{2.13}$$

For ease of exposition, from now on, we use the notations

$$\begin{aligned}
e_n^\delta &:= u_n^\delta - u^\dagger, & G_n^\delta &:= G_{u_n^\delta}, & A_n^\delta &:= G_n^{\delta *} G_n^\delta, & B_n^\delta &:= G_n^\delta G_n^{\delta *},\\
e_n &:= u_n - u^\dagger, & G_n &:= G_{u_n}, & A_n &:= G_n^* G_n, & B_n &:= G_n G_n^*,\\
& & G_\dagger &:= G_{u^\dagger}, & A &:= G_\dagger^* G_\dagger, & B &:= G_\dagger G_\dagger^*.
\end{aligned}$$

Remark 2.2. If F is smooth and if $G_u = F'(u)$, then (2.9) coincides with the classical Levenberg–Marquardt method; see, e.g., [15, 25, 6, 9].

2.1 WELL-POSEDNESS

We first show the well-posedness of the proposed iterative method as well as the logarithmic estimate of the stopping index N_δ.

The first lemma gives a useful tool to estimate the difference between iterates.

Lemma 2.3 (cf. [9, Lem. 2]). *Assume that (2.4) and (2.5) are fulfilled. Let $\rho \in (0, \rho_0]$. For any $u, v \in \overline{B}_U(u^\dagger, \rho)$, let $A_u := G_u^* G_u$ and $A_v := G_v^* G_v$. Then for any $\alpha > 0$, the following identities hold*

$$(\alpha I + A_u)^{-1} - (\alpha I + A_v)^{-1} = (\alpha I + A_v)^{-1} G_v^* R_\alpha(u, v) \tag{2.14}$$

and

$$(\alpha I + A_u)^{-1} G_u^* - (\alpha I + A_v)^{-1} G_v^* = (\alpha I + A_v)^{-1} G_v^* S_\alpha(u, v), \tag{2.15}$$

where $R_\alpha(u, v) : U \to Y$ and $S_\alpha(u, v) : Y \to Y$ are bounded linear operators satisfying

$$\|R_\alpha(u, v)\|_{\mathbb{L}(U,Y)} \le \alpha^{-1/2}\kappa(\rho) \quad \textit{and} \quad \|S_\alpha(u, v)\|_{\mathbb{L}(Y)} \le 3\kappa(\rho). \tag{2.16}$$

Proof. The proof is analogous to that in [9]. We see from (2.4) that

$$\begin{aligned}
&(\alpha I + A_u)^{-1} - (\alpha I + A_v)^{-1} \\
&\qquad = (\alpha I + A_v)^{-1} (A_v - A_u) (\alpha I + A_u)^{-1} \\
&\qquad = (\alpha I + A_v)^{-1} \left(G_v^*(G_v - G_u) + (G_v^* - G_u^*)G_u\right) (\alpha I + A_u)^{-1} \\
&\qquad = (\alpha I + A_v)^{-1} \left(G_v^*(Q(v,u) - I)G_u + G_v^*(I - Q(u,v)^*)G_u\right) (\alpha I + A_u)^{-1} \\
&\qquad = (\alpha I + A_v)^{-1} G_v^* \left((Q(v,u) - I) + (I - Q(u,v)^*)\right) G_u (\alpha I + A_u)^{-1} \\
&\qquad = (\alpha I + A_v)^{-1} G_v^* R_\alpha(u,v)
\end{aligned}$$

with

$$R_\alpha(u,v) := \left((Q(v,u) - I) + (I - Q(u,v)^*)\right) G_u (\alpha I + A_u)^{-1}.$$

Easily, (2.5) and Lemma A.4 ensure that $R_\alpha(u,v)$ satisfies the first inequality in (2.16). On the other hand, using (2.4) and the above representation yields

$$\begin{aligned}
(\alpha I + A_u)^{-1} G_u^* - (\alpha I + A_v)^{-1} G_v^* &= (\alpha I + A_v)^{-1} (G_u^* - G_v^*) + \left[(\alpha I + A_u)^{-1} - (\alpha I + A_v)^{-1}\right] G_u^* \\
&= (\alpha I + A_v)^{-1} G_v^* (Q(u,v)^* - I) + (\alpha I + A_v)^{-1} G_v^* R_\alpha(u,v) G_u^* \\
&= (\alpha I + A_v)^{-1} G_v^* S_\alpha(u,v)
\end{aligned}$$

with

$$\begin{aligned}
S_\alpha(u,v) &:= Q(u,v)^* - I + R_\alpha(u,v)G_u^* \\
&= Q(u,v)^* - I + \left((Q(v,u) - I) + (I - Q(u,v)^*)\right) (\alpha I + B_u)^{-1} B_u
\end{aligned}$$

and $B_u := G_u G_u^*$. From the definition of S_α, (2.5) and Lemma A.3 lead to the second inequality in (2.16). ☐

To simplify the notation in the following proofs, we introduce the constants

$$c_0 := \frac{1}{\sqrt{r}}, \quad c_1 := \frac{1}{1-\sqrt{r}}, \quad c_2 := \frac{1}{\sqrt{r}(1-\sqrt{r})}, \quad c_3 := \frac{\sqrt{r}}{1-r}, \quad c_4 := \frac{1}{1-r}, \tag{2.17}$$

as well as

$$K_0(r,\nu) := \frac{1}{\sqrt{r}\left(r^{\nu-1/2} - 1\right)}, \quad K_1(r,\nu) := \frac{1}{r\left(r^{\nu-1/2} - 1\right)} \tag{2.18}$$

for $0 \le \nu < \frac{1}{2}$.

Let now $\tilde{N}_\delta \in \mathbb{N}$ be such that

$$\alpha_{\tilde{N}_\delta} \le \left(\frac{\delta}{\gamma_0 \|e_0\|_U}\right)^2 < \alpha_n \quad \text{for all } 0 \le n < \tilde{N}_\delta, \tag{2.19}$$

for a constant

$$\gamma_0 > \frac{2c_0}{(1-\eta_0)(\tau - \tau_0)}$$

with $\tau > \tau_0 > 1$. We can now prove a logarithmic estimate for $\tilde{N}_\delta$, which will later be used to obtain the corresponding estimate for the actual stopping index N_δ.

Lemma 2.4. *Let $\tilde{N}_\delta$ be defined by (2.19). Then there holds*

$$\tilde{N}_\delta = O(1 + |\log(\delta)|).$$

Proof. From (2.19) and (2.10), we conclude that

$$\left(\frac{\delta}{\gamma_0\|e_0\|_U}\right)^2 < \alpha_0 r^{\tilde{N}_\delta - 1},$$

which, together with the fact that $0 < r < 1$, directly gives

$$\tilde{N}_\delta < 1 + 2\log_r \delta - 2\log_r(\gamma_0\|e_0\|_U) - \log_r \alpha_0$$

and hence the desired estimate. □

We now show a uniform bound on the iterates and the error by, if necessary, further restricting the radius ρ of the neighborhood of $u^\dagger$.

Lemma 2.5 (cf. [9, Lem. 4]). *Let $\{\alpha_n\}$ be defined by (2.10) and (2.11). Assume that Assumption (A1) holds. Assume further that there exists a positive constant $\rho_1 \le \rho_0$ such that*

$$\begin{cases} 2\kappa(\rho_1) + \dfrac{c_0}{1-\eta_0}(1+3\kappa_0)\eta(\rho_1) \le \dfrac{c_0}{2c_2} \\ (c_1+3)\kappa(\rho_1) + \dfrac{2c_2}{1-\eta_0}(1+3\kappa_0)\eta(\rho_1) \le 1. \end{cases} \tag{2.20}$$

Let $\rho \in (0, \rho_1]$ be arbitrary and let $u_0 \in U$ be such that $(2 + c_1\gamma_0)\|e_0\|_U < \rho$. Then there hold

(i) $u_n^\delta \in \overline{B}_U(u^\dagger, \rho)$;

(ii) $\|e_n^\delta\|_U \le (2 + c_1\gamma_0)\|e_0\|_U < \rho$;

(iii) $\|G_\dagger e_n^\delta\|_Y \le (c_0 + 2c_2\gamma_0)\|e_0\|_U \alpha_n^{1/2}$

for all $0 \le n \le \tilde{N}_\delta$, where the constants c_i, $i = 0, 1, 2$, are given by (2.17).

Proof. It is sufficient to show (ii) and (iii) by induction on n with $0 \le n \le \tilde{N}_\delta$. Obviously, (ii) and (iii) are fulfilled with $n = 0$. Now for any fixed $0 \le l < \tilde{N}_\delta$, we assume that (ii) and (iii) hold true for all $0 \le n \le l$. We shall prove these assertions for $n = l + 1$. To this end, we set for any $0 \le m \le l$

$$z_m^\delta := F(u_m^\delta) - y^\dagger - G_m^\delta e_m^\delta. \tag{2.21}$$

Moreover, we see from (2.9) and the identity $I = \alpha(\alpha I + T)^{-1} + (\alpha I + T)^{-1}T$ that

$$\begin{aligned} e_{m+1}^\delta &= u_{m+1}^\delta - u^\dagger \\ &= e_m^\delta + \left(\alpha_n I + A_m^\delta\right)^{-1} G_m^{\delta*}\left(y^\delta - F(u_m^\delta)\right) \\ &= \alpha_m\left(\alpha_m I + A_m^\delta\right)^{-1} e_m^\delta + \left(\alpha_m I + A_m^\delta\right)^{-1} A_m^\delta e_m^\delta + \left(\alpha_m I + A_m^\delta\right)^{-1} G_m^{\delta*}\left(y^\delta - F(u_m^\delta)\right) \\ &= \alpha_m\left(\alpha_m I + A_m^\delta\right)^{-1} e_m^\delta + \left(\alpha_m I + A_m^\delta\right)^{-1} G_m^{\delta*}\left(y^\delta - F(u_m^\delta) + G_m^\delta e_m^\delta\right) \\ &= \alpha_m\left(\alpha_m I + A_m^\delta\right)^{-1} e_m^\delta + \left(\alpha_m I + A_m^\delta\right)^{-1} G_m^{\delta*}\left(y^\delta - y^\dagger - z_m^\delta\right), \end{aligned}$$

which together with Lemma 2.3 gives

$$\begin{aligned} e_{m+1}^\delta = \alpha_m(\alpha_m I + A)^{-1}\left[I + G_\dagger^* R_{\alpha_m}(u_m^\delta, u^\dagger)\right] e_m^\delta \\ + (\alpha_m I + A)^{-1} G_\dagger^*\left[I + S_{\alpha_m}(u_m^\delta, u^\dagger)\right]\left(y^\delta - y^\dagger - z_m^\delta\right). \end{aligned}$$

Consequently, it holds that

$$e^\delta_{m+1} = \alpha_m \left(\alpha_m I + A\right)^{-1} e^\delta_m + \left(\alpha_m I + A\right)^{-1} G_\dagger^* w^\delta_m \tag{2.22}$$

with

$$w^\delta_m := \alpha_m R_{\alpha_m}(u^\delta_m, u^\dagger) e^\delta_m + \left[I + S_{\alpha_m}(u^\delta_m, u^\dagger)\right]\left(y^\delta - y^\dagger - z^\delta_m\right). \tag{2.23}$$

The definition of w^δ_m and the estimates (2.16) imply that

$$\|w^\delta_m\|_Y \leq \kappa(\rho)\alpha_m^{1/2}\|e^\delta_m\|_U + (1 + 3\kappa(\rho))\left(\delta + \|z^\delta_m\|_Y\right).$$

Furthermore, (GTCC) and (2.7) give

$$\|z^\delta_m\|_Y \leq \frac{\eta(\rho)}{1-\eta_0}\|G_\dagger e^\delta_m\|_Y. \tag{2.24}$$

We thus have

$$\|w^\delta_m\|_Y \leq \kappa(\rho)\alpha_m^{1/2}\|e^\delta_m\|_U + (1 + 3\kappa(\rho))\left(\delta + \frac{\eta(\rho)}{1-\eta_0}\|G_\dagger e^\delta_m\|_Y\right). \tag{2.25}$$

By telescoping (2.22), we obtain

$$e^\delta_{l+1} = \prod_{m=0}^{l} \alpha_m \left(\alpha_m I + A\right)^{-1} e_0 + \sum_{m=0}^{l} \alpha_m^{-1} \prod_{j=m}^{l} \alpha_j \left(\alpha_j I + A\right)^{-1} G_\dagger^* w^\delta_m \tag{2.26}$$

and thus

$$G_\dagger e^\delta_{l+1} = \prod_{m=0}^{l} \alpha_m \left(\alpha_m I + B\right)^{-1} G_\dagger e_0 + \sum_{m=0}^{l} \alpha_m^{-1} \prod_{j=m}^{l} \alpha_j \left(\alpha_j I + B\right)^{-1} B w^\delta_m, \tag{2.27}$$

where we have used that identity $G_\dagger\left(\alpha I + A\right)^{-1} = \left(\alpha I + B\right)^{-1} G_\dagger$. Applying Lemmas A.3 and A.4 to (2.26) and using (2.25) yields

$$\begin{aligned}
\|e^\delta_{l+1}\|_U &\leq \|e_0\|_U + \frac{1}{2}\sum_{m=0}^{l} \alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \|w^\delta_m\|_Y \\
&\leq \|e_0\|_U + \frac{1}{2}\sum_{m=0}^{l} \alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \left[\kappa(\rho)\alpha_m^{1/2}\|e^\delta_m\|_U + (1 + 3\kappa(\rho))\left(\delta + \frac{\eta(\rho)}{1-\eta_0}\|G_\dagger e^\delta_m\|_Y\right)\right].
\end{aligned}$$

We now use the induction hypothesis to deduce that

$$\begin{aligned}
\|e^\delta_{l+1}\|_U \leq \|e_0\|_U &+ \frac{1}{2}\sum_{m=0}^{l} \alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \kappa(\rho)\alpha_m^{1/2}\left(2 + c_1\gamma_0\right)\|e_0\|_U \\
&+ \frac{1}{2}\sum_{m=0}^{l} \alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \left[(1 + 3\kappa(\rho))\left(\delta + \frac{\eta(\rho)}{1-\eta_0}\left(c_0 + 2c_2\gamma_0\right)\|e_0\|_U\alpha_m^{1/2}\right)\right].
\end{aligned}$$

From the choice of $\tilde{N}_\delta$, there holds that $\delta \leq \gamma_0 \|e_0\|_U \alpha_m^{1/2}$ for all $0 \leq m \leq l < \tilde{N}_\delta$. The above estimates and Lemma A.1 imply that

$$
\begin{aligned}
\|e^\delta_{l+1}\|_U &\leq \|e_0\|_U + \frac{1}{2}\sum_{m=0}^{l} \alpha_m^{-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \kappa(\rho)\,(2+c_1\gamma_0)\,\|e_0\|_U \\
&\quad + \frac{1}{2}\sum_{m=0}^{l} \alpha_m^{-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \left[(1+3\kappa(\rho))\left(\gamma_0 + \frac{\eta(\rho)}{1-\eta_0}\,(c_0+2c_2\gamma_0)\right)\right]\|e_0\|_U \\
&\leq \|e_0\|_U + \frac{1}{2}c_1\|e_0\|_U\left[\kappa(\rho)\,(2+c_1\gamma_0) + (1+3\kappa(\rho))\left(\gamma_0+\frac{\eta(\rho)}{1-\eta_0}\,(c_0+2c_2\gamma_0)\right)\right] \\
&= \|e_0\|_U\left[1+\frac{c_1}{2}H_1(\rho) + \frac{c_1}{2}\gamma_0(1+H_2(\rho))\right]
\end{aligned}
$$

with

$$
H_1(\rho) := 2\kappa(\rho) + (1+3\kappa(\rho))\frac{c_0\eta(\rho)}{1-\eta_0}, \quad H_2(\rho) := (c_1+3)\kappa(\rho) + \frac{2c_2\eta(\rho)}{1-\eta_0}(1+3\kappa(\rho)).
$$

Combining this with the monotonic growth of κ, η on $(0, \rho_0]$, the fact that $c_0/(2c_2) \leq 2/c_1$, and (2.20) yields

$$
\|e^\delta_{l+1}\|_U \leq \|e_0\|_U\,(2+c_1\gamma_0)\,. \tag{2.28}
$$

On the other hand, (2.27) along with Lemmas A.3 and A.4 gives

$$
\|G_\dagger e^\delta_{l+1}\|_Y \leq \frac{1}{2}\left(\sum_{j=0}^{l}\alpha_j^{-1}\right)^{-1/2}\|e_0\|_U + \sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\|w^\delta_m\|_Y\,.
$$

From this, Lemma A.1, and (2.25), the induction hypothesis and the choice of $\tilde{N}_\delta$ satisfying (2.19) lead to

$$
\begin{aligned}
\|G_\dagger e^\delta_{l+1}\|_Y &\leq \frac{1}{2}c_0\alpha_{l+1}^{1/2}\|e_0\|_U + \|e_0\|_U \sum_{m=0}^{l}\alpha_m^{-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1} \\
&\qquad \cdot \left[\kappa(\rho)(2+c_1\gamma_0) + (1+3\kappa(\rho))\left(\gamma_0+\frac{\eta(\rho)}{1-\eta_0}\,(c_0+2c_2\gamma_0)\right)\right] \\
&\leq \|e_0\|_U\alpha_{l+1}^{1/2}\left\{\frac{1}{2}c_0 + c_2\left[\kappa(\rho)(2+c_1\gamma_0) + (1+3\kappa(\rho))\left(\gamma_0+\frac{\eta(\rho)}{1-\eta_0}\,(c_0+2c_2\gamma_0)\right)\right]\right\} \\
&\leq \|e_0\|_U\alpha_{l+1}^{1/2}\left[\frac{1}{2}c_0 + c_2H_1(\rho) + c_2\gamma_0\,(1+H_2(\rho))\right].
\end{aligned}
$$

From this, the monotonic growth of κ, η on $(0, \rho_0]$ and (2.20), we have that

$$
\|G_\dagger e^\delta_{l+1}\|_Y \leq (c_0+2c_2\gamma_0)\,\|e_0\|_U\alpha_{l+1}^{1/2},
$$

which together with (2.28) implies that (ii) and (iii) are fulfilled with $n = l+1$. □

By using a similar argument for $\{u_n\}$ defined by (2.13), we obtain the following result.

Lemma 2.6 (cf. [9, Lem. 5]). *Let $\{\alpha_n\}$ be defined by (2.10) and (2.11). Assume that Assumption (A1) and the first condition in (2.20) hold. Then, for any $u_0 \in U$ such that $2\|e_0\|_U < \rho$ with $\rho \in (0, \rho_1]$, there hold*

(i) $u_n \in \overline{B}_U(u^\dagger, \rho)$;

(ii) $\|e_n\|_U \leq 2\|e_0\|_U < \rho$;

(iii) $\|G_\dagger e_n\|_Y \leq c_0\|e_0\|_U \alpha_n^{1/2}$

for all $n \geq 0$.

The next lemma is a crucial tool in our analysis to prove the well-posedness of the method as well as the asymptotic stability estimates.

Lemma 2.7. *Assume that all assumptions in Lemma 2.5 are satisfied. Then*

$$u_{n+1}^\delta - u_{n+1} = \alpha_n (\alpha_n I + A)^{-1} (u_n^\delta - u_n) + \sum_{i=1}^{5} s_n^{(i)} \quad \text{for all } 0 \leq n < \tilde{N}_\delta, \tag{2.29}$$

where

$$s_n^{(i)} = (\alpha_k I + A)^{-1} G_\dagger^* \xi_n^{(i)}, \quad i = 1, \dots, 5 \tag{2.30}$$

with some $\xi_n^{(i)} \in Y$ *satisfying*

$$\sum_{i=1}^{4} \|\xi_n^{(i)}\|_Y \leq L_1(\rho) \left(\alpha_n^{1/2}\|e_n\|_U + \alpha_n^{1/2}\|u_n^\delta - u_n\|_U + \|G_\dagger e_n\|_Y + \|G_\dagger(u_n^\delta - u_n)\|_Y\right) \tag{2.31}$$

and

$$\xi_n^{(5)} = (y^\delta - y^\dagger) + \tilde{\xi}_n^{(5)} \quad \text{with} \quad \|\tilde{\xi}_n^{(5)}\|_Y \leq L_1(\rho)\delta \tag{2.32}$$

for

$$L_1(\rho) := \max\left\{3\kappa(\rho), (1+3\kappa_0)\left(2 + \frac{3\eta_0}{1-\eta_0}\right)\kappa(\rho), \frac{(1+3\kappa_0)(1+\kappa_0)}{1-\eta_0}\eta(\rho)\right\}. \tag{2.33}$$

Proof. For any $m \in \mathbb{N}$, we define

$$z_m := F(u_m) - y^\dagger - G_m e_m. \tag{2.34}$$

We thus obtain from (2.9), (2.13), (2.21), and (2.34) that

$$\begin{aligned}
u_{n+1}^\delta - u_{n+1} &= u_n^\delta - u_n + \left(\alpha_n I + A_n^\delta\right)^{-1} G_n^{\delta*}\left(y^\delta - F(u_n^\delta)\right) - (\alpha_n I + A_n)^{-1} G_n^*\left(y^\dagger - F(u_n)\right)\\
&= u_n^\delta - u_n + \left(\alpha_n I + A_n^\delta\right)^{-1} G_n^{\delta*}\left(y^\delta - y^\dagger - z_n^\delta - G_n^\delta e_n^\delta\right)\\
&\quad + (\alpha_n I + A_n)^{-1} G_n^*(z_n + G_n e_n)\\
&= u_n^\delta - u_n + \left(\alpha_n I + A_n^\delta\right)^{-1} G_n^{\delta*}\left(y^\delta - y^\dagger\right) - \left(\alpha_n I + A_n^\delta\right)^{-1} G_n^{\delta*} z_n^\delta\\
&\quad + (\alpha_n I + A_n)^{-1} G_n^* z_n + \left[(\alpha_n I + A_n)^{-1} A_n e_n - \left(\alpha_n I + A_n^\delta\right)^{-1} A_n^\delta e_n^\delta\right].
\end{aligned}$$

Furthermore, using the identity $(\alpha I + T)^{-1}T = I - \alpha(\alpha I + T)^{-1}$ gives

$$\begin{aligned}
&(\alpha_n I + A_n)^{-1} A_n e_n - \left(\alpha_n I + A_n^\delta\right)^{-1} A_n^\delta e_n^\delta\\
&\qquad = (\alpha_n I + A_n)^{-1} A_n(u_n - u_n^\delta) + \left[(\alpha_n I + A_n)^{-1} A_n - \left(\alpha_n I + A_n^\delta\right)^{-1} A_n^\delta\right] e_n^\delta\\
&\qquad = (\alpha_n I + A)^{-1} A(u_n - u_n^\delta) + \alpha_n\left[(\alpha_n I + A_n)^{-1} - (\alpha_n I + A)^{-1}\right](u_n^\delta - u_n)\\
&\qquad\quad + \alpha_n\left[\left(\alpha_n I + A_n^\delta\right)^{-1} - (\alpha_n I + A_n)^{-1}\right] e_n^\delta
\end{aligned}$$

and thus

$$(\alpha_n I + A_n)^{-1} A_n e_n - \left(\alpha_n I + A_n^\delta\right)^{-1} A_n^\delta e_n^\delta = (\alpha_n I + A)^{-1} A(u_n - u_n^\delta) \\ + \alpha_n \left[\left(\alpha_n I + A_n^\delta\right)^{-1} - (\alpha_n I + A)^{-1}\right](u_n^\delta - u_n) + \alpha_n \left[\left(\alpha_n I + A_n^\delta\right)^{-1} - (\alpha_n I + A_n)^{-1}\right] e_n.$$

Defining

$$\begin{aligned} s_n^{(1)} &:= \alpha_n \left[\left(\alpha_n I + A_n^\delta\right)^{-1} - (\alpha_n I + A)^{-1}\right](u_n^\delta - u_n), \\ s_n^{(2)} &:= \alpha_n \left[\left(\alpha_n I + A_n^\delta\right)^{-1} - (\alpha_n I + A_n)^{-1}\right] e_n, \\ s_n^{(3)} &:= \left[(\alpha_n I + A_n)^{-1} G_n^* - \left(\alpha_n I + A_n^\delta\right)^{-1} G_n^{\delta *}\right] z_n, \\ s_n^{(4)} &:= \left(\alpha_n I + A_n^\delta\right)^{-1} G_n^{\delta *}(z_n - z_n^\delta), \\ s_n^{(5)} &:= \left(\alpha_n I + A_n^\delta\right)^{-1} G_n^{\delta *}(y^\delta - y^\dagger) \end{aligned}$$

yields (2.29). We now verify (2.30), (2.31), and (2.32). To this end, we use Lemma 2.3 and obtain that

$$s_n^{(1)} = \alpha_n (\alpha_n I + A)^{-1} G_\dagger^* R_{\alpha_n}(u_n^\delta, u^\dagger)(u_n^\delta - u_n),$$

and so (2.30) holds for $i = 1$ with

$$\xi_n^{(1)} := \alpha_n R_{\alpha_n}(u_n^\delta, u^\dagger)(u_n^\delta - u_n).$$

Note that $u_n^\delta, u_n \in \overline{B}(u^\dagger, \rho)$, according to Lemmas 2.5 and 2.6. We thus deduce from (2.16) that

$$\|\xi_n^{(1)}\|_Y \le \kappa(\rho)\alpha_n^{1/2}\|u_n^\delta - u_n\|_U. \tag{2.35}$$

Similarly, we obtain

$$\begin{cases} \xi_n^{(2)} := \alpha_n \left[I + S_{\alpha_n}(u_n, u^\dagger)\right] R_{\alpha_n}(u_n^\delta, u_n)e_n, \\ \xi_n^{(3)} := -\left[I + S_{\alpha_n}(u_n, u^\dagger)\right] S_{\alpha_n}(u_n^\delta, u_n)z_n, \\ \xi_n^{(4)} := \left[I + S_{\alpha_n}(u_n^\delta, u^\dagger)\right](z_n - z_n^\delta), \\ \xi_n^{(5)} := \left[I + S_{\alpha_n}(u_n^\delta, u^\dagger)\right](y^\delta - y^\dagger) \end{cases} \tag{2.36}$$

and (2.30) then follows. Obviously, (2.32) is verified with $\tilde{\xi}_n^{(5)} := S_{\alpha_n}(u_n^\delta, u^\dagger)(y^\delta - y^\dagger)$. It remains to prove the estimate (2.31). First, it is easy to see from (2.16) and the definition of $\xi_n^{(2)}$ in (2.36) that

$$\|\xi_n^{(2)}\|_Y \le (1 + 3\kappa(\rho))\kappa(\rho)\alpha_n^{1/2}\|e_n\|_U. \tag{2.37}$$

As a result of (GTCC), we have

$$\|z_n\|_Y \le \frac{\eta(\rho)}{1 - \eta_0}\|G_\dagger e_n\|_Y,$$

which together with (2.16) and the definition of $\xi_n^{(3)}$ yields

$$\|\xi_n^{(3)}\|_Y \le \frac{3\kappa(\rho)\eta(\rho)(1 + 3\kappa(\rho))}{1 - \eta_0}\|G_\dagger e_n\|_Y. \tag{2.38}$$

Furthermore, we can conclude from the definitions of z_n^δ and z_n, (GTCC), (2.4), and (2.5) that

$$\begin{aligned}\|z_n^\delta - z_n\|_Y &\le \|F(u_n) - F(u_n^\delta) - G_n^\delta(u_n - u_n^\delta)\|_Y + \|(G_n - G_n^\delta)e_n\|Y\\ &\le \frac{\eta(\rho)}{1-\eta_0}\|G_n(u_n - u_n^\delta)\|_Y + \|Q(u_n,u^\dagger) - Q(u_n^\delta,u^\dagger)\|_{\mathbb{L}(Y)}\|G_\dagger e_n\|_Y\\ &\le \frac{\eta(\rho)}{1-\eta_0}\|Q(u_n,u^\dagger)G_\dagger(u_n - u_n^\delta)\|_Y + \|Q(u_n,u^\dagger) - Q(u_n^\delta,u^\dagger)\|_{\mathbb{L}(Y)}\|G_\dagger e_n\|_Y\\ &\le \frac{\eta(\rho)}{1-\eta_0}(1+\kappa(\rho))\|G_\dagger(u_n - u_n^\delta)\|_Y + 2\kappa(\rho)\|G_\dagger e_n\|_Y.\end{aligned}$$

This, the definition of $\xi_n^{(4)}$, and (2.16) therefore imply that

$$\|\xi_n^{(4)}\|_Y \le (1+3\kappa(\rho))\left[\frac{1+\kappa(\rho)}{1-\eta_0}\eta(\rho)\|G_\dagger(u_n - u_n^\delta)\|_Y + 2\kappa(\rho)\|G_\dagger e_n\|_Y\right].$$

From this, (2.35), (2.37), (2.38), and the monotonic growth of κ on $(0,\rho_0]$, we obtain (2.31). □

Lemma 2.8. *Let all assumptions of Lemma 2.7 be satisfied. Then there hold*

$$\|u_{l+1}^\delta - u_{l+1}\|_U \le \frac{1}{2}c_3\left(1+L_1(\rho)\right)\frac{\delta}{\sqrt{\alpha_{l+1}}} + \frac{L_1(\rho)}{2}\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}\sigma_m \tag{2.39}$$

and

$$\|G_\dagger(u_{l+1}^\delta - u_{l+1}) - y^\delta + y^\dagger\|_Y \le \delta\left(1+c_4L_1(\rho)\right) + L_1(\rho)\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\sigma_m \tag{2.40}$$

for all $0 \le l < \tilde N_\delta$ with $L_1(\rho)$ defined as (2.33) and

$$\sigma_m := \alpha_m^{1/2}\|e_m\|_U + \alpha_m^{1/2}\|u_m^\delta - u_m\|_U + \|G_\dagger e_m\|_Y + \|G_\dagger(u_m^\delta - u_m)\|_Y.$$

Proof. Telescoping (2.29) and (2.30) gives

$$u_{l+1}^\delta - u_{l+1} = \sum_{m=0}^{l}\alpha_m^{-1}\prod_{j=m}^{l}\alpha_j\left(\alpha_jI + A\right)^{-1}G_\dagger^*\sum_{i=1}^{5}\xi_m^{(i)} \tag{2.41}$$

and thus

$$\begin{aligned}&G_\dagger(u_{l+1}^\delta - u_{l+1}) - (y^\delta - y^\dagger) \\ &= \sum_{m=0}^{l}\alpha_m^{-1}\prod_{j=m}^{l}\alpha_j\left(\alpha_jI+B\right)^{-1}B\left[\sum_{i=1}^{4}\xi_m^{(i)} + \tilde\xi_m^{(5)}\right] + \left[I - \sum_{m=0}^{l}\alpha_m^{-1}\prod_{j=m}^{l}\alpha_j\left(\alpha_jI+B\right)^{-1}B\right](y^\dagger - y^\delta)\\ &= \sum_{m=0}^{l}\alpha_m^{-1}\prod_{j=m}^{l}\alpha_j\left(\alpha_jI+B\right)^{-1}B\left[\sum_{i=1}^{4}\xi_m^{(i)} + \tilde\xi_m^{(5)}\right] + \prod_{j=0}^{l}\alpha_j\left(\alpha_jI+B\right)^{-1}(y^\dagger - y^\delta),\end{aligned} \tag{2.42}$$

where we have used the identity

$$I - \sum_{m=0}^{l}\alpha_m^{-1}\prod_{j=m}^{l}\alpha_j\left(\alpha_jI+B\right)^{-1}B = \prod_{j=0}^{l}\alpha_j\left(\alpha_jI+B\right)^{-1}$$

to obtain the last equality. Applying Lemma A.4 to (2.41) and exploiting the estimates (2.31) as well as (2.32) yields

$$\begin{aligned}
&\|u^\delta_{l+1} - u_{l+1}\|_U \\
&\quad\le \frac{1}{2}\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}\sum_{i=1}^{5}\|\xi_m^{(i)}\|_Y \\
&\quad\le \frac{L_1(\rho)}{2}\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}\left(\alpha_m^{1/2}\|e_m\|_U + \alpha_m^{1/2}\|u_m^\delta - u_m\|_U + \|G_\dagger e_m\|_Y + \|G_\dagger(u_m^\delta - u_m)\|_Y\right) \\
&\qquad + \frac{1}{2}\delta\,(1 + L_1(\rho))\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}.
\end{aligned}$$

The estimate (2.39) then follows from the above estimate and Lemma A.1. Similarly, applying Lemma A.3 to (2.42), using Lemma A.1, and exploiting the estimates (2.31) as well as (2.32) yield

$$\begin{aligned}
&\|G_\dagger(u^\delta_{l+1} - u_{l+1}) - y^\delta + y^\dagger\|_Y \\
&\quad\le \delta + \sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\left[\sum_{i=1}^{4}\|\xi_m^{(i)}\|_Y + \|\tilde\xi_m^{(5)}\|_Y\right] \\
&\quad\le L_1(\rho)\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\left(\alpha_m^{1/2}\|e_m\|_U + \alpha_m^{1/2}\|u_m^\delta - u_m\|_U + \|G_\dagger e_m\|_Y + \|G_\dagger(u_m^\delta - u_m)\|_Y\right) \\
&\qquad + \delta\,(1 + c_4 L_1(\rho)),
\end{aligned}$$

which gives (2.40). □

Corollary 2.9. *Under the assumptions in Lemma 2.8, there hold that*

$$\|G_\dagger(u_l^\delta - u_l) - y^\delta + y^\dagger\|_Y \le \delta\,(1 + c_4L_1(\rho)) + L_1(\rho)c_2\,[(6+3c_0) + \gamma_0(c_1 + 2c_2)]\,\|e_0\|_U\,\alpha_l^{1/2} \tag{2.43}$$

and

$$\|F(u_l^\delta) - F(u_l) - y^\delta + y^\dagger\|_Y \le \delta\,(1 + c_4L_1(\rho)) + (L_2(\rho) + \gamma_0 L_3(\rho))\,\|e_0\|_U\,\alpha_l^{1/2} \tag{2.44}$$

for all $0 \le l \le \tilde N_\delta$ *with*

$$\begin{cases} L_2(\rho) := c_2(6+3c_0)L_1(\rho) + 2c_0\left[\dfrac{(1+\kappa_0)}{1-\eta_0}\eta(\rho) + \kappa(\rho)\right], \\ L_3(\rho) := c_2(c_1+2c_2)L_1(\rho) + 2c_2\left[\dfrac{(1+\kappa_0)}{1-\eta_0}\eta(\rho) + \kappa(\rho)\right]. \end{cases} \tag{2.45}$$

Proof. It suffices to prove (2.43) and (2.44) for all $1 \le l \le \tilde N_\delta$. According to (2.40), Lemmas 2.5 and 2.6, we have

$$\begin{aligned}
\|G_\dagger(u_l^\delta - u_l) - y^\delta + y^\dagger\|_Y &\le \delta\,(1 + c_4L_1(\rho)) \\
&\quad + L_1(\rho)\,[(6+3c_0) + \gamma_0(c_1+2c_2)]\,\|e_0\|_U\sum_{m=0}^{l-1}\alpha_m^{-1/2}\left(\sum_{j=m}^{l-1}\alpha_j^{-1}\right)^{-1},
\end{aligned}$$

which along with Lemma A.1 gives (2.43). On the other hand, we can deduce from Assumption (A1) and (2.8) that

$$\begin{aligned}\|F(u_l^\delta) - F(u_l) - y^\delta + y^\dagger\|_Y &\le \|G_\dagger(u_l^\delta - u_l) - y^\delta + y^\dagger\|_Y \\ &\quad + \|F(u_l^\delta) - F(u_l) - G_l(u_l^\delta - u_l)\|_Y + \|(G_l - G_\dagger)(u_l^\delta - u_l)\|_Y \\ &\le \|G_\dagger(u_l^\delta - u_l) - y^\delta + y^\dagger\|_Y \\ &\quad + \frac{\eta(\rho)}{1-\eta_0}\|G_l(u_l^\delta - u_l)\|_Y + \|(Q(u_l, u^\dagger) - I)G_\dagger(u_l^\delta - u_l)\|_Y \\ &\le \|G_\dagger(u_l^\delta - u_l) - y^\delta + y^\dagger\|_Y \\ &\quad + \frac{\eta(\rho)}{1-\eta_0}\|Q(u_l, u^\dagger)G_\dagger(u_l^\delta - u_l)\|_Y + \kappa(\rho)\|G_\dagger(u_l^\delta - u_l)\|_Y \\ &\le \|G_\dagger(u_l^\delta - u_l) - y^\delta + y^\dagger\|_Y \\ &\quad + \left[\frac{(1+\kappa_0)}{1-\eta_0}\eta(\rho) + \kappa(\rho)\right]\|G_\dagger(u_l^\delta - u_l)\|_Y.\end{aligned}$$

From this, (2.43), Lemmas 2.5 and 2.6, and the monotonic growth of κ, a simple computation verifies (2.44). □

We finish this subsection by providing the logarithmic estimate of the stopping index N_δ, where we again may have to further restrict the radius ρ of the neighborhood of $u^\dagger$.

Lemma 2.10. *Let Assumption (A1) and (2.20) be fulfilled and let $\{\alpha_n\}$ be defined by (2.10) and (2.11). Assume that $\tau > \tau_0 > 1$, $\gamma_0 \ge \frac{2c_0}{(1-\eta_0)(\tau-\tau_0)}$. Assume further that there exists a positive constant $\rho_2 \le \rho_1$, with ρ_1 given as in Lemma 2.5, such that*

$$c_4 L_1(\rho_2) + L_3(\rho_2) \le \tau_0 - 1 \quad \text{and} \quad L_2(\rho_2) \le \frac{c_0}{1-\eta_0} \tag{2.46}$$

with L_i, $1 \le i \le 3$, defined as in (2.33) and (2.45). Let $\rho \in (0, \rho_2]$ and $u_0 \in U$ be arbitrary such that $(2 + c_1\gamma_0)\|e_0\|_U < \rho$. Then the modified Levenberg–Marquardt iteration (2.9)–(2.12) terminates after N_δ steps with

$$N_\delta = O(1 + |\log(\delta)|).$$

Proof. As a result of Lemma 2.4, it suffices to prove $N_\delta \le \tilde{N}_\delta$. If $\tilde{N}_\delta = 0$, then by definition we have $\alpha_0^{1/2}\|e_0\|_U \le \frac{\delta}{\gamma_0}$. The estimate (2.7) thus gives

$$\begin{aligned}\|F(u_0) - y^\delta\|_Y &\le \|y^\dagger - y^\delta\|_Y + \|F(u_0) - y^\dagger\|_Y \\ &\le \delta + \frac{1}{1-\eta_0}\|G_\dagger e_0\|_Y \\ &\le \delta + \frac{1}{1-\eta_0}\|e_0\|_U \alpha_0^{1/2} \\ &\le \delta\left(1 + \frac{1}{(1-\eta_0)\gamma_0}\right) \\ &< \tau\delta.\end{aligned}$$

In the following we shall assume $\tilde{N}_\delta > 0$. We deduce from (2.44) for $l = \tilde{N}_\delta$ that

$$\|F(u_{\tilde{N}_\delta}^\delta) - F(u_{\tilde{N}_\delta}) - y^\delta + y^\dagger\|_Y \le \delta\,(1 + c_4 L_1(\rho)) + (L_2(\rho) + \gamma_0 L_3(\rho))\,\|e_0\|_U \alpha_{\tilde{N}_\delta}^{1/2}.$$

Using (2.7), Lemma 2.6, and noting that $\alpha_{\tilde{N}_\delta}^{1/2}\|e_0\|_U \leq \frac{\delta}{\gamma_0}$, we derive

$$\begin{aligned}\|y^\delta - F(u^\delta_{\tilde{N}_\delta})\|_Y &\leq \|F(u^\delta_{\tilde{N}_\delta}) - F(u_{\tilde{N}_\delta}) - y^\delta + y^\dagger\|_Y + \|F(u_{\tilde{N}_\delta}) - y^\dagger\|_Y \\ &\leq \|F(u^\delta_{\tilde{N}_\delta}) - F(u_{\tilde{N}_\delta}) - y^\delta + y^\dagger\|_Y + \frac{1}{1-\eta_0}\|G_\dagger e_{\tilde{N}_\delta}\|_Y \\ &\leq \delta\left(1 + c_4L_1(\rho)\right) + \left(L_2(\rho) + \gamma_0 L_3(\rho)\right)\|e_0\|_U\alpha_{\tilde{N}_\delta}^{1/2} + \frac{c_0}{1-\eta_0}\|e_0\|_U\alpha_{\tilde{N}_\delta}^{1/2} \\ &\leq \delta\left[1 + (c_4L_1(\rho) + L_3(\rho)) + \frac{1}{\gamma_0}\left(\frac{c_0}{1-\eta_0} + L_2(\rho)\right)\right].\end{aligned}$$

Combining this with (2.46), the definitions of L_i, $1 \leq i \leq 3$, and the monotonic growth of κ, η, we obtain

$$\|y^\delta - F(u^\delta_{\tilde{N}_\delta})\|_Y \leq \delta\left(\tau_0 + \frac{1}{\gamma_0}\frac{2c_0}{1-\eta_0}\right) \leq \delta\tau.$$

From this and the definition of N_δ, we have $N_\delta \leq \tilde{N}_\delta$. □

2.2 CONVERGENCE IN THE NOISE FREE SETTING

In this subsection we will show the convergence of the sequence $\{u_n\}$ defined via (2.13), provided that $e_0 \in \mathcal{N}(G_\dagger)^\perp$ and that the parameter $\eta(\rho)$ and $\kappa(\rho)$ are small enough if the radius ρ can be chosen accordingly.

We first derive some estimates on e_n and $G_\dagger e_n$ under the *generalized source condition*

$$e_0 = A^\nu w \tag{2.47}$$

for some $\nu \in (0, \frac{1}{2})$ and some $w \in U$, where $A = G_\dagger^* G_\dagger$. Again, we may have to restrict ρ further.

Lemma 2.11. *Let all assumptions in Lemma 2.6 hold. Assume further that there exists a constant $\bar\rho_1 \in (0, \rho_1]$, with ρ_1 given as in Lemma 2.5, satisfying*

$$\kappa(\bar\rho_1) + c_0\frac{1+3\kappa_0}{1-\eta_0}\eta(\bar\rho_1) \leq \frac{c_0}{2K_1(r,\nu)} \tag{2.48}$$

for some constant $\nu \in (0, \frac{1}{2})$. Let $\rho \in (0, \bar\rho_1]$ be arbitrary and let $u_0 \in U$ satisfy $2\|e_0\|_U < \rho$ and $e_0 = A^\nu w$ for some $w \in U$. Then there hold

$$\|e_n\|_U \leq 2c_0\alpha_n^\nu\|w\|_U \quad \text{and} \quad \|G_\dagger e_n\|_Y \leq 2c_0^2\alpha_n^{\nu+1/2}\|w\|_U \tag{2.49}$$

for all $n \geq 0$.

Proof. We shall prove the lemma by induction on n. Obviously, (2.49) is valid for $n = 0$. We now assume that (2.49) holds for all $0 \leq n \leq l$ and prove it is also true for $n = l+1$. An argument similar to the one used to obtain (2.26) and (2.27) yields

$$e_{l+1} = \prod_{m=0}^{l}\alpha_m\left(\alpha_m I + A\right)^{-1}e_0 + \sum_{m=0}^{l}\alpha_m^{-1}\prod_{j=m}^{l}\alpha_j\left(\alpha_j I + A\right)^{-1}G_\dagger^* w_m \tag{2.50}$$

and

$$G_\dagger e_{l+1} = \prod_{m=0}^{l}\alpha_m\left(\alpha_m I + B\right)^{-1}G_\dagger e_0 + \sum_{m=0}^{l}\alpha_m^{-1}\prod_{j=m}^{l}\alpha_j\left(\alpha_j I + B\right)^{-1}Bw_m, \tag{2.51}$$

where, analogous to (2.23),

$$w_m := \alpha_m R_{\alpha_m}(u_m, u^\dagger)e_m - \left[I + S_{\alpha_m}(u_m, u^\dagger)\right] z_m$$

for all $0 \le m \le l$ with z_m defined via (2.34). Similarly to (2.25), we have

$$\|w_m\|_Y \le \kappa(\rho)\alpha_m^{1/2}\|e_m\|_U + (1 + 3\kappa(\rho))\,\frac{\eta(\rho)}{1-\eta_0}\|G_\dagger e_m\|_Y.$$

This and the induction hypothesis yield

$$\|w_m\|_Y \le Q_1(\rho)\alpha_m^{\nu+1/2}\|w\|_U \tag{2.52}$$

for all $0 \le m \le l$ with

$$Q_1(\rho) := 2c_0\kappa(\rho) + 2c_0^2\frac{1 + 3\kappa(\rho)}{1-\eta_0}\eta(\rho).$$

Inserting $e_0 = A^\nu w$ into (2.50) and then applying Lemmas A.3 and A.4, we deduce

$$\begin{aligned}
\|e_{l+1}\|_U &\le \|w\|_U\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-\nu} + \frac{1}{2}\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}\|w_m\|_Y\\
&\le c_0^{2\nu}\alpha_{l+1}^{\nu}\|w\|_U + \frac{1}{2}Q_1(\rho)\|w\|_U\sum_{m=0}^{l}\alpha_m^{\nu-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}\\
&\le c_0\alpha_{l+1}^{\nu}\|w\|_U + \frac{1}{2}Q_1(\rho)K_0(r,\nu)\alpha_{l+1}^{\nu}\|w\|_U,
\end{aligned}$$

where we used (2.52) and Lemma A.1 to obtain the second inequality and exploited Lemma A.2 to obtain the last inequality. By virtue of (2.48), the fact that $c_0/(2K_1(r,\nu)) \le 1/K_0(r,\nu)$, and the monotonic growth of κ and η, it holds that

$$\|e_{l+1}\|_U \le 2c_0\alpha_{l+1}^{\nu}\|w\|_U. \tag{2.53}$$

Moreover, by inserting $e_0 = A^\nu w$ into (2.51), Corollary A.5 and Lemma A.3 and (2.52) reveal that

$$\begin{aligned}
\|G_\dagger e_{l+1}\|_Y &\le \|w\|_U\left(\sum_{j=0}^{l}\alpha_j^{-1}\right)^{-\nu-1/2} + \sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\|w_m\|_Y\\
&\le c_0^{2\nu+1}\alpha_{l+1}^{\nu+1/2}\|w\|_U + Q_1(\rho)\|w\|_U\sum_{m=0}^{l}\alpha_m^{\nu-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\\
&\le c_0^2\alpha_{l+1}^{\nu+1/2}\|w\|_U + Q_1(\rho)K_1(r,\nu)\alpha_{l+1}^{\nu+1/2}\|w\|_U.
\end{aligned}$$

Here the second estimate is derived using Lemma A.1 while the last estimate is obtained using Lemma A.2. Then there holds

$$\|G_\dagger e_{l+1}\|_Y \le 2c_0^2\alpha_{l+1}^{\nu+1/2}\|w\|_U. \tag{2.54}$$

From (2.53) and (2.54), we conclude that (2.49) is fulfilled with $n = l + 1$. □

We now take $\hat{u}_0 \in U$ to be a perturbation of $u_0 \in U$ and denote by $\{\hat{u}_n\}$ the iterates given by (2.13) with u_0 replaced by $\hat{u}_0$, that is,

$$\hat{u}_{n+1} = \hat{u}_n + \left(\alpha_n I + G_{\hat{u}_n}^* G_{\hat{u}_n}\right)^{-1} G_{\hat{u}_n}^*\left(y^\dagger - F(\hat{u}_n)\right), \quad n = 0, 1, \ldots \tag{2.55}$$

For ease of exposition, from now on, we use the notations

$$\hat{e}_n := \hat{u}_n - u^\dagger, \qquad \hat{G}_n := G_{\hat{u}_n}, \qquad \hat{A}_n := \hat{G}_n^* \hat{G}_n, \qquad \hat{B}_n := \hat{G}_n \hat{G}_n^*, \qquad \hat{z}_n := F(\hat{u}_n) - y^\dagger - \hat{G}_n \hat{e}_n.$$

The next lemma is analogous to Lemma 2.7.

Lemma 2.12. *Let all assumptions in Lemma 2.6 be fulfilled and let $\bar{\rho}_1 \in (0, \rho_1]$ and $\nu \in (0, \frac{1}{2})$ satisfy (2.48). Assume that $\rho \in (0, \bar{\rho}_1]$ and that $u_0, \hat{u}_0 \in U$ satisfy $\min\{2\|e_0\|_U, 2\|\hat{e}_0\|_U\} < \rho$ and $\hat{e}_0 = A^\nu w$ for some $w \in U$. Then for all $k \geq 0$, there holds*

$$u_{k+1} - \hat{u}_{k+1} = \alpha_k (\alpha_k I + A)^{-1} (u_k - \hat{u}_k) + \sum_{i=1}^{4} t_k^{(i)}, \tag{2.56}$$

where

$$t_k^{(i)} = (\alpha_k I + A)^{-1} G_\dagger^* h_k^{(i)}, \quad i = 1, 2, 3, 4 \tag{2.57}$$

with some $h_k^{(i)} \in Y$ satisfying

$$\sum_{i=1}^{4} \|h_k^{(i)}\|_Y \leq C(\rho) \left[\|w\|_U \alpha_k^{\nu+1/2} + \|u_k - \hat{u}_k\|_U \alpha_k^{1/2} + \|G_\dagger(u_k - \hat{u}_k)\|_Y \right] \tag{2.58}$$

for

$$C(\rho) := (1 + 3\kappa_0) \max \left\{ 2c_0 \left(\frac{3c_0\eta_0}{1 - \eta_0} + 1 + 2c_0(1 + \kappa_0) \right) \kappa(\rho), \frac{(1 + \kappa_0)}{1 - \eta_0} \eta(\rho) \right\}. \tag{2.59}$$

Proof. Analogous to (2.29), we see from (2.13), (2.55), (2.34), and the definition of $\hat{z}_k$ that (2.56) is satisfied with

$$\begin{aligned}
t_k^{(1)} &:= \alpha_k \left[(\alpha_k I + A_k)^{-1} - (\alpha_k I + A)^{-1} \right] (u_k - \hat{u}_k), \\
t_k^{(2)} &:= \alpha_k \left[(\alpha_k I + A_k)^{-1} - \left(\alpha_k I + \hat{A}_k \right)^{-1} \right] \hat{e}_k, \\
t_k^{(3)} &:= \left[\left(\alpha_k I + \hat{A}_k \right)^{-1} \hat{G}_k^* - (\alpha_k I + A_k)^{-1} G_k^* \right] \hat{z}_k, \\
t_k^{(4)} &:= (\alpha_k I + A_k)^{-1} G_k^* (\hat{z}_k - z_k).
\end{aligned}$$

We now prove (2.57) and (2.58). To verify these relations, we use Lemma 2.3 to obtain

$$t_k^{(1)} = \alpha_k (\alpha_k I + A)^{-1} G_\dagger^* R_{\alpha_k}(u_k, u^\dagger)(u_k - \hat{u}_k)$$

and thus $h_k^{(1)} := \alpha_k R_{\alpha_k}(u_k, u^\dagger)(u_k - \hat{u}_k)$ verifies (2.57) for $i = 1$. The estimate (2.16) then implies that

$$\|h_k^{(1)}\|_Y \leq \kappa(\rho) \|u_k - \hat{u}_k\|_U \alpha_k^{1/2}. \tag{2.60}$$

Furthermore, we have

$$\begin{aligned}
t_k^{(2)} &= \alpha_k \left(\alpha_k I + \hat{A}_k \right)^{-1} \hat{G}_k^* R_{\alpha_k}(u_k, \hat{u}_k) \hat{e}_k \\
&= \alpha_k (\alpha_k I + A)^{-1} G_\dagger^* \left[I + S_{\alpha_k}(\hat{u}_k, u^\dagger) \right] R_{\alpha_k}(u_k, \hat{u}_k) \hat{e}_k,
\end{aligned}$$

and so (2.57) is valid for $i = 2$ with

$$h_k^{(2)} := \alpha_k \left[I + S_{\alpha_k}(\hat{u}_k, u^\dagger) \right] R_{\alpha_k}(u_k, \hat{u}_k) \hat{e}_k.$$

This and (2.16) yield

$$\begin{aligned}\|h_k^{(2)}\|_Y &\le (1+3\kappa(\rho))\kappa(\rho)\alpha_k^{1/2}\|\hat{e}_k\|_U \\ &\le 2c_0(1+3\kappa_0)\kappa(\rho)\|w\|_U\alpha_k^{\nu+1/2},\end{aligned} \tag{2.61}$$

where we have used Lemma 2.11 and the monotonic growth of κ to obtain the last estimate. Noting that $u_k, \hat{u}_k \in \overline{B}(u^\dagger, \rho)$, according to Lemma 2.6, we have

$$t_k^{(3)} = (\alpha_k I + A)^{-1} G_\dagger^* \left[I + S_{\alpha_k}(u_k, u^\dagger)\right] S_{\alpha_k}(\hat{u}_k, u_k)\hat{z}_k$$

and therefore $h_k^{(3)} := \left[I + S_{\alpha_k}(u_k, u^\dagger)\right] S_{\alpha_k}(\hat{u}_k, u_k)\hat{z}_k$. The estimate (2.16) then yields

$$\|h_k^{(3)}\|_Y \le 3(1+3\kappa_0)\kappa(\rho)\|\hat{z}_k\|_Y.$$

On the other hand, as a result of (GTCC) and Lemma 2.11, we have

$$\|\hat{z}_k\|_Y \le \frac{\eta(\rho)}{1-\eta_0}\|G_\dagger \hat{e}_k\|_Y \le 2c_0^2\frac{\eta(\rho)}{1-\eta_0}\|w\|_U\alpha_k^{\nu+1/2}.$$

The two estimates above show that $h_k^{(3)}$ satisfies

$$\|h_k^{(3)}\|_Y \le 6c_0^2(1+3\kappa_0)\kappa(\rho)\frac{\eta(\rho)}{1-\eta_0}\|w\|_U\alpha_k^{\nu+1/2}. \tag{2.62}$$

Finally,

$$\begin{aligned}t_k^{(4)} &= (\alpha_k I + A)^{-1} G_\dagger^* \left[I + S_{\alpha_k}(u_k, u^\dagger)\right](\hat{z}_k - z_k) \\ &= (\alpha_k I + A)^{-1} G_\dagger^* h_k^{(4)}\end{aligned}$$

with $h_k^{(4)} := \left[I + S_{\alpha_k}(u_k, u^\dagger)\right](\hat{z}_k - z_k)$. From this and (2.16), we obtain

$$\|h_k^{(4)}\|_Y \le (1+3\kappa_0)\,\|\hat{z}_k - z_k\|_Y. \tag{2.63}$$

From the definitions of z_k and $\hat{z}_k$, it follows that

$$\begin{aligned}\|\hat{z}_k - z_k\|_Y &= \|F(\hat{u}_k) - F(u_k) - G_k(\hat{u}_k - u_k) + (G_k - \hat{G}_k)\hat{e}_k\|_Y \\ &\le \|F(\hat{u}_k) - F(u_k) - G_k(\hat{u}_k - u_k)\|_Y + \|(G_k - \hat{G}_k)\hat{e}_k\|_Y \\ &\le \frac{\eta(\rho)}{1-\eta_0}\|G_k(\hat{u}_k - u_k)\|_Y + \|(G_k - \hat{G}_k)\hat{e}_k\|_Y.\end{aligned}$$

Here we used (2.6). Combining this with (2.4), (2.5), and (2.8), we obtain

$$\begin{aligned}\|\hat{z}_k - z_k\|_Y &\le \frac{\eta(\rho)}{1-\eta_0}(1+\kappa_0)\|G_\dagger(\hat{u}_k - u_k)\|_Y + 2\kappa(\rho)(1+\kappa_0)\|G_\dagger\hat{e}_k\|_Y \\ &\le (1+\kappa_0)\left[\frac{\eta(\rho)}{1-\eta_0}\|G_\dagger(\hat{u}_k - u_k)\|_Y + 4c_0^2\kappa(\rho)\|w\|_U\alpha_k^{\nu+1/2}\right],\end{aligned}$$

where we have used (2.49) to get the last inequality. This together with (2.63) shows that

$$\|h_k^{(4)}\|_Y \le (1+3\kappa_0)\,(1+\kappa_0)\left[\frac{\eta(\rho)}{1-\eta_0}\|G_\dagger(\hat{u}_k - u_k)\|_Y + 4c_0^2\kappa(\rho)\|w\|_U\alpha_k^{\nu+1/2}\right]. \tag{2.64}$$

Summing up from (2.60)–(2.62) to (2.64) yields (2.58). □

Lemma 2.13. *Let all assumptions in Lemma 2.6 be fulfilled and let $\bar\rho_1, \nu$ be defined as in Lemma 2.11. Assume that there exists a constant $\bar\rho_2 \in (0, \bar\rho_1]$ satisfying*

$$C(\bar\rho_2) \le \min\left\{\frac{c_0}{2c_2(2+c_0)}, \frac{1}{K_0(r,\nu)+2K_1(r,\nu)}\right\}, \tag{2.65}$$

where $C(\rho)$ is defined by (2.59). Let $\rho \in (0, \bar\rho_2]$ be arbitrary. Assume in addition that $u_0, \hat u_0 \in U$ are such that $\min\{2\|\hat e\|_U, 2\|\hat e_0\|_U\} < \rho$ and $\hat e_0 = A^\nu w$ for some $w \in U$. Then there hold

$$\|u_n - \hat u_n\|_U \le 2\|u_0 - \hat u_0\|_U + \pi_1(\rho)\|w\|_U \alpha_n^\nu \tag{2.66}$$

and

$$\|G_\dagger(u_n - \hat u_n)\|_U \le c_0\alpha_n^{1/2}\|u_0 - \hat u_0\|_U + \pi_2(\rho)\|w\|_U\alpha_n^{\nu+1/2} \tag{2.67}$$

for all $n \ge 0$. Here

$$\pi_1(\rho) := C(\rho)K_0(r,\nu), \qquad \pi_2(\rho) := 2C(\rho)K_1(r,\nu).$$

Proof. We show (2.66) and (2.67) by induction on n. Easily, these estimates hold for $n = 0$. Assume that (2.66) and (2.67) are satisfied for all $0 \le n \le l$. We shall prove these estimates also hold for $n = l+1$. To that purpose, we apply Lemma 2.12 to obtain

$$u_{l+1} - \hat u_{l+1} = \prod_{m=0}^{l} \alpha_m (\alpha_m I + A)^{-1}(u_0 - \hat u_0) + \sum_{m=0}^{l} \alpha_m^{-1} \prod_{j=m}^{l} \alpha_j \left(\alpha_j I + A\right)^{-1} G_\dagger^* \sum_{i=1}^{4} h_m^{(i)} \tag{2.68}$$

and

$$G_\dagger(u_{l+1} - \hat u_{l+1}) = \prod_{m=0}^{l} \alpha_m (\alpha_m I + B)^{-1} G_\dagger(u_0 - \hat u_0) + \sum_{m=0}^{l} \alpha_m^{-1} \prod_{j=m}^{l} \alpha_j \left(\alpha_j I + B\right)^{-1} B \sum_{i=1}^{4} h_m^{(i)}. \tag{2.69}$$

Applying Lemma A.3 and Lemma A.4 to (2.68) and using Lemma 2.12, we obtain

$$\begin{aligned}
\|u_{l+1} - \hat u_{l+1}\|_U &\le \|u_0 - \hat u_0\|_U + \frac12 \sum_{m=0}^{l} \alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \sum_{i=1}^{4}\|h_m^{(i)}\|_Y \\
&\le \|u_0 - \hat u_0\|_U + \frac12 C(\rho)\sum_{m=0}^{l} \alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \\
&\quad \cdot \left[\alpha_m^{\nu+1/2}\|w\|_U + \alpha_m^{1/2}\|u_m - \hat u_m\|_U + \|G_\dagger(u_m - \hat u_m)\|_Y\right],
\end{aligned}$$

which together with the induction hypothesis as well as Lemmas A.1 and A.2 shows that

$$\begin{aligned}
\|u_{l+1} - \hat u_{l+1}\|_U &\le \|u_0 - \hat u_0\|_U + \frac12 C(\rho)(2+c_0)\|u_0 - \hat u_0\|_U \sum_{m=0}^{l}\alpha_m^{-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \\
&\quad + \frac12 C(\rho)(1+\pi_1(\rho)+\pi_2(\rho))\|w\|_U \sum_{m=0}^{l}\alpha_m^{\nu-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2} \\
&\le \|u_0 - \hat u_0\|_U\left[1 + \frac12 C(\rho)(2+c_0)c_1\right] + \frac12 C(\rho)K_0(r,\nu)(1+\pi_1(\rho)+\pi_2(\rho))\|w\|_U\alpha_{l+1}^{\nu}.
\end{aligned}$$

Thanks to (2.65), the definition of $C(\rho)$, the fact that $c_0/(2c_2) \le 2/c_1$, and the monotonic growth of κ, η, we obtain

$$\|u_{l+1} - \hat u_{l+1}\|_U \le 2\|u_0 - \hat u_0\|_U + C(\rho)K_0(r,\nu)\|w\|_U\alpha_{l+1}^{\nu}.$$

This verifies (2.66) for $n = l + 1$. It remains to prove (2.67) for $n = l + 1$. To this end, using similar argument as above, we obtain from (2.69), (2.58), Lemmas A.3 and A.4 that

$$\begin{aligned}\|G_\dagger(u_{l+1} - \hat{u}_{l+1})\|_Y &\le \frac{1}{2}\left(\sum_{j=0}^{l} \alpha_j^{-1}\right)^{-1/2} \|u_0 - \hat{u}_0\|_U + \sum_{m=0}^{l} \alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1} \sum_{i=1}^{4} \|h_m^{(i)}\|_Y \\ &\le \frac{1}{2} c_0 \alpha_{l+1}^{1/2}\|u_0 - \hat{u}_0\|_U + C(\rho)\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1} \\ &\quad \cdot \left[\alpha_m^{\nu+1/2}\|w\|_U + \alpha_m^{1/2}\|u_m - \hat{u}_m\|_U + \|G_\dagger(u_m - \hat{u}_m)\|_Y\right].\end{aligned}$$

The induction hypothesis as well as Lemmas A.1 and A.2 then imply that

$$\begin{aligned}\|G_\dagger(u_{l+1} - \hat{u}_{l+1})\|_Y &\le \frac{1}{2} c_0 \alpha_{l+1}^{1/2}\|u_0 - \hat{u}_0\|_U + C(\rho)(2 + c_0)\|u_0 - \hat{u}_0\|_U \sum_{m=0}^{l}\alpha_m^{-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1} \\ &\quad + C(\rho)(1 + \pi_1(\rho) + \pi_2(\rho))\|w\|_U \sum_{m=0}^{l}\alpha_m^{\nu-1/2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1} \\ &\le \|u_0 - \hat{u}_0\|_U \alpha_{l+1}^{1/2}\left[\frac{1}{2}c_0 + C(\rho)(2 + c_0)c_2\right] \\ &\quad + C(\rho)K_1(r,\nu)(1 + \pi_1(\rho) + \pi_2(\rho))\|w\|_U \alpha_{l+1}^{\nu+1/2} \\ &\le c_0\|u_0 - \hat{u}_0\|_U \alpha_{l+1}^{1/2} + 2C(\rho)K_1(r,\nu)\|w\|_U \alpha_{l+1}^{\nu+1/2},\end{aligned}$$

where the last inequality follows from (2.65), the definition of $C(\rho)$, and the monotonic growth of κ, η. We thus obtain the desired conclusion. □

The following corollary is a direct consequence of Lemmas 2.11 and 2.13.

Corollary 2.14. *Under the assumptions of Lemma 2.13, there hold*

$$\|e_n\|_U \le 2\|u_0 - \hat{u}_0\|_U + (2c_0 + \pi_1(\rho))\,\alpha_n^{\nu}\|w\|_U \tag{2.70}$$

and

$$\|G_\dagger e_n\|_Y \le c_0\alpha_n^{1/2}\|u_0 - \hat{u}_0\|_U + \left(2c_0^2 + \pi_2(\rho)\right)\alpha_n^{\nu+1/2}\|w\|_U \tag{2.71}$$

for all $n \ge 0$.

In the remainder of this subsection, we show the convergence to $u^\dagger$ of the sequence $\{u_n\}$.

Theorem 2.15. *Let $\{\alpha_n\}$ be defined by (2.10) and (2.11). Assume that Assumption (A1) holds and that there exists a constant $\bar{\rho}_2$ satisfying (2.65) corresponding to $\nu = \frac{1}{4}$. Let $\rho \in (0, \bar{\rho}_2]$ and $u_0 \in U$ satisfy $4\|e_0\|_U < \rho$ and $e_0 \in \mathcal{N}(G_\dagger)^\perp$. Then, there holds*

$$\|e_n\|_U \to 0 \quad \text{and} \quad \frac{\|G_\dagger e_n\|_Y}{\sqrt{\alpha_n}} \to 0 \quad \text{as } n \to \infty. \tag{2.72}$$

Proof. Let $\varepsilon > 0$ be such that $4\varepsilon < \rho$. Since $e_0 \in \mathcal{N}(G_\dagger)^\perp$ and

$$\mathcal{N}(G_\dagger)^\perp = \overline{\mathcal{R}(G_\dagger^*)} = \overline{\mathcal{R}(A^{1/2})} \subset \overline{\mathcal{R}(A^{1/4})},$$

there exists an element $\hat{u} \in U$ such that $\|\hat{u}_0 - u_0\| < \varepsilon$ and $\hat{u}_0 - u^\dagger = A^{1/4}w$ for some $w \in U$. Obviously, $2\|\hat{e}_0\|_U < \rho$ with $\hat{e}_0 := \hat{u}_0 - u^\dagger$. Applying Corollary 2.14 to the case $\nu = 1/4$ leads to the estimates

$$\|e_n\|_U \le 2\|u_0 - \hat{u}_0\|_U + (2c_0 + \pi_1(\rho))\,\alpha_n^{1/4}\|w\|_U \tag{2.73}$$

and

$$\|G_\dagger e_n\|_Y \leq c_0\alpha_n^{1/2}\|u_0 - \hat{u}_0\|_U + \left(2c_0^2 + \pi_2(\rho)\right)\alpha_n^{3/4}\|w\|_U \tag{2.74}$$

are satisfied. Since $\alpha_n \to 0$ as $n \to \infty$, there exists a number $n_0 := n_0(\varepsilon, \|w\|_U)$ such that

$$(2c_0 + \pi_1(\rho))\,\alpha_n^{1/4}\|w\|_U \leq \varepsilon \quad \text{for all } n \geq n_0.$$

This and (2.73) give $\|e_n\|_U \leq 3\varepsilon$ for all $n \geq n_0$. The first limit in (2.72) then follows. The second limit in (2.72) is similarly obtained from (2.74). □

2.3 ASYMPTOTIC STABILITY ESTIMATES

This subsection provides some estimates on $\|u_n^\delta - u_n\|_U$ and $\|G_\dagger(u_n^\delta - u_n)\|_Y$ with $0 \leq n \leq \tilde{N}_\delta$ that are crucial to prove the regularization property of the modified Levenberg–Marquardt method.

Proposition 2.16. *Let all assumptions in Lemma 2.13 hold true. Assume furthermore that a positive constant $\rho_3 \leq \min\{\rho_2, \bar{\rho}_2\}$ exists such that*

$$\begin{cases} T_1(\rho_3) + T_2(\rho_3) \leq 2 + c_0, \\ \pi_1(\rho_3) + \pi_2(\rho_3) + T_1(\rho_3) + T_2(\rho_3) \leq 2c_0(1 + c_0), \\ L_1(\rho_3)(3 + c_3 + T_3(\rho_3)) \leq 1, \\ T_3(\rho_3) \leq 3 + c_3 \end{cases} \tag{2.75}$$

with

$$T_1(\rho) := (2c_0 + 2c_0^2)K_0(r,\nu)L_1(\rho), \quad T_2(\rho) := 4c_0(1 + c_0)K_1(r,\nu)L_1(\rho), \quad T_3(\rho) := 2c_4(3 + c_3)L_1(\rho),$$

$L_1(\rho)$ defined as in Lemma 2.7, and $\pi_1(\rho)$ and $\pi_2(\rho)$ given in Lemma 2.13. Let $\rho \leq \rho_3$ and $u_0, \hat{u}_0 \in U$ be such that $\min\{2\|e_0\|_U, 2\|\hat{e}_0\|_U\} < \rho$ *and $\hat{e}_0 = A^\nu w$ for some $w \in U$. Then there hold*

$$\|u_n^\delta - u_n\|_U \leq T_1(\rho)\left(\|u_0 - \hat{u}_0\|_U + \alpha_n^\nu\|w\|_U\right) + c_3\frac{\delta}{\sqrt{\alpha_n}} \tag{2.76}$$

and

$$\|G_\dagger(u_n^\delta - u_n)\|_Y \leq T_2(\rho)\left(\|u_0 - \hat{u}_0\|_U\alpha_n^{1/2} + \alpha_n^{\nu+1/2}\|w\|_U\right) + (2 + T_3(\rho))\delta. \tag{2.77}$$

Proof. We show (2.76) and (2.77) by induction on $0 \leq n \leq \tilde{N}_\delta$. It is easy to see that these estimates are valid for $n = 0$. Now for any fixed $0 \leq l < \tilde{N}_\delta$ we assume that (2.76) and (2.77) are fulfilled for all $0 \leq n \leq l$ and show that these estimates also hold true for $n = l + 1$. To this end, using (2.39), the induction hypothesis, and Corollary 2.14, we estimate

$$\begin{aligned}
&\|u_{l+1}^\delta - u_{l+1}\|_U \\
&\quad\leq \frac{L_1(\rho)}{2}\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}\|u_0 - \hat{u}_0\|_U\alpha_m^{1/2}\left(2 + c_0 + T_1(\rho) + T_2(\rho)\right) \\
&\qquad + \frac{L_1(\rho)}{2}\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}\|w\|_U\alpha_m^{\nu+1/2}\left(2c_0 + 2c_0^2 + \pi_1(\rho) + \pi_2(\rho) + T_1(\rho) + T_2(\rho)\right) \\
&\qquad + \frac{L_1(\rho)}{2}\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}\delta(2 + c_3 + T_3(\rho)) + \frac{1}{2}c_3\left(1 + L_1(\rho)\right)\frac{\delta}{\sqrt{\alpha_{l+1}}},
\end{aligned}$$

which together with Lemmas A.1 and A.2 leads to

$$\begin{aligned}\|u^\delta_{l+1} - u_{l+1}\|_U \le{}& \frac{1}{2}L_1(\rho)\|u_0 - \hat{u}_0\|_U \left(2 + c_0 + T_1(\rho) + T_2(\rho)\right) c_1\\ &+ \frac{1}{2}L_1(\rho)\|w\|_U \left(2c_0 + 2c_0^2 + \pi_1(\rho) + \pi_2(\rho) + T_1(\rho) + T_2(\rho)\right) K_0(r,\nu)\alpha_{l+1}^{\nu}\\ &+ \frac{1}{2}L_1(\rho)(2 + c_3 + T_3(\rho))c_3\delta\alpha_{l+1}^{-1/2} + \frac{1}{2}c_3\left(1 + L_1(\rho)\right)\frac{\delta}{\sqrt{\alpha_{l+1}}},\end{aligned}$$

or, equivalently,

$$\begin{aligned}\|u^\delta_{l+1} - u_{l+1}\|_U \le{}& \frac{1}{2}c_3\left[1 + L_1(\rho)(3 + c_3 + T_3(\rho))\right]\frac{\delta}{\sqrt{\alpha_{l+1}}}\\ &+ \frac{1}{2}L_1(\rho)\|u_0 - \hat{u}_0\|_U \left(2 + c_0 + T_1(\rho) + T_2(\rho)\right) c_1\\ &+ \frac{1}{2}L_1(\rho)\|w\|_U \left(2c_0 + 2c_0^2 + \pi_1(\rho) + \pi_2(\rho) + T_1(\rho) + T_2(\rho)\right) K_0(r,\nu)\alpha_{l+1}^{\nu}.\end{aligned}$$

From this and (2.75), the monotonic growth of κ and η gives

$$\|u^\delta_{l+1} - u_{l+1}\|_U \le T_1(\rho)\left(\|u_0 - \hat{u}_0\|_U + \alpha_{l+1}^{\nu}\|w\|_U\right) + c_3\frac{\delta}{\sqrt{\alpha_{l+1}}}.$$

The estimate (2.76) is thus verified for $n = l + 1$. Similarly, from (2.40), the induction hypothesis, and Corollary 2.14, we obtain

$$\begin{aligned}&\|G_\dagger(u^\delta_{l+1} - u_{l+1}) - y^\delta + y^\dagger\|_Y\\ &\quad\le L_1(\rho)\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\|u_0 - \hat{u}_0\|_U\alpha_m^{1/2}\left(2 + c_0 + T_1(\rho) + T_2(\rho)\right)\\ &\qquad+ L_1(\rho)\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\|w\|_U\alpha_m^{\nu+1/2}\left(2c_0 + 2c_0^2 + \pi_1(\rho) + \pi_2(\rho) + T_1(\rho) + T_2(\rho)\right)\\ &\qquad+ L_1(\rho)\sum_{m=0}^{l}\alpha_m^{-1}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1}\delta(2 + c_3 + T_3(\rho)) + \delta\left(1 + c_4L_1(\rho)\right),\end{aligned}$$

which together with Lemmas A.1 and A.2 leads to

$$\begin{aligned}&\|G_\dagger(u^\delta_{l+1} - u_{l+1}) - y^\delta + y^\dagger\|_Y\\ &\qquad\le \delta\left[1 + L_1(\rho)c_4(3 + c_3 + T_3(\rho))\right] + L_1(\rho)\|u_0 - \hat{u}_0\|_U\left(2 + c_0 + T_1(\rho) + T_3(\rho)\right)c_2\alpha_{l+1}^{1/2}\\ &\qquad\quad+ L_1(\rho)\|w\|_U\left(2c_0 + 2c_0^2 + \pi_1(\rho) + \pi_2(\rho) + T_1(\rho) + T_2(\rho)\right)K_1(r,\nu)\alpha_{l+1}^{\nu+1/2}.\end{aligned}$$

By virtue of (2.75) and the monotonicity of κ and η, we have that

$$\begin{aligned}\|G_\dagger(u^\delta_{l+1} - u_{l+1}) - y^\delta + y^\dagger\|_Y \le{}& L_1(\rho)\|u_0 - \hat{u}_0\|_U\left(4 + 2c_0\right)c_2\alpha_{l+1}^{1/2}\\ &+ L_1(\rho)\|w\|_U\left(4c_0 + 4c_0^2\right)K_1(r,\nu)\alpha_{l+1}^{\nu+1/2}\\ &+ \delta\left[1 + 2L_1(\rho)c_4(3 + c_3)\right]\\ \le{}& T_2(\rho)\left(\|u_0 - \hat{u}_0\|_U\alpha_{l+1}^{1/2} + \|w\|_U\alpha_{l+1}^{\nu+1/2}\right) + \delta(1 + T_3(\rho)).\end{aligned}\tag{2.78}$$

Noting that $L_1(\rho)(4 + 2c_0)c_2 \le L_1(\rho)(4c_0 + 4c_0^2)K_1(r,\nu) =: T_2(\rho)$, the estimate (2.77) is therefore satisfied for $n = l + 1$. ◻

As a result of (2.78) and Proposition 2.16, we have the following corollary, whose proof is similar to that of (2.44).

Corollary 2.17. *Let all assumptions of Proposition 2.16 be satisfied. Then there holds*

$$\|F(u_n^\delta) - F(u_n) - y^\delta + y^\dagger\|_Y \leq T_4(\rho)\left(\|u_0 - \hat{u}_0\|_U \alpha_n^{1/2} + \|w\|_U \alpha_n^{\nu+1/2}\right) + \delta(1 + T_5(\rho))$$

for all $0 \leq n \leq \tilde{N}_\delta$, *where*

$$T_4(\rho) := T_2(\rho)\left(1 + \kappa(\rho) + \frac{1+\kappa_0}{1-\eta_0}\eta(\rho)\right), \quad T_5(\rho) := T_3(\rho) + \left(\kappa(\rho) + \frac{1+\kappa_0}{1-\eta_0}\eta(\rho)\right)(2 + T_3(\rho))$$

and $T_2(\rho)$ *and* $T_3(\rho)$ *are defined as in Proposition 2.16.*

2.4 REGULARIZATION PROPERTY

This subsection is concerned with the convergence of the sequence $\{u^\delta_{N_\delta}\}$ as $\delta \to 0$, provided that $e_0 \in \mathcal{N}(G_\dagger)^\perp$ and that u_0 is sufficiently close to $u^\dagger$. Let $\{\delta_k\}$ be a positive zero sequence. To simplify the notation, from now on we write $N_k := N_{\delta_k}$. The next lemma will be used to show the convergence of subsequences of $\{u^{\delta_k}_{N_k}\}$ for the case where $\{N_k\}$ is bounded.

Lemma 2.18. *Assume that all assumptions of Lemma 2.5 are satisfied. Let* $\overline{N} \in \mathbb{N}$ *be arbitrary but fixed and let* $\{\delta_k\}$ *be a positive zero sequence such that* $\tilde{N}_{\delta_k} \geq \overline{N}$ *for all* $k \geq 1$. *Assume in addition that Assumption (A2) holds. Then for any subsequence of* $\{\delta_k\}$ *there exist a subsequence* $\{\delta_{k_i}\}$ *and elements* $\tilde{u}_j \in \overline{B}_U(u^\dagger, \rho)$ *for* $0 \leq j \leq \overline{N}$ *such that*

$$u_j^{\delta_{k_i}} \to \tilde{u}_j \quad as \quad i \to \infty \tag{2.79}$$

for all $0 \leq j \leq \overline{N}$.

Proof. We shall show by induction on j the existence of a subsequence $\{\delta_{k_i}\}$ and elements $\tilde{u}_j \in \overline{B}_U(u^\dagger, \rho)$ for $0 \leq j \leq \overline{N}$ that satisfy (2.79).

First, (2.79) holds for $j = 0$ with $\tilde{u}_0 := u_0$. By a slight abuse of notation, we assume $\{\delta_{k_i}\}$ itself is a subsequence satisfying $u_j^{\delta_{k_i}} \to \tilde{u}_j$ as $i \to \infty$ for some $\tilde{u}_j \in \overline{B}_U(u^\dagger, \rho)$ and some $0 \leq j < \overline{N}$. To simplify the notation, we write

$$u_j^{(i)} := u_j^{\delta_{k_i}}, \qquad u_{j+1}^{(i)} := u_{j+1}^{\delta_{k_i}}, \qquad A_j^{(i)} := A_j^{\delta_{k_i}}, \qquad \text{and} \quad G_j^{(i)} := G_j^{\delta_{k_i}}.$$

It follows from (2.9) and Lemma 2.3 that

$$\begin{aligned} u_{j+1}^{(i)} &= u_j^{(i)} + \left(\alpha_j I + A_j^{(i)}\right)^{-1} G_j^{(i)*}\left(y^{\delta_{k_i}} - F(u_j^{(i)})\right) \\ &= u_j^{(i)} + (\alpha_j I + A)^{-1} G_\dagger^*\left[I + S_{\alpha_j}\left(u_j^{(i)}, u^\dagger\right)\right]\left(y^{\delta_{k_i}} - F(u_j^{(i)})\right) \\ &= u_j^{(i)} + (\alpha_j I + A)^{-1} G_\dagger^*\left[I + S_{\alpha_j}\left(u_j^{(i)}, u^\dagger\right)\right]\left(y^\dagger - F(\tilde{u}_j)\right) \\ &\quad + (\alpha_j I + A)^{-1} G_\dagger^*\left[I + S_{\alpha_j}\left(u_j^{(i)}, u^\dagger\right)\right]\left(y^{\delta_{k_i}} - F(u_j^{(i)}) - y^\dagger + F(\tilde{u}_j)\right) \end{aligned}$$

and thus

$$u_{j+1}^{(i)} = u_j^{(i)} + a_i + b_i, \tag{2.80}$$

where

$$a_i := \left(\alpha_j I + A\right)^{-1} G_\dagger^* h_i,$$
$$b_i := \left(\alpha_j I + A\right)^{-1} G_\dagger^* \left[I + S_{\alpha_j}\left(u_j^{(i)}, u^\dagger\right)\right]\left(y^{\delta_{k_i}} - F(u_j^{(i)}) - y^\dagger + F(\tilde{u}_j)\right)$$

with

$$h_i := \left[I + S_{\alpha_j}\left(u_j^{(i)}, u^\dagger\right)\right]\left(y^\dagger - F(\tilde{u}_j)\right).$$

Applying Lemma A.4 with $m = l$ and using (2.16) gives

$$\|b_i\|_U \le \frac{1}{2\sqrt{\alpha_j}}\left(1 + 3\kappa(\rho)\right)\|y^{\delta_{k_i}} - F(u_j^{(i)}) - y^\dagger + F(\tilde{u}_j)\|_Y.$$

Letting $i \to \infty$ and employing the continuity of F yields

$$b_i \to 0 \quad \text{as} \quad i \to \infty. \tag{2.81}$$

Furthermore, (2.16) ensures the boundedness of sequence $\{h_i\}$ in Y. Moreover, as a result of Assumption (A2), the operator $\left(\alpha_j I + A\right)^{-1} G_\dagger^*$ is compact. This implies that $\{a_i\}$ is compact in U. There thus exist a subsequence of $\{a_i\}$, denoted by the same symbol, and an element $a \in U$ such that

$$a_i \to a \quad \text{as} \quad i \to \infty. \tag{2.82}$$

From (2.80), (2.81), (2.82), and the induction hypothesis, we deduce $u_{j+1}^{(i)} \to \tilde{u}_j + a =: \tilde{u}_{j+1}$. Consequently, (2.79) holds for $j + 1$. From Lemma 2.5, we have $u_{j+1}^{(i)} \in \overline{B}(u^\dagger, \rho)$ for all $i \ge 0$ and so $\tilde{u}_{j+1} \in \overline{B}(u^\dagger, \rho)$. The proof is complete. □

Before representing our main theorem, we give a result on the *asymptotic stability* of the modified Levenberg–Marquardt method. The definition of this notion in the following proposition generalizes that in [3, Def. 2.1].

Proposition 2.19. *Let Assumptions (A1) and (A2) be fulfilled. Assume that there exists a constant ρ_3 satisfying (2.75) in Proposition 2.16 corresponding to $\nu = \frac{1}{4}$. Let $\rho \in (0, \rho_3]$ and $u_0 \in U$ satisfy $e_0 \in \mathcal{N}(G_\dagger)^\perp$ and $2(2 + c_1\gamma_0)\|e_0\|_U < \rho$. Then the modified Levenberg–Marquardt method (2.9)–(2.12) is* asymptotically stable *in the following sense: For any subsequence of a positive zero sequence $\{\delta_k\}$, there exist a subsequence $\{\delta_{k_i}\}$ and elements $\tilde{u}_n \in \overline{B}_U(u^\dagger, \rho)$ for all $0 \le n \le \overline{N} := \lim_{i\to\infty} N_{k_i}$ (where the last inequality is strict if $\overline{N} = \infty$) such that*

$$\lim_{n\to\overline{N}}\left(\limsup_{i\to\infty}\|u_n^{\delta_{k_i}} - \tilde{u}_n\|_U\right) = 0 \tag{2.83}$$

and

$$\tilde{u}_n \to u^* \quad \textit{as} \quad n \to \overline{N} \tag{2.84}$$

for some $u^* \in S_\rho(u^\dagger)$.

Proof. Let $\{\delta_k\}$ itself be a subsequence. Since $\{N_k\}$ is a sequence of integers, there exists a subsequence $\{N_{k_i}\}$ such that either it is a constant sequence or it tends to infinity. For the first case where $N_{k_i} = \overline{N}$ for some integer $\overline{N}$ and for all $i \ge 0$, Lemma 2.18 and the discrepancy principle (2.12) give the conclusion of the proposition. For the second case where $N_{k_i} \to \infty$, we shall show that the elements $\tilde{u}_n := u_n$, $n \ge 0$, any subsequence $\{\delta_{k_i}\}$, and $u^* := u^\dagger$ satisfy (2.83) and (2.84). To this end, we first see that Theorem 2.15 implies (2.84). Let $\varepsilon > 0$ be arbitrary small but fixed such that $2(2 + c_1\gamma_0)\varepsilon \le \rho$. Since $e_0 \in \mathcal{N}(G_\dagger)^\perp$

and $\mathcal{N}(G_\dagger)^\perp = \overline{\mathcal{R}(G_\dagger^*)} = \overline{\mathcal{R}(A^{1/2})} \subset \overline{\mathcal{R}(A^{1/4})}$, there is an element $\hat u \in U$ such that $\|\hat u_0 - u_0\| < \varepsilon$ and $\hat e_0 := \hat u_0 - u^\dagger = A^{1/4}w$ for some $w \in U$. Obviously, we have $(2 + c_1\gamma_0)\|\hat e_0\|_U < \rho$. From this and the choice of ρ, we thus can apply Proposition 2.16 to obtain the estimate

$$\|u_n^{\delta_{k_i}} - u_n\|_U \leq T_1(\rho)\left(\|u_0 - \hat u_0\|_U + \alpha_n^{1/4}\|w\|_U\right) + c_3\frac{\delta_{k_i}}{\sqrt{\alpha_n}}$$

for all $0 \leq n \leq N_{k_i}$ and for all $i \geq 0$. By letting $i \to \infty$ and then $n \to \infty$, we therefore have

$$\limsup_{n\to\infty}\left(\limsup_{i\to\infty}\|u_n^{\delta_{k_i}} - u_n\|_U\right) \leq T_1(\rho)\varepsilon.$$

The limit (2.83) then follows. □

We are now well prepared to derive the main result of the paper, where some lines in the proof follow the ones in [8].

Theorem 2.20 (regularization property). *Let $\{\alpha_n\}$ be defined by (2.10) and (2.11) and let $\{\delta_k\}$ be a positive zero sequence. Assume that Assumptions (A1) and (A2) hold and that $\tau > \tau_0 > 1$, $\gamma_0 > \frac{2c_0}{(1-\eta_0)(\tau-\tau_0)}$. Assume further that a constant $\rho_3 \leq \rho_0$ exists and satisfies (2.75) corresponding to $\nu = \frac{1}{4}$ as well as*

$$T_5(\rho_3) < \tau - 1 \tag{2.85}$$

with $T_5(\rho)$ defined as in Corollary 2.17.

Let $\rho \in (0, \rho_3]$ and $u_0 \in U$ satisfy $2(2 + c_1\gamma_0)\|u_0 - u^\dagger\|_U < \rho$. Then the method (2.9)–(2.12) is well-defined and the integer N_{δ_k} defined by the discrepancy principle (2.12) satisfies

$$N_{\delta_k} = O\left(1 + |\log(\delta_k)|\right). \tag{2.86}$$

Moreover, if $u_0 - u^\dagger \in \mathcal{N}(G_\dagger)^\perp$, then

$$u_{N_{\delta_k}}^{\delta_k} \to u^\dagger \quad \text{as} \quad k \to \infty. \tag{2.87}$$

Proof. Under the assumptions, the well-posedness of the method follows from Lemma 2.5, while the logarithmic estimate (2.86) is shown in Lemma 2.10. It is therefore sufficient to prove (2.87).

To this end, we first assume that there exists a subsequence δ_{k_i} such that $N_{k_i} = \overline{N}$ for all $i \geq 0$. By virtue of Lemma 2.18, there exist a subsequence $\{k_m\}$ of $\{k_i\}$ and elements $\tilde u_j \in \overline{B}_U(u^\dagger, \rho)$ with $j = 0, 1, \dots, \overline{N}$ such that

$$u_j^{\delta_{k_m}} \to \tilde u_j \quad \text{as} \quad m \to \infty \tag{2.88}$$

for all $0 \leq j \leq \overline{N}$. Moreover, from the discrepancy principle (2.12), we obtain

$$\|F(u_{\overline{N}}^{\delta_{k_m}}) - y^{\delta_{k_m}}\|_Y \leq \tau\delta_{k_m}.$$

Letting $m \to \infty$ and using the continuity of F yields

$$F(\tilde u_{\overline{N}}) = y^\dagger,$$

which together with (2.88) yields that

$$u_{\overline{N}}^{\delta_{k_m}} \to \tilde u_{\overline{N}} \quad \text{as } m \to \infty \tag{2.89}$$

with $\tilde{u}_{\overline{N}} \in S_\rho(u^\dagger)$. We now show that $\tilde{u}_{\overline{N}} = u^\dagger$. According to (2.9), it holds for all $0 \le n \le \overline{N} - 1$ and $m \ge 0$ that

$$u_{n+1}^{\delta_{k_m}} - u_n^{\delta_{k_m}} = G_n^{\delta_{k_m}*} \left(\alpha_n I + G_n^{\delta_{k_m}} G_n^{\delta_{k_m}*}\right)^{-1} (y^{\delta_{k_m}} - F(u_n^{\delta_{k_m}})) \subset \mathcal{R}\left(G_n^{\delta_{k_m}*}\right)$$

Combining this with (2.4), we have $u_{n+1}^{\delta_{k_m}} - u_n^{\delta_{k_m}} \subset \mathcal{R}(G_\dagger^*) \subset \mathcal{N}(G_\dagger)^\perp$ for all $m \ge 0$ and $0 \le n \le \overline{N} - 1$. Consequently, using $u_0 - u^\dagger \in \mathcal{N}(G_\dagger)^\perp$, there holds $u_{\overline{N}}^{\delta_{k_m}} - u^\dagger \subset \mathcal{R}(G_\dagger^*) \subset \mathcal{N}(G_\dagger)^\perp$. From this and the limit (2.89), we have $\tilde{u}_{\overline{N}} - u^\dagger \in \mathcal{N}(G_\dagger)^\perp$. On the other hand, as a result of (GTCC) and the fact that $u^\dagger, \tilde{u}_{\overline{N}} \in S_\rho(u^\dagger)$, it holds that $\tilde{u}_{\overline{N}} - u^\dagger \in \mathcal{N}(G_\dagger)$. We thus have $\tilde{u}_{\overline{N}} - u^\dagger \in \mathcal{N}(G_\dagger) \cap \mathcal{N}(G_\dagger)^\perp = \{0\}$. Therefore, a subsequence-subsequence argument can conclude that

$$u_{\overline{N}}^{\delta_{k_i}} \to u^\dagger \quad \text{as } i \to \infty. \tag{2.90}$$

We next assume that there exists a subsequence δ_{k_i} such that $N_{k_i} \to \infty$ as $i \to \infty$. In this case, let $\varepsilon > 0$ be arbitrary but fixed such that $0 < 2(2 + c_1\gamma_0)\varepsilon < \rho$. Since $e_0 \in \mathcal{N}(G_\dagger)^\perp$ and $\mathcal{N}(G_\dagger)^\perp = \overline{\mathcal{R}(G_\dagger^*)} = \overline{\mathcal{R}(A^{1/2})} \subset \overline{\mathcal{R}(A^{1/4})}$, there exists an element $\hat{u} \in U$ such that $\|\hat{u}_0 - u_0\| < \varepsilon$ and $\hat{u}_0 - u^\dagger = A^{1/4}w$ for some $w \in U$. Easily, $2\|\hat{e}_0\|_U \le (2 + c_1\gamma_0)\|\hat{e}_0\|_U < \rho$ with $\hat{e}_0 := \hat{u}_0 - u^\dagger$. From Proposition 2.16 and Corollary 2.17, we have

$$\|u_j^{\delta_{k_i}} - u_j\|_U \le T_1(\rho)\left(\|u_0 - \hat{u}_0\|_U + \alpha_j^{1/4}\|w\|_U\right) + c_3\frac{\delta_{k_i}}{\sqrt{\alpha_j}} \tag{2.91}$$

and

$$\|F(u_j^{\delta_{k_i}}) - F(u_j) - y^{\delta_{k_i}} + y^\dagger\|_Y \le T_4(\rho)\left(\|u_0 - \hat{u}_0\|_U\alpha_j^{1/2} + \alpha_j^{3/4}\|w\|_U\right) + (1 + T_5(\rho))\delta_{k_i} \tag{2.92}$$

for all $0 \le j \le N_{k_i}$ and for all $i \ge 0$. On the other hand, we can conclude from the discrepancy principle (2.12) and the estimate (2.7) for all $0 \le j < N_{k_i}$ that

$$\begin{aligned}
\tau\delta_{k_i} &< \|F(u_j^{\delta_{k_i}}) - y^{\delta_{k_i}}\|_Y \\
&\le \|F(u_j^{\delta_{k_i}}) - F(u_j) - y^{\delta_{k_i}} + y^\dagger\|_Y + \|F(u_j) - y^\dagger\|_Y \\
&\le \|F(u_j^{\delta_{k_i}}) - F(u_j) - y^{\delta_{k_i}} + y^\dagger\|_Y + \frac{1}{1 - \eta_0}\|G_\dagger e_j\|_Y.
\end{aligned}$$

Combining this with (2.92) yields for all $0 \le j < N_{k_i}$ that

$$\delta_{k_i}(\tau - (1 + T_5(\rho))) \le T_4(\rho)\left(\|u_0 - \hat{u}_0\|_U\alpha_j^{1/2} + \alpha_j^{3/4}\|w\|_U\right) + \frac{1}{1 - \eta_0}\|G_\dagger e_j\|_Y$$

and thus

$$(\tau - (1 + T_5(\rho)))\frac{\delta_{k_i}}{\sqrt{\alpha_{N_{k_i}}}} \le T_4(\rho)\left(\frac{1}{r^{1/2}}\|u_0 - \hat{u}_0\|_U + \frac{1}{r^{3/4}}\alpha_{N_{k_i}}^{1/4}\|w\|_U\right) + \frac{1}{\sqrt{r}(1 - \eta_0)}\frac{\|G_\dagger e_{N_{k_i}-1}\|_Y}{\sqrt{\alpha_{N_{k_i}-1}}}.$$

Letting $i \to \infty$, employing (2.85), and using the second limit in (2.72) gives

$$(\tau - (1 + T_5(\rho)))\limsup_{i\to\infty}\frac{\delta_{k_i}}{\sqrt{\alpha_{N_{k_i}}}} \le \frac{T_4(\rho)}{r^{1/2}}\|u_0 - \hat{u}_0\|_Y \le \frac{T_4(\rho)}{r^{1/2}}\varepsilon.$$

Noting that $N_{k_i} \to \infty$ as $i \to \infty$, this implies that

$$\limsup_{i\to\infty}\frac{\delta_{k_i}}{\sqrt{\alpha_{N_{k_i}}}} = 0.$$

This and (2.91) yield

$$\limsup_{i\to\infty} \|u_{N_{k_i}}^{\delta_{k_i}} - u_{N_{k_i}}\|_U \le T_1(\rho)\|u_0 - \hat{u}_0\|_U \le T_1(\rho)\varepsilon$$

and hence, since $\varepsilon > 0$ was arbitrary,

$$\limsup_{i\to\infty} \|u_{N_{k_i}}^{\delta_{k_i}} - u_{N_{k_i}}\|_U = 0.$$

Together with (2.72), this implies that

$$u_{N_{k_i}}^{\delta_{k_i}} \to u^\dagger \quad \text{as } i \to \infty.$$

From this, (2.90), and a subsequence-subsequence argument, we obtain (2.87). □

3 ITERATIVE REGULARIZATION FOR A NON-SMOOTH FORWARD OPERATOR

In this section, we study the solution operator to (1.2) based on previous results from [2, 3]. In particular, we show that this operator together with one of its Bouligand subderivatives satisfies the assumptions in Section 2.

3.1 WELL-POSEDNESS AND DIRECTIONAL DIFFERENTIABILITY

Let $\Omega \subset \mathbb{R}^d$, $2 \le d \le 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. For $u \in L^2(\Omega)$, we consider the equation

$$\begin{cases} -\Delta y + y^+ = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $y^+(x) := \max(y(x), 0)$ for all $x \in \Omega$. From [27, Thm. 4.7], we obtain for each $u \in L^2(\Omega)$ a unique weak solution y_u belonging to $H_0^1(\Omega) \cap C(\overline{\Omega})$ and satisfying the a priori estimate

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\overline{\Omega})} \le c_\infty \|u\|_{L^2(\Omega)}$$

for some constant $c_\infty > 0$ independent of u.

Let us denote by $F : L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega}) \hookrightarrow L^2(\Omega)$ the solution operator of (3.1). As shown in [3, Prop. 3.1] (see also [2, Prop. 2.1]), F is Lipschitz continuous as a function from $L^2(\Omega)$ to $H_0^1(\Omega) \cap C(\overline{\Omega})$, that is,

$$\|F(u) - F(v)\|_{H_0^1(\Omega)} + \|F(u) - F(v)\|_{C(\overline{\Omega})} \le C_F \|u - v\|_{L^2(\Omega)} \tag{3.2}$$

for all $u, v \in L^2(\Omega)$ and for some constant C_F. Moreover, F is completely continuous as a function from $L^2(\Omega)$ to $H_0^1(\Omega)$ and from $L^2(\Omega)$ to itself. However, F is in general not Gâteaux differentiable, but it is Gâteaux differentiable at u if and only if $|\{F(u) = 0\}| = 0$.

Similarly to [3], we shall use as a replacement for the Fréchet derivative a Bouligand subderivative of F as the operator G_u in Section 2. We first define the set of *Gâteaux points* of F as

$$D := \{v \in L^2(\Omega) : F : L^2(\Omega) \to H_0^1(\Omega) \text{ is Gâteaux differentiable in } v\}.$$

Denoting the Gâteaux derivative of F at $u \in D$ by $F'(u) \in \mathbb{L}(L^2(\Omega), H_0^1(\Omega))$ by $F'(u)$, the (strong-strong) *Bouligand subdifferential* at $u \in L^2(\Omega)$ is then defined as

$$\begin{aligned} \partial_B F(u) := \{G_u \in \mathbb{L}(L^2(\Omega), H_0^1(\Omega)) : \text{there exists } \{u_n\}_{n\in\mathbb{N}} \subset D \text{ such that} \\ u_n \to u \text{ in } L^2(\Omega) \text{ and } F'(u_n)h \to G_u h \text{ in } H_0^1(\Omega) \text{ for all } h \in L^2(\Omega)\}. \end{aligned}$$

We have the following convenient characterization of a specific Bouligand subderivative of F.

Proposition 3.1 ([2, Prop. 3.16]). *Given $u \in L^2(\Omega)$, let $G_u : L^2(\Omega) \to H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ be the solution operator mapping $h \in L^2(\Omega)$ to the unique solution $\zeta \in H_0^1(\Omega)$ of*

$$\begin{cases} -\Delta\zeta + \mathbb{1}_{\{y_u>0\}}\zeta = h & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where $y_u := F(u)$. Then $G_u \in \partial_B F(u)$.

In general, for a given $h \in L^2(\Omega)$, the mapping $L^2(\Omega) \ni u \mapsto G_u h \in L^2(\Omega)$ is not continuous (see, e.g., [3, Exam. 3.8]), and the mapping $L^2(\Omega) \ni u \mapsto G_u \in \mathbb{L}(L^2(\Omega))$ is thus not continuous.

3.2 VERIFICATION OF ASSUMPTIONS

We now verify that the solution mapping for our example together with the mapping G_u defined as in Proposition 3.1 satisfies Assumption (A1) as well as allowing ρ to be taken sufficiently small to satisfy the conditions of Theorems 2.15 and 2.20. We begin with the verification of the generalized tangential cone condition (GTCC).

Proposition 3.2. *Let $\bar u \in L^2(\Omega)$, $\bar y := F(\bar u)$, and $\rho > 0$. Then there holds*

$$\|F(\hat u) - F(u) - G_u(\hat u - u)\|_{L^2(\Omega)} \le \eta(\rho)\|F(\hat u) - F(u)\|_{L^2(\Omega)}$$

for all $u, \hat u \in \overline{B}_{L^2(\Omega)}(\bar u, \rho)$ with

$$\eta(\rho) := C_\Omega \left|\{|\bar y| \le C_F\rho\}\right|^{1/14} \tag{3.4}$$

for some constant $C_\Omega > 0$.

Proof. Applying to [3, Lem. 3.9] for $p = \frac{7}{4}$ yields

$$\|F(\hat u) - F(u) - G_u(\hat u - u)\|_{L^2(\Omega)} \le C_\Omega M(u,\hat u)^{1/14}\|F(\hat u) - F(u)\|_{L^2(\Omega)} \tag{3.5}$$

for some constant C_Ω and $M(u,\hat u) := |\{y_u \le 0, y_{\hat u} > 0\} \cup \{y_u > 0, y_{\hat u} \le 0\}|$. According to (3.2), we thus have that

$$\|\bar y - y_u\|_{C(\overline{\Omega})} \le C_F\|\bar u - u\|_{L^2(\Omega)} \le C_F\rho =: \varepsilon$$

for all $u \in \overline{B}_{L^2(\Omega)}(\bar u, \rho)$ and $y_u := F(u)$. This implies, for any $u \in \overline{B}_{L^2(\Omega)}(\bar u, \rho)$, that

$$-\varepsilon + y_u(x) \le \bar y \le \varepsilon + y_u(x)$$

for all $x \in \overline{\Omega}$ with $y_u := F(u)$. We then have for any $u, \hat u \in \overline{B}_{L^2(\Omega)}(\bar u, \rho)$ that

$$\begin{aligned} \{y_u > 0, y_{\hat u} \le 0\} &\subset \{-\varepsilon \le \bar y \le \varepsilon\}, \\ \{y_u \le 0, y_{\hat u} > 0\} &\subset \{-\varepsilon \le \bar y \le \varepsilon\} \end{aligned}$$

with $y_u := F(u)$ and $y_{\hat u} := F(\hat u)$. It therefore holds that

$$\left|\mathbb{1}_{\{y_u>0\}} - \mathbb{1}_{\{y_{\hat u}>0\}}\right| = \left|\mathbb{1}_{\{y_u>0, y_{\hat u}\le 0\}} - \mathbb{1}_{\{y_{\hat u}>0, y_u\le 0\}}\right| \le \mathbb{1}_{\{-\varepsilon\le\bar y\le\varepsilon\}}. \tag{3.6}$$

From this, we have

$$M(u,\hat u) \le |\{|\bar y| \le \varepsilon\}| = |\{|\bar y| \le C_F\rho\}|,$$

which together with (3.5) deduces the desired result. □

We next construct, for any $u_1, u_2 \in L^2(\Omega)$, a bounded linear operator $Q(u_1,u_2) : L^2(\Omega) \to L^2(\Omega)$ that satisfies (2.4) and (2.5).

Lemma 3.3. *Let $u_1, u_2 \in L^2(\Omega)$ be arbitrary and let G_{u_i}, $i = 1, 2$, be defined as in Proposition 3.1. Then there exists a bounded linear operator $Q(u_1, u_2) : L^2(\Omega) \to L^2(\Omega)$ such that*

$$G_{u_1} = Q(u_1, u_2)G_{u_2} \tag{3.7}$$

and

$$\|I - Q(u_1, u_2)\|_{\mathbb{L}(L^2(\Omega))} \leq C(u_1, u_2), \tag{3.8}$$

where

$$C(u_1, u_2) := C_* \|\mathbb{1}_{\{F(u_1)>0\}} - \mathbb{1}_{\{F(u_2)>0\}}\|_{L^3(\Omega)}$$

with some constant $C_* > 0$ independent of u_1 and u_2.

Proof. To prove the existence of the bounded linear operator $Q(u_1, u_2)$, we first construct this operator on $H_0^1(\Omega)$ and then extend it to $L^2(\Omega)$ by density. To this end, we set $y_i := F(u_i)$ with $i = 1, 2$. We now define the linear operator $Q(u_1, u_2) : H_0^1(\Omega) \to H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ as follows: for any $v \in H_0^1(\Omega)$, we set $w := Q(u_1, u_2)v$ defined as the unique solution in $H_0^1(\Omega)$ to

$$-\Delta w + \mathbb{1}_{\{y_1>0\}} w = -\Delta v + \mathbb{1}_{\{y_2>0\}} v \quad \text{in } \Omega.$$

We now show that

$$\|Q(u_1, u_2)v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega) \tag{3.9}$$

and for some constant C independent of v. First, we have for any $v \in H_0^1(\Omega)$ that

$$-\Delta(w - v) + \mathbb{1}_{\{y_1>0\}}(w - v) = \left[\mathbb{1}_{\{y_2>0\}} - \mathbb{1}_{\{y_1>0\}}\right] v \quad \text{in } \Omega. \tag{3.10}$$

It follows that

$$\|w - v\|_{H_0^1(\Omega)} \leq C\|v\|_{L^2(\Omega)}$$

for some constant $C > 0$. This and the continuous embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ give

$$\|w - v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)},$$

which along with the triangle inequality yields (3.9). From the estimate (3.9) and the density of $H_0^1(\Omega)$ in $L^2(\Omega)$, the operator $Q(u_1, u_2)$ has a unique continuous extension, also denoted by $Q(u_1, u_2)$, from $L^2(\Omega)$ to $L^2(\Omega)$.

It remains to show (3.7) and (3.8). It is easy to obtain the identity (3.7) from the definition of $Q(u_1, u_2)$ and the uniqueness of solutions to (3.3). By density, to prove (3.8) we only need to show that

$$\|v - Q(u_1, u_2)v\|_{L^2(\Omega)} \leq C(u_1, u_2)\|v\|_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega). \tag{3.11}$$

Since Ω is bounded in $\mathbb{R}^d$ with $d \in \{2, 3\}$, one has $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. Testing (3.10) by $w - v$ and exploiting the Hölder inequality yield

$$\begin{aligned}\|\nabla(w - v)\|_{L^2(\Omega)}^2 &\leq \int_\Omega \left[\mathbb{1}_{\{y_2>0\}} - \mathbb{1}_{\{y_1>0\}}\right] v(w - v)dx \\ &\leq \|\mathbb{1}_{\{y_2>0\}} - \mathbb{1}_{\{y_1>0\}}\|_{L^3(\Omega)}\|v\|_{L^2(\Omega)}\|w - v\|_{L^6(\Omega)}.\end{aligned}$$

From this and the continuous embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\|\nabla(w - v)\|_{L^2(\Omega)} \leq C\|\mathbb{1}_{\{y_2>0\}} - \mathbb{1}_{\{y_1>0\}}\|_{L^3(\Omega)}\|v\|_{L^2(\Omega)}$$

for some constant C independent of u_1 and u_2. The Poincaré inequality thus implies that

$$\|w - v\|_{L^2(\Omega)} \leq C_*\|\mathbb{1}_{\{y_2>0\}} - \mathbb{1}_{\{y_1>0\}}\|_{L^3(\Omega)}\|v\|_{L^2(\Omega)},$$

which is identical to (3.11). □

Proposition 3.4. *Let $Q : L^2(\Omega)^2 \to \mathbb{L}(L^2(\Omega))$ be the mapping defined as in Lemma 3.3, let $\bar{u} \in L^2(\Omega)$ be arbitrary, and let ρ be a positive number. Then, for any $u_1, u_2 \in \overline{B}_{L^2(\Omega)}(\bar{u}, \rho)$, there holds*

$$\|I - Q(u_1, u_2)\|_{\mathbb{L}(L^2(\Omega))} \leq \kappa(\rho),$$

where

$$\kappa(\rho) := C_* |\{|\bar{y}| \leq C_F \rho\}|^{1/3} \tag{3.12}$$

with $\bar{y} := F(\bar{u})$ and C_* defined as in Lemma 3.3.

Proof. Set $\bar{y} = F(\bar{u})$. According to (3.2), we thus have that

$$\|\bar{y} - y_u\|_{C(\overline{\Omega})} \leq C_F \|\bar{u} - u\|_{L^2(\Omega)} \leq C_F \rho =: \varepsilon$$

for all $u \in \overline{B}_{L^2(\Omega)}(\bar{u}, \rho)$ and $y_u := F(u)$. Similar to (3.6), it holds that

$$\left|\mathbb{1}_{\{y_1 > 0\}} - \mathbb{1}_{\{y_2 > 0\}}\right| \leq \mathbb{1}_{\{-\varepsilon \leq \bar{y} \leq \varepsilon\}},$$

which together with the definition of $C(u_1, u_2)$ yields

$$C(u_1, u_2) \leq C_* |\{|\bar{y}| \leq \varepsilon\}|^{1/3}.$$

This and Lemma 3.3 give the desired conclusion. □

From (3.4) and (3.12), we immediately obtain that $\kappa(\rho)$ and $\eta(\rho)$ can be made arbitrarily small provided that $|\{\bar{y} = 0\}|$ is small enough. In particular, we deduce that (GTCC) holds with the required bound on the constant $\eta(\rho)$.

Corollary 3.5. *Let functions κ and η be, respectively, defined by (3.12) and (3.4). Let $\bar{u} \in L^2(\Omega)$ be such that $|\{F(\bar{u}) = 0\}|$ is sufficiently small. Then (2.3) holds.*

3.3 BOULIGAND–LEVENBERG–MARQUARDT ITERATION

The results obtained so far indicate that the solution mapping F of (3.1) and the mapping $u \mapsto G_u$ with G_u the Bouligand subderivative defined as in Proposition 3.1 satisfy Assumption (A1), provided that $\left|\{F(u^\dagger) = 0\}\right|$ is small enough. We note that in this case F is injective, i.e., $u^\dagger$ is the unique solution to (2.1). We can therefore exploit G_u in the Levenberg–Marquardt method (2.9)–(2.12) to obtain a convergent *Bouligand–Levenberg–Marquardt iteration* for the iterative regularization of the non-smooth ill-posed problem $F(u) = y$.

Corollary 3.6. *Let $u^\dagger \in L^2(\Omega)$ be such that $\left|\{y^\dagger = 0\}\right|$ is small enough with $y^\dagger := F(u^\dagger)$. Let $\{\alpha_n\}$ be defined by (2.10) with $\alpha_0^{1/2} \geq \|G_{u^\dagger}\|_{\mathbb{L}(L^2(\Omega))}$. Then there exists $\rho^* > 0$ such that for all starting points $u_0 \in \overline{B}_{L^2(\Omega)}(u^\dagger, \rho^*)$, the Bouligand–Levenberg–Marquardt iteration (2.9) stopped according to the discrepancy principle (2.12) is a well-posed and strongly convergent regularization method.*

Proof. Take $U = Y = L^2(\Omega)$ and note that $\mathcal{N}(G_{u^\dagger}) = \{0\}$ and so $\mathcal{N}(G_{u^\dagger})^\perp = L^2(\Omega)$. Then, Assumption (A1) is satisfied according to Propositions 3.2 and 3.4, Lemma 3.3, and Corollary 3.5. Assumption (A2) follows directly from Proposition 3.1 together with the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. Finally, the various requirements on the smallness of constants involving $\eta(\rho)$ and $\kappa(\rho)$ are satisfied due to Proposition 3.4. The claim now follows from Theorem 2.20. □

We point out that the assumption on the support of $F(u^\dagger)$ does *not* entail a similar requirement on $F(u_n^\delta)$, and that this non-differentiability of F at the iterates is the primary source of difficulty in showing convergence.

To close this section, we comment on the practical implementation of the Bouligand–Levenberg–Marquardt iteration (2.9) for the non-smooth PDE (3.1). Let $y^\delta \in L^2(\Omega)$. For any $u_n^\delta \in L^2(\Omega)$, we set $y_n^\delta := F(u_n^\delta)$ and define the *correction step*

$$s_n^\delta := \left(\alpha_n I + (G_n^\delta)^* G_n^\delta\right)^{-1} (G_n^\delta)^* \left(y^\delta - y_n^\delta\right). \tag{3.13}$$

From this, (2.9) can be rewritten as $u_{n+1}^\delta = u_n^\delta + s_n^\delta$ with

$$\alpha_n s_n^\delta = (G_n^\delta)^* \left(-G_n^\delta s_n^\delta + y^\delta - y_n^\delta\right).$$

By introducing $z_n^\delta := G_n^\delta s_n^\delta$ and $b_n^\delta := y^\delta - y_n^\delta$, we deduce that s_n^δ and z_n^δ satisfy

$$\begin{cases} -\Delta z_n^\delta + \mathbb{1}_{\{y_n^\delta > 0\}} z_n^\delta = s_n^\delta & \text{in } \Omega, \quad z_n^\delta = 0 \text{ on } \partial\Omega, \\ -\Delta s_n^\delta + \mathbb{1}_{\{y_n^\delta > 0\}} s_n^\delta = \dfrac{1}{\alpha_n}\left(-z_n^\delta + b_n^\delta\right) & \text{in } \Omega, \quad s_n^\delta = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.14}$$

A Bouligand–Levenberg–Marquardt step can thus be performed by solving a coupled system of two elliptic equations.

4 NUMERICAL EXPERIMENTS

This section provides numerical results that illustrate the performance of the Bouligand–Levenberg–Marquardt iteration. In the first subsection, we give a short description of our discretization scheme and the solution of the non-smooth PDE using a semismooth Newton (SSN) method. The second subsection reports the results of numerical examples.

4.1 DISCRETIZATION

In the following, we restrict ourselves to the case where Ω is an open bounded convex polygonal domain in $\mathbb{R}^2$. We shall use the standard continuous piecewise linear finite elements (FE), see, e.g., [17, 5], to discretize the non-smooth semilinear elliptic equation (3.1) as well as the linear system (3.14). In [2, 3], the discrete version of (3.1) as well as its equivalent nonlinear algebraic system were obtained by employing a mass lumping scheme for the non-smooth nonlinearity. We shall use the same technique to discretize the system (3.14). Let $\mathcal{T}_h$ stand for the triangulation of Ω corresponding to parameter h, where h denotes the maximum length of the edges of all the triangles of $\mathcal{T}_h$. For each triangulation $\mathcal{T}_h$, let $V_h \subset H_0^1(\Omega)$ be the space of piecewise linear finite elements on Ω. We denote by d_h and $\{\varphi_j\}_{j=1}^{d_h}$, respectively, the dimension and the basis of V_h corresponding to the set of nodes $\mathcal{N}_h := \{x_1, \dots, x_{d_h}\}$. For each $T \in \mathcal{T}_h$, we write $\overline{T}$ for the closure of T (i.e., the inner sum is over all vertices of the triangle T).

We first consider the nonlinear equation (3.1). Let y_h and $u_h \in V_h$ be the FE approximations of y and u, respectively, with y and u satisfying (3.1). As shown in [3, 2], the discrete equation of (3.1) is given by

$$\int_\Omega \nabla y_h \cdot \nabla v_h \,dx + \frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{x_i \in \overline{T} \cap \mathcal{N}_h} \max(0, y_h(x_i)) v_h(x_i) = \int_\Omega u_h v_h \,dx, \quad v_h \in V_h, \tag{4.1}$$

and its equivalent nonlinear algebraic system is defined as

$$\mathbf{A}y + \mathbf{D}\max(y, 0) = \mathbf{M}u, \tag{4.2}$$

where $\mathbf{A} := ((\nabla\varphi_j, \nabla\varphi_i)_{L^2(\Omega)})_{i,j=1}^{d_h}$ is the stiffness matrix, $\mathbf{M} := ((\varphi_j, \varphi_i)_{L^2(\Omega)})_{i,j=1}^{d_h}$ is the mass matrix, $\mathbf{D} := \frac{1}{3}\operatorname{diag}(\omega_1, \dots, \omega_{d_h})$ with $\omega_i := |\{\varphi_i \neq 0\}|$ is the lumped mass matrix, and $\max(\cdot, 0) : \mathbb{R}^{d_h} \to \mathbb{R}^{d_h}$ is the componentwise max-function. According to [3], the equation (4.2) is semismooth in $\mathbb{R}^{d_h}$ and can be solved via a SSN method. Here, with a slight abuse of notation, we write $y \in \mathbb{R}^{d_h}$ and $u \in \mathbb{R}^{d_h}$, respectively, instead of $(y_h(x_i))_{i=1}^{d_h}$ and $(u_h(x_i))_{i=1}^{d_h}$.

We now turn to the system (3.14). According to [5, Sec. 2.5] (see also [28, Sec. 9.1.3]), for a fixed $\delta > 0$, the discrete linear system of (3.14) is given by

$$\begin{cases} \int_\Omega \nabla z_h \cdot \nabla v_h \, dx + \frac{1}{3}\sum_{T\in\mathcal{T}_h} |T| \sum_{x_i \in \overline{T}\cap\mathcal{N}_h} \mathbb{1}_{\{y_n^\delta > 0\}}(x_i) z_h(x_i) v_h(x_i) = \int_\Omega s_h v_h \, dx, \\ \int_\Omega \nabla s_h \cdot \nabla w_h \, dx + \frac{1}{3}\sum_{T\in\mathcal{T}_h} |T| \sum_{x_i \in \overline{T}\cap\mathcal{N}_h} \mathbb{1}_{\{y_n^\delta > 0\}}(x_i) s_h(x_i) w_h(x_i) = -\frac{1}{\alpha_n}\int_\Omega (z_h - b_h)\, w_h \, dx \end{cases}$$

for all $v_h, w_h \in V_h$, where z_h, s_h, and b_h stand for the FE approximations of z_n^δ, s_n^δ, and b_n^δ, respectively. By standard computations, the above variational system can be reformulated as

$$\begin{cases} \mathbf{A}z + \mathbf{K}_\mathbf{y} z = \mathbf{M}s, \\ \mathbf{A}s + \mathbf{K}_\mathbf{y} s = -\frac{1}{\alpha_n}\mathbf{M}(z - b) \end{cases} \tag{4.3}$$

with

$$\mathbf{K}_\mathbf{y} = \frac{1}{3}\operatorname{diag}\left(\omega_i \mathbb{1}_{\{\mathbf{y}_i > 0\}}\right) \in \mathbb{R}^{d_h \times d_h}, \quad \mathbf{y}_i := y_n^\delta(x_i) \quad \text{for all } 1 \le i \le d_h.$$

Here, again, we denote the coefficient vectors $(z_h(x_i))_{i=1}^{d_h}$, $(s_h(x_i))_{i=1}^{d_h}$, and $(b_h(x_i))_{i=1}^{d_h}$ by $z \in \mathbb{R}^{d_h}$, $s \in \mathbb{R}^{d_h}$, and $b \in \mathbb{R}^{d_h}$, respectively. A standard argument shows that (4.3) is uniquely solvable.

4.2 NUMERICAL EXAMPLES

In this subsection, we consider $\Omega := (0, 1) \times (0, 1) \subset \mathbb{R}^2$ and employ a uniform triangular Friedrichs–Keller triangulation with $n_h \times n_h$ vertices for $n_h = 512$ unless noted otherwise. A direct sparse solver is used to solve the SSN system (4.2) and the linear system (4.3). The SSN iteration for solving (4.2) is initiated at $y^0 = 0$ and terminated if the active sets $AC^k := \{i : y_i^k > 0\}$ at two consecutive iterates coincide. The Python implementation used to generate the following results (as well as a Julia implementation) can be downloaded from https://github.com/clason/bouligandlevenbergmarquardt. The timings reported in the following were obtained using an Intel Core i7-7600U CPU (2.80 GHz) and 16 GByte RAM.

As in [3], we choose the exact solution

$$\begin{aligned} u^\dagger(x_1, x_2) &:= \max(y^\dagger(x_1, x_2), 0) \\ &\quad + \left[4\pi^2 y^\dagger(x_1, x_2) - 2\left((2x_1 - 1)^2 + 2(x_1 - 1 + \beta)(x_1 - \beta)\right)\sin(2\pi x_2)\right]\mathbb{1}_{[\beta, 1-\beta]}(x_1) \end{aligned}$$

where

$$y^\dagger(x_1, x_2) := \left[(x_1 - \beta)^2 (x_1 - 1 + \beta)^2 \sin(2\pi x_2)\right]\mathbb{1}_{[\beta, 1-\beta]}(x_1)$$

for some $\beta \in [0, 0.5]$ is the corresponding exact state. Obviously, $y^\dagger \in H^2(\Omega) \cap H_0^1(\Omega)$ and satisfies (3.1) for the right-hand side $u^\dagger$. Moreover, $y^\dagger$ vanishes on a set of measure 2β. The forward operator

$F : L^2(\Omega) \to L^2(\Omega)$ is therefore not Gâteaux differentiable at $u^\dagger$ whenever $\beta \in (0, 0.5]$; see, e.g., [3, Prop. 3.4]. Let us denote by $y_h^\dagger$ the discrete projection of $y^\dagger$ to V_h. We now add a random Gaussian noise componentwise to $y_h^\dagger$ to create noisy data y_h^δ corresponding to the noise level

$$\delta := \|y_h^\dagger - y_h^\delta\|_{L^2(\Omega)}.$$

Here and below, all norms for discrete functions v_h are computed exactly by $\|v_h\|^2_{L^2(\Omega)} = v_h^T \mathbf{M} v_h$ (identifying again the function v_h with its vector of expansion coefficients). From now on, to simplify the notation, we omit the subscript h. In the following, we consider different choices of the parameter β and two different choices of starting points: the trivial point $u_0 \equiv 0$ and the discrete projection of

$$\bar{u} := u^\dagger - 20\sin(\pi x_1)\sin(2\pi x_2). \tag{4.4}$$

We point out that for the second starting point, $u^\dagger$ satisfies the generalized source condition

$$u^\dagger - \bar{u} \in \mathcal{R}\left[\left(G^*_{u^\dagger} G_{u^\dagger}\right)^{1/2}\right] \subset \mathcal{R}\left[\left(G^*_{u^\dagger} G_{u^\dagger}\right)^{\nu}\right] \tag{4.5}$$

for some $\nu \in (0, 1/2)$. Note also that $\bar{u}$ is far from the exact solution $u^\dagger$ and that $u_0 \equiv 0$ is not close to $u^\dagger$ when the parameter β is far from 0.5. For the case $\beta = 0.005$, the exact solution $u^\dagger$ and the starting point $\bar{u}$ are shown in Figure 1 and Figure 2, respectively. The corresponding noisy data y^δ and the reconstructions $u^\delta_{N_\delta}$ with respect to the noise level $\delta \in \{1.056 \cdot 10^{-2}, 1.058 \cdot 10^{-4}\}$ are presented in Figure 3 for parameters $\alpha_0 = 1$, $r = 0.5$, $\beta = 0.005$, $\tau = 1.5$ and for the starting point $u_0 = \bar{u}$.

We now address the regularization property of the Bouligand–Levenberg–Marquardt iteration from Corollary 3.6. We first illustrate the effects of the starting guess on the convergence of the iteration. Table 1 displays for the same parameters $\alpha_0 = 1$, $r = 0.5$, $\beta = 0.005$, $\tau = 1.5$, a decreasing sequence of noise levels, and both starting points (for the same realization of the random data) the stopping index $N_\delta = N(\delta, y^\delta)$, the logarithmic rate of the stopping index

$$LR_\delta := \frac{N_\delta}{1 + |\log(\delta)|}, \tag{4.6}$$

the relative error

$$E^\delta := \frac{\|u^\dagger - u^\delta_{N_\delta}\|_{L^2(\Omega)}}{\|u^\dagger\|_{L^2(\Omega)}}, \tag{4.7}$$

the empirical convergence rate

$$R^\delta := \frac{\|u^\dagger - u^\delta_{N_\delta}\|_{L^2(\Omega)}}{\sqrt{\delta}}, \tag{4.8}$$

as well as the final Tikhonov parameter α_{N_δ} from (1.7). This table indicates that the speed of convergence of the iteration for the starting point $u_0 = \bar{u}$ is faster than that for the trivial starting point $u_0 \equiv 0$. While the growth of the stopping index N_δ for the trivial starting point is slightly faster than that for $\bar{u}$, the logarithmic rates (4.6) for both starting points are stable. This fits Theorem 2.20. For the starting guess $u_0 = \bar{u}$, the empirical convergence rate R^δ is not greater than 0.4 as δ is small enough. This agrees with the convergence rate $O(\sqrt{\delta})$ expected from the classical source condition $u^\dagger - u_0 \in \mathcal{R}\left[(F'(u^\dagger)^* F'(u^\dagger))^{1/2}\right]$.

To show the dependence on parameter β of the performance of the Bouligand–Levenberg–Marquardt iteration, we summarize in Table 2 the results obtained for $\beta \in \{0, 0.15, 0.3\}$, $\alpha_0 = 1$, $r = 0.5$, $\tau = 1.5$, and $u_0 = \bar{u}$. Table 2 indicates that the stopping index seems not to be significantly influenced by the

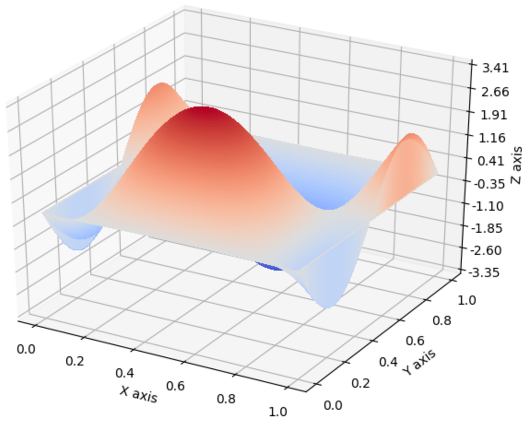


Figure 1: exact solution $u^\dagger$ for $\beta = 0.005$

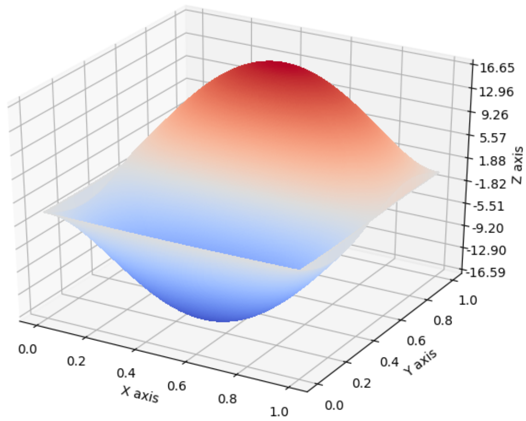
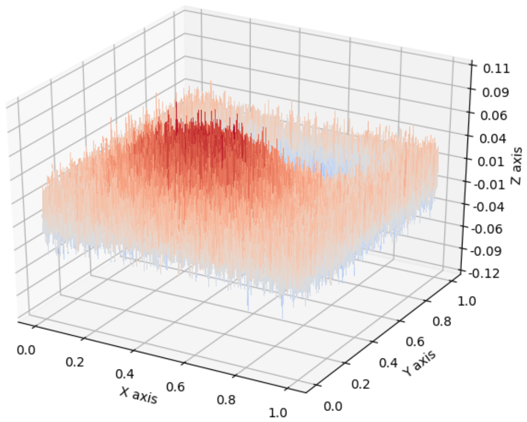
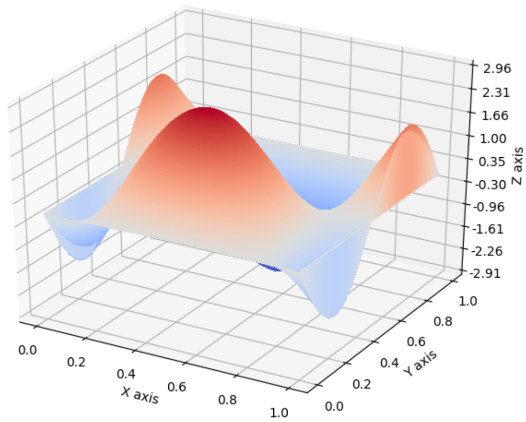


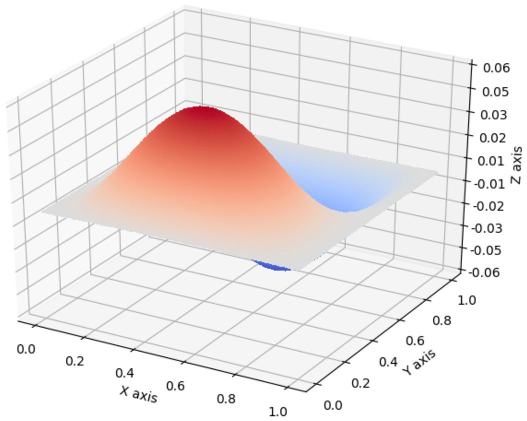
Figure 2: starting point $u_0 = \bar{u}$ for $\beta = 0.005$



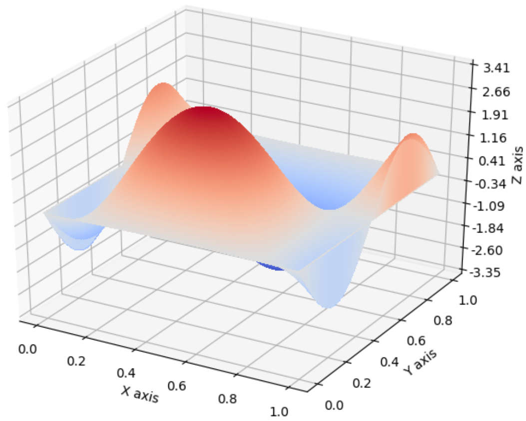
(a) y^δ, $\delta = 1.056 \cdot 10^{-2}$



(b) $u^\delta_{N_\delta}$, $N_\delta = 14$



(c) y^δ, $\delta = 1.058 \cdot 10^{-4}$



(d) $u^\delta_{N_\delta}$, $N_\delta = 16$

Figure 3: noisy data y^δ and reconstructions $u^\delta_{N_\delta}$ for $u_0 = \bar{u}$ and $\alpha_0 = 1$, $r = 0.5$, $\beta = 0.005$, $\tau = 1.5$

Table 1: regularization property for $\alpha_0 = 1$, $r = 0.5$, $\beta = 0.005$, $\tau = 1.5$: noise level δ; stopping index N_δ; logarithmic rate LR_δ from (4.6); relative error E^δ from (4.7); empirical convergence rate R^δ from (4.8); final Tikhonov parameter α_{N_δ}

	$u_0 \equiv 0$				$u_0 = \bar{u}$				
δ	N_δ	LR_δ	E^δ	α_{N_δ}	N_δ	LR_δ	E^δ	R^δ	α_{N_δ}
$1.06 \cdot 10^{-2}$	12	2.2	$4.70 \cdot 10^{-1}$	$4.1 \cdot 10^{-9}$	14	2.5	$1.55 \cdot 10^{-1}$	2.3	$1.6 \cdot 10^{-10}$
$1.06 \cdot 10^{-3}$	16	2.0	$2.36 \cdot 10^{-1}$	$6.6 \cdot 10^{-12}$	15	1.9	$2.07 \cdot 10^{-2}$	1.0	$3.3 \cdot 10^{-11}$
$1.06 \cdot 10^{-4}$	20	2.0	$1.44 \cdot 10^{-1}$	$1.0 \cdot 10^{-14}$	16	1.6	$1.57 \cdot 10^{-3}$	0.2	$6.6 \cdot 10^{-12}$
$1.05 \cdot 10^{-5}$	25	2.0	$7.33 \cdot 10^{-2}$	$3.4 \cdot 10^{-18}$	17	1.4	$3.57 \cdot 10^{-4}$	0.2	$1.3 \cdot 10^{-12}$
$1.06 \cdot 10^{-6}$	30	2.0	$3.55 \cdot 10^{-2}$	$1.1 \cdot 10^{-21}$	18	1.2	$1.85 \cdot 10^{-4}$	0.3	$2.6 \cdot 10^{-13}$
$1.06 \cdot 10^{-7}$	34	2.0	$2.77 \cdot 10^{-2}$	$1.7 \cdot 10^{-24}$	21	1.2	$6.47 \cdot 10^{-5}$	0.3	$2.1 \cdot 10^{-15}$

Table 2: regularization property for $\alpha_0 = 1$, $r = 0.5$, $\tau = 1.5$, $u_0 = \bar{u}$: noise level δ; stopping index $N_\delta = N(\delta, y^\delta)$; relative error E^δ from (4.7)

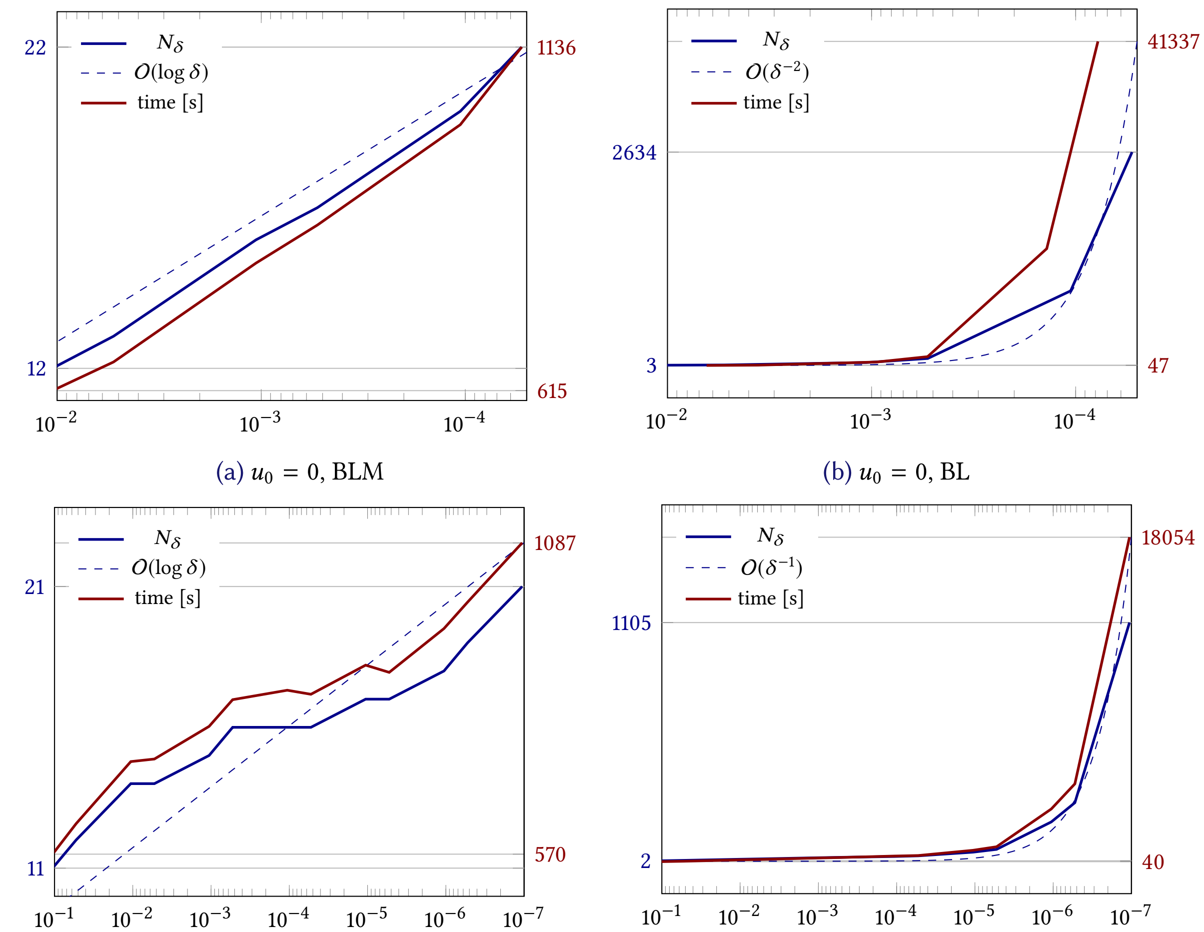
	$\beta = 0$		$\beta = 0.15$		$\beta = 0.3$	
δ	N_δ	E^δ	N_δ	E^δ	N_δ	E^δ
$1.06 \cdot 10^{-1}$	11	3.10	11	$1.13 \cdot 10^{1}$	11	$6.41 \cdot 10^{1}$
$1.06 \cdot 10^{-2}$	14	$1.50 \cdot 10^{-1}$	14	$5.47 \cdot 10^{-1}$	14	3.10
$1.06 \cdot 10^{-3}$	15	$1.96 \cdot 10^{-2}$	15	$7.11 \cdot 10^{-2}$	15	$4.11 \cdot 10^{-1}$
$1.06 \cdot 10^{-4}$	16	$1.50 \cdot 10^{-3}$	16	$5.75 \cdot 10^{-3}$	16	$3.52 \cdot 10^{-2}$
$1.06 \cdot 10^{-5}$	17	$3.43 \cdot 10^{-4}$	17	$3.56 \cdot 10^{-3}$	17	$1.02 \cdot 10^{-2}$
$1.06 \cdot 10^{-6}$	18	$1.81 \cdot 10^{-4}$	19	$3.74 \cdot 10^{-3}$	18	$5.29 \cdot 10^{-3}$

parameter β. However, it is not surprising that the relative error E^δ increases with respect to β since $|\{y^\dagger = 0\}| \to 0$ as $\beta \to 0^+$.

Finally, the stopping index as well as the total CPU time (in seconds) of the proposed Bouligand–Levenberg–Marquardt (BLM) iteration and of the Bouligand–Landweber (BL) iteration from [3] are compared in Figure 4. Recall that the BL iteration is defined by

$$u_{n+1}^\delta = u_n^\delta + w_n G^*_{u_n^\delta} \left(y^\delta - F(u_n^\delta) \right), \quad n \geq 0 \tag{4.9}$$

with parameter $w_n > 0$ and is terminated via the discrepancy principle (2.12). To compare the numerical results, we set $\alpha_0 = 1$, $r = 0.5$, $\beta = 0.005$, $\tau = 1.5$, $w_n = (2 - 2\mu)/\bar{L}^2$ for all $n \geq 0$ with $\mu = 0.1$ and $\bar{L} = 0.05$. Figure 4 shows the stopping index of the two iterative methods versus the noise level δ for both $u_0 = \bar{u}$ and $u_0 \equiv 0$. Figures 4a and 4c indicate that for the BLM iteration, in both cases $N_\delta = O(1 + |\log(\delta)|)$ as $\delta \to 0$, as expected from Theorem 2.20. On the other hand, Figures 4b and 4d show that for the BL iteration, $N_\delta = O(\delta^{-1})$ for $u_0 = \bar{u}$ and $N_\delta = O(\delta^{-2})$ for $u_0 \equiv 0$ as $\delta \to 0$. As also shown in these figures, the total CPU time to run each method is almost directly proportional to their stopping indices (approximately 52 seconds per step for the BLM iteration and 16 seconds per for the BL iteration, corresponding to the size of (4.3) compared to that of the discretization of (4.9)). For $u_0 \equiv 0$ and $\delta \approx 5 \cdot 10^{-5}$, the total CPU time of the BLM iteration is only 1136 seconds while that of the BL iteration is nearly 41337 seconds. Similarly, for $u_0 = \bar{u}$ and $\delta \approx 10^{-7}$, it takes 1087 seconds for the BLM iteration and approximately 18054 seconds for the BL iteration to terminate. Hence even though the cost of each step of the two iterations is different, the BLM iteration is significantly faster also in terms of CPU time for small values of δ.



(a) $u_0 = 0$, BLM

(b) $u_0 = 0$, BL

(c) $u_0 = \bar{u}$, BLM

(d) $u_0 = \bar{u}$, BL

Figure 4: comparison of stopping index N_δ and total CPU time (in seconds) for Bouligand–Levenberg–Marquardt (BLM) and Bouligand–Landweber (BL) iterations

5 CONCLUSION

We have proposed a novel Newton-type regularization method for non-smooth ill-posed inverse problems that extends the classical Levenberg–Marquardt iteration. Using a family of bounded operators $\{G_u\}$ to replace the Fréchet derivative in the classical Levenberg–Marquardt iteration, we proved under a generalized tangential cone condition the asymptotic stability of the iterates and from this derived the regularization property of the iteration. In particular, when considering ill-posed inverse problem where the forward operator corresponds to the solution of a non-smooth semilinear elliptic PDE, we can take G_u from the Bouligand subdifferential of the forward operator. If the non-differentiability of the forward mapping is sufficiently "weak" at the exact solution, these operators satisfy the required assumptions, and the resulting Bouligand–Levenberg–Marquardt iteration thus provides a convergent regularization method. As the numerical example illustrates, this iteration requires significantly less iterations and can be much faster than first-order methods such as the Bouligand–Landweber iteration from [3].

This work can be extended in several directions. First, it would be interesting to derive convergence rates under the generalized source condition (2.47). Of particular interest would be the extension of the proposed iteration for non-smooth ill-posed inverse problems with additional constraints such as non-negativity of the unknown parameter. Finally, similar non-smooth extensions of other Newton-type methods such as the iteratively regularized Gauss-Newton method could be derived.

APPENDIX A AUXILIARY LEMMAS

This section provides some estimates on the sequence of parameters defined by (2.10) and on bounded linear operators between Hilbert spaces.

Lemma A.1 ([9]). *Let $\{\alpha_n\}$ be defined via (2.10). Then there hold for all $k \geq 0$*

$$\left(\sum_{j=0}^{k} \alpha_j^{-1}\right)^{-1} \leq c_0^2 \alpha_{k+1}, \qquad \sum_{m=0}^{k} \alpha_m^{-1/2} \left(\sum_{j=m}^{k} \alpha_j^{-1}\right)^{-1/2} \leq c_1, \qquad \sum_{m=0}^{k} \alpha_m^{-1/2} \left(\sum_{j=m}^{k} \alpha_j^{-1}\right)^{-1} \leq c_2 \alpha_{k+1}^{1/2},$$

$$\sum_{m=0}^{k} \alpha_m^{-1} \left(\sum_{j=m}^{k} \alpha_j^{-1}\right)^{-1/2} \leq c_3 \alpha_{k+1}^{-1/2}, \qquad \sum_{m=0}^{k} \alpha_m^{-1} \left(\sum_{j=m}^{k} \alpha_j^{-1}\right)^{-1} \leq c_4,$$

where the constants c_i, $i = 0, \dots, 4$, are defined in (2.17).

The next lemma provides some more estimates on sequence $\{\alpha_n\}$ defined via (2.10) with exponent $\nu \in [0, 1/2)$. Its proof is standard and thus is omitted.

Lemma A.2. *Let $\{\alpha_n\}$ be defined via (2.10) and let $0 \leq \nu < \frac{1}{2}$. Then there hold for all $k \geq 0$*

$$\sum_{m=0}^{k} \alpha_m^{\nu-1/2} \left(\sum_{j=m}^{k} \alpha_j^{-1}\right)^{-1/2} \leq K_0(r, \nu) \alpha_{k+1}^{\nu} \tag{A.1}$$

and

$$\sum_{m=0}^{k} \alpha_m^{\nu-1/2} \left(\sum_{j=m}^{k} \alpha_j^{-1}\right)^{-1} \leq K_1(r, \nu) \alpha_{k+1}^{\nu+1/2} \tag{A.2}$$

with $K_0(r, \nu)$ and $K_1(r, \nu)$ defined in (2.18).

The next lemmas give useful estimates of a bounded linear operator between Hilbert spaces and generalize the corresponding results in [7]. Their proofs are based on the spectral theory and functional calculus of self-adjoint operators; see, e.g. [4, 26].

Lemma A.3 ([9, Lem. 2]). *Let $\{\alpha_k\}$ be a sequence of positive numbers and let $T : H_1 \to H_2$ be a bounded linear operator between Hilbert spaces. Then, for any $0 \leq \nu \leq 1$ and any integers $0 \leq m \leq l$, there holds*

$$\left\| \prod_{j=m}^{l} \alpha_j \left(\alpha_j I + T^*T\right)^{-1} (T^*T)^{\nu} \right\|_{\mathbb{L}(H_1)} \leq \left(\sum_{j=m}^{l} \alpha_j^{-1}\right)^{-\nu}.$$

This result can be improved for the specific case $\nu = \frac{1}{2}$ to be sharp as shown by the choice $m = l$.

Lemma A.4. *Let $\{\alpha_k\}$ be a sequence of positive numbers and let $T : H_1 \to H_2$ be a bounded linear operator between Hilbert spaces. Then, for any integers $0 \leq m \leq l$, there holds*

$$\left\| \prod_{j=m}^{l} \alpha_j \left(\alpha_j I + T^*T\right)^{-1} T^* \right\|_{\mathbb{L}(H_2, H_1)} \leq \frac{1}{2} \left(\sum_{j=m}^{l} \alpha_j^{-1}\right)^{-1/2}.$$

Proof. Set $t_0 := \|T\|^2_{\mathbb{L}(H_1, H_2)}$ and define the continuous function

$$g(t) := \prod_{j=m}^{l} \alpha_j \left(\alpha_j + t\right)^{-1}, \qquad t \geq 0.$$

From the spectral theory of self-adjoint operators, see, e.g. [4, Chap. 2], we have that

$$\|g(T^*T)T^*\|_{\mathbb{L}(H_2,H_1)} \leq \sup\left\{\sqrt{t}|g(t)| : 0 \leq t \leq t_0\right\}. \tag{A.3}$$

On the other hand, a simple computation and the Cauchy–Schwarz inequality give

$$\prod_{j=m}^{l}(\alpha_j + t) \geq \prod_{j=m}^{l}\alpha_j\left(1 + t\sum_{j=m}^{l}\alpha_j^{-1}\right) \geq 2\sqrt{t}\prod_{j=m}^{l}\alpha_j\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{1/2},$$

which leads to

$$\sqrt{t}|g(t)| \leq \frac{1}{2}\left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-1/2}$$

for all $t \geq 0$. Combining this with (A.3) yields the desired estimate. □

Finally, we have the following direct consequence of Lemmas A.3 and A.4.

Corollary A.5. *Let $\{\alpha_k\}$ be a sequence of positive numbers and let $T : H_1 \to H_2$ be a bounded linear operator between Hilbert spaces. Then, for any $0 \leq \nu \leq \frac{1}{2}$ and any integers $0 \leq m \leq l$, there holds*

$$\left\|\prod_{j=m}^{l}\alpha_j\left(\alpha_j I + T^*T\right)^{-1}(T^*T)^{\nu}T^*\right\|_{\mathbb{L}(H_2,H_1)} \leq \left(\sum_{j=m}^{l}\alpha_j^{-1}\right)^{-\nu-1/2}.$$

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