Multigrid solution of optimality systems discretized by high-order schemes

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Outline of the talk

- Control-constrained nonlinear elliptic optimal control problems
- Optimality systems

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- Optimality systems
- Finite-difference framework
- Discretization of the optimality system

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- Discretization of the optimality system
- Smoothing and multigrid methods
- Numerical experiments
- A state-constrained elliptic optimal control problem

Control-constrained nonlinear elliptic optimal control problems

Consider a two-dimensional material plate Ω whose state is described by the temperature distribution y.

Assume thermal radiation $(G(y) < 0)$ or positive temperature feedback $(G(y) > 0)$ due to chemical reactions.

We may control y to come close to a given target profile $z \in L^2(\Omega)$, by acting with a (boundary or) distributed source term u, the control function.

$$
\left\{
\begin{array}{rcll}
\min_{u \in U_{ad}} J(y,u) & := & \frac{1}{2}\|y-z\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} & \\
\Delta y + G(y) & = & u + f & \text{in } \Omega \\
y & = & 0 & \text{on } \partial\Omega
\end{array}
\right.
$$

$$U_{ad} = \{u \in L^2(\Omega) \,|\, u_L(\mathbf{x}) \le u(\mathbf{x}) \le u_H(\mathbf{x}) \ \text{a.e. in } \Omega\}$$

Optimality system

Optimal solutions are characterized by the following optimality system

$$\begin{aligned} \Delta y + G(y) - u =& \quad f && \text{in } \Omega, \\ y =& \quad 0 && \text{on } \partial\Omega, \\ \Delta p + G'(y)\, p + y =& \quad z && \text{in } \Omega, \\ p =& \quad 0 && \text{on } \partial\Omega, \\ (\nu u - p,\ v - u) \geq& \quad 0 && \text{for all } v \in U_{ad}. \end{aligned}$$

The last equation gives the optimality condition. It is equivalent to

$$u = \max\{u_L, \min\{u_H, \frac{1}{\nu} p(u)\}\} \text{ in } \Omega, \quad \nu > 0$$

Nondifferentiability !

Optimality system (continue)

The case with $\nu = 0$ is characterized by the following system

$$
\begin{aligned}
\Delta y + G(y) - u &= f && \text{in } \Omega, \\
y &= 0 && \text{on } \partial\Omega, \\
\Delta p + G'(y)\, p + y &= z && \text{in } \Omega, \\
p &= 0 && \text{on } \partial\Omega, \\
p &= \min\{0, p + u - u_L\} + \max\{0, p + u - u_H\} && \text{in } \Omega.
\end{aligned}
$$

Nondifferentiability prevents the use of classical Newton or gradient techniques, requiring more sophisticated methods based on generalized differentiability concepts.

We propose an alternative MG approach where differentiation is not required.

Finite-difference framework

Consider the finite-difference framework [Hackbusch,Süli].
Let Ω be rectangular domain. Introduce the discrete L^2_h-scalar product $(v_h, w_h)_{L^2_h} = h^2 \sum_{\mathbf{x}\in\Omega_h} v_h(\mathbf{x})\, w_h(\mathbf{x})$, with norm $|v_h|_0 = (v_h, v_h)_{L^2_h}^{1/2}$.

First-order backward and forward partial derivatives of v_h in the x_i direction are denoted by ∂_i^- and ∂_i^+, respectively.
Assume sufficiently smooth functions $v \in C^k(\bar{\Omega})$, $k = 0, 1, \ldots$, and denote with $(R_h v)(x) = v(x)$ the restriction operator on $\bar{\Omega}$. We have
The second-order five-point Laplacian

$$\tilde{\Delta}_h = \partial_1^+ \partial_1^- + \partial_2^+ \partial_2^-$$

The fourth-order nine-point Laplacian

$$\Delta_h = (1 - \frac{h^2}{12} \partial_1^+ \partial_1^-)\, \partial_1^+ \partial_1^- + (1 - \frac{h^2}{12} \partial_2^+ \partial_2^-)\, \partial_2^+ \partial_2^- .$$

A priori accuracy estimate

In the linear case, $G(y) = g\,y$ with $g \leq 0$ and $U_{ad} = L^2(\Omega)$. We have the following discrete optimality system

$$\begin{aligned} \Delta_h y_h + g\,y_h - p_h/\nu &= f_h \\ \Delta_h p_h + g\,p_h + y_h &= z_h \end{aligned}$$

Theorem

Let $y \in C^{k+2}(\bar{\Omega})$, $k = 2, 4$, and $p \in C^{l+2}(\bar{\Omega})$, $l = 2, 4$, be solutions to the optimality system, and let y_h and p_h be solutions to the discrete optimality system. Then there exists a constant c, depending on Ω, and independent of h, such that

$$|y_h - R_h y|_0^2 + \frac{1}{\nu}\,|p_h - R_h p|_0^2 \leq c\,(h^{2k}\,\|y\|^2_{C^{k+2}(\bar{\Omega})} + h^{2l}\,\frac{1}{\nu}\,\|p\|^2_{C^{l+2}(\bar{\Omega})}).$$

Results of numerical experiments give evidence that it appears to hold also in the presence of nonlinearity and of constraints.

Discretization of the optimality system

The one-dimensional expanded form of $\Delta_h v(x)$ is

$$\frac{1}{12h^2}(-v(x-2h)+16\,v(x-h)-30\,v(x)+16\,v(x+h)-v(x+2h)).$$

This scheme results in a system which is neither diagonally dominant nor of non-negative type [Bramble Hubbard]. Nevertheless it satisfies a max principle.

We can express the action of Δ (resp. $\tilde{\Delta}$) on the function v_h in the following compact form

$$\Delta_h v_h|_{ij} = \frac{1}{h^2}\Big(\sum_{s,t\in\omega_{ij},\,s,t\neq i,j} c_{st}\,v_{st} - c_{ij}\,v_{ij}\Big).$$

and for convenience set

$$A_{ij} = \sum_{s,t\in\omega_{ij},\,s,t\neq i,j} c^y_{st}\,y_{st} - h^2 f_{ij} \quad \text{and} \quad B_{ij} = \sum_{s,t\in\omega_{ij},\,s,t\neq i,j} c^p_{st}\,p_{st} - h^2 z_{ij}$$

Discretization of the optimality system (continue)

We have the following set of equations at i, j for the three scalar variables y_{ij}, p_{ij}, and u_{ij}:

$$\begin{aligned} A_{ij} - c^y_{ij}\, y_{ij} + h^2\, G(y_{ij}) - h^2\, u_{ij} &= 0 \\ B_{ij} - c^p_{ij}\, p_{ij} + h^2\, G'(y_{ij})\, p_{ij} + h^2\, y_{ij} &= 0 \\ (\nu u_{ij} - p_{ij}) \cdot (v_{ij} - u_{ij}) &\geq 0 \quad \text{for all } v_h \in U_{ad\,h} \end{aligned}$$

Solving these equations at each grid point in a given order results in a robust smoother.

Smoothing

- Compute the inverse of the Jacobian for the y, p system

$$J_{ij}^{-1} = \frac{1}{\det J_{ij}} \begin{pmatrix} -c_{ij}^{p} + h^2 G'(y_{ij}) & 0 \\ -h^2(1 + G''(y_{ij})\, p_{ij}) & -c_{ij}^{y} + h^2 G'(y_{ij}) \end{pmatrix}$$

Smoothing

- Compute the inverse of the Jacobian for the y, p system

$$J_{ij}^{-1} = \frac{1}{\det J_{ij}} \begin{pmatrix} -c_{ij}^p + h^2 G'(y_{ij}) & 0 \\ -h^2(1 + G''(y_{ij})\, p_{ij}) & -c_{ij}^y + h^2 G'(y_{ij}) \end{pmatrix}$$

- Define a local Newton update for y_{ij} and p_{ij} at i, j

$$\begin{pmatrix} y_{ij}(u_{ij}) \\ p_{ij}(u_{ij}) \end{pmatrix} = \begin{pmatrix} y_{ij} \\ p_{ij} \end{pmatrix} + J_{ij}^{-1} \begin{pmatrix} r_{ij}^y(u_{ij}) \\ r_{ij}^p \end{pmatrix},$$

Where

$$r_{ij}^y(u_{ij}) = -(A_{ij} - c_{ij}^y\, y_{ij} + h^2\, G(y_{ij}) - h^2\, u_{ij})$$

$$r_{ij}^p = -(B_{ij} - c_{ij}^p\, p_{ij} + h^2\, G'(y_{ij})\, p_{ij} + h^2\, y_{ij})$$

Smoothing (continue)

- Find u^*_{ij} such that $J'(y(u),u) = \nu u^*_{ij} - p_{ij}(u^*_{ij}) = 0$.

$$
\begin{aligned}
u^*_{ij} \quad = \quad & \left(\nu + \frac{(1 + G''(y_{ij})\,p_{ij})\,h^4}{\det J_{ij}}\right)^{-1} \quad \times \\
& [p_{ij} - \frac{1}{\det J_{ij}} \ \left(h^2(1 + G''(y_{ij})\,p_{ij})\,(A_{ij} - c^y_{ij}\,y_{ij} + h^2\,G(y_{ij}))\right) \\
& + \frac{1}{\det J_{ij}} \ \left((c^y_{ij} - h^2 G'(y_{ij}))\,(B_{ij} - c^p_{ij}\,p_{ij} + h^2\,G'(y_{ij})\,p_{ij} + h^2\,y_{ij})\right)].
\end{aligned}
$$

Smoothing (continue)

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& [p_{ij} - \frac{1}{\det J_{ij}} \, \left(h^2(1 + G''(y_{ij})\, p_{ij})\, (A_{ij} - c^y_{ij}\, y_{ij} + h^2\, G(y_{ij}))\right) \\
& + \frac{1}{\det J_{ij}} \, \left((c^y_{ij} - h^2 G'(y_{ij}))\, (B_{ij} - c^p_{ij}\, p_{ij} + h^2\, G'(y_{ij})\, p_{ij} + h^2\, y_{ij})\right)].
\end{aligned}
$$

- Set (projection)

$$
u_{ij} = \begin{cases} u_{H\,ij} & \text{if} \quad u^*_{ij} \geq u_{H\,ij} \\ u^*_{ij} & \text{if} \quad u_{L\,ij} < u^*_{ij} < u_{H\,ij} \\ u_{L\,ij} & \text{if} \quad u^*_{ij} \leq u_{L\,ij}. \end{cases}
$$

- Update state and adjoint variables $y_{ij} = y_{ij}(u_{ij})$ and $p_{ij} = p_{ij}(u_{ij})$.

Multigrid FAS-V(m_1, m_2)-Cycle [Brandt]

Set $B_1(w_1^{(0)}) \approx A_1^{-1}$ (e.g., iterating with S_1 starting with $w_1^{(0)}$). For $k = 2, \ldots, L$ define B_k in terms of B_{k-1} as follows.

1. Set the starting approximation $w_k^{(0)}$.
2. Pre-smoothing. Define $w_k^{(l)}$ for $l = 1, \ldots, m_1$, by
$$w_k^{(l)} = S_k(w_k^{(l-1)}, f_k).$$
3. Coarse grid correction.
Set $w_k^{(m_1+1)} = w_k^{(m_1)} + I_{k-1}^k(w_{k-1} - \hat{I}_k^{k-1} w_k^{(m_1)})$ where
$$w_{k-1} = B_{k-1}(\hat{I}_k^{k-1} w_k^{(m_1)}) \left[I_k^{k-1}(f_k - A_k(w_k^{(m_1)})) + A_{k-1}(\hat{I}_k^{k-1} w_k^{(m_1)}) \right].$$
4. Post-smoothing. Define $w_k^{(l)}$ for $l = m_1 + 2, \cdots, m_1 + m_2 + 1$, by
$$w_k^{(l)} = S_k(w_k^{(l-1)}, f_k).$$
5. Set $B_k(w_k^{(0)})\, f_k = w_k^{(m_1+m_2+1)}$.

Local Fourier analysis

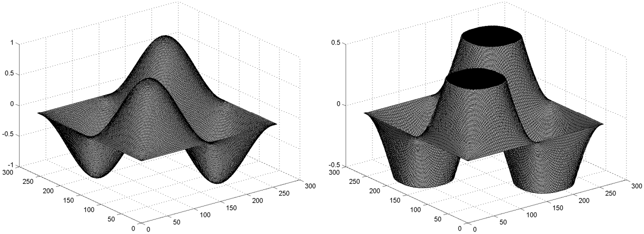
Estimates for convergence factors and smoothing factors (linear case)

		$\Delta_h y$, $\Delta_h p$		$\tilde{\Delta}_h y$, $\tilde{\Delta}_h p$	
(m_1, m_2)	ν	μ	$\rho(TG_k^{k-1})$	μ	$\rho(TG_k^{k-1})$
(1,1)	10^{-4}	0.5362	0.2429	0.5020	0.1939
(2,2)	10^{-4}	0.5362	0.1233	0.5020	0.0851
(1,1)	10^{-8}	0.6089	0.3457	0.5491	0.2772
(2,2)	10^{-8}	0.6089	0.1933	0.5491	0.1255
		$\Delta_h y$, $\tilde{\Delta}_h p$		$\tilde{\Delta}_h y$, $\Delta_h p$	
(m_1, m_2)	ν	μ	$\rho(TG_k^{k-1})$	μ	$\rho(TG_k^{k-1})$
(1,1)	10^{-4}	0.5346	0.2413	0.5346	0.2413
(2,2)	10^{-4}	0.5346	0.1215	0.5346	0.1215
(1,1)	10^{-8}	0.5787	0.3094	0.5787	0.3094
(2,2)	10^{-8}	0.5787	0.1566	0.5787	0.1566

A control-constrained nonlinear optimal control problem

$$\begin{aligned} \Delta y + y^4 &= u + f, \\ \Delta p + 4y^3 p + y &= z \\ (\nu u - p,\ v - u) &\geq 0 \quad \text{for all } v \in U_{ad} \end{aligned}$$

$$U_{ad} = \{u \in L^2(\Omega) \,|\, -1/2 \leq u(\mathbf{x}) \leq 1/2 \ \text{a.e. in } \Omega\}$$



Solution for $\nu = 10^{-6}$. The state (left) and the control (right). $z(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2) + O(\nu)$

Results of experiments

Fourth-order scheme versus second-order scheme for both equations

$\Delta_h y,\ \Delta_h p$						
ν	Mesh	$\lvert y_h - R_h y\rvert_0$	$\lvert u_h - R_h u\rvert_0$	$\lvert y_h - R_h z\rvert_0$	ρ	CPU s
10^{-3}	256^2	0.19(-8)	0.11(-7)	0.39(-1)	0.100	1.23
10^{-3}	1024^2	0.76(-11)	0.43(-10)	0.39(-1)	0.115	13.59
10^{-6}	256^2	0.55(-9)	0.39(-6)	0.39(-4)	0.116	1.28
10^{-6}	1024^2	0.21(-11)	0.15(-8)	0.39(-4)	0.117	13.50
$\tilde{\Delta}_h y,\ \tilde{\Delta}_h p$						
ν	Mesh	$\lvert y_h - R_h y\rvert_0$	$\lvert u_h - R_h u\rvert_0$	$\lvert y_h - R_h z\rvert_0$	ρ	CPU s
10^{-3}	256^2	0.24(-4)	0.13(-3)	0.39(-1)	0.04	0.81
10^{-3}	1024^2	0.15(-5)	0.86(-5)	0.39(-1)	0.03	10.79
10^{-6}	256^2	0.71(-5)	0.48(-2)	0.42(-4)	0.03	0.82
10^{-6}	1024^2	0.43(-6)	0.30(-3)	0.39(-4)	0.04	13.29

Results of experiments

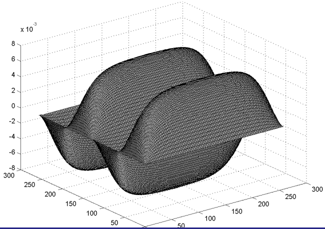
Mixed fourth-order/second-order schemes

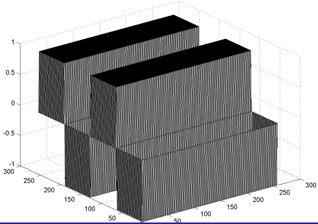
$\Delta_h y,\ \tilde{\Delta}_h p$						
ν	Mesh	$\lvert y_h - R_h y\rvert_0$	$\lvert u_h - R_h u\rvert_0$	$\lvert y_h - R_h z\rvert_0$	ρ	CPU s
10^{-3}	256^2	0.54(-7)	0.99(-5)	0.39(-1)	0.115	1.26
10^{-3}	1024^2	0.35(-8)	0.62(-6)	0.39(-1)	0.114	12.68
10^{-6}	256^2	0.18(-8)	0.77(-6)	0.39(-4)	0.116	1.26
10^{-6}	1024^2	0.11(-9)	0.25(-7)	0.39(-4)	0.117	12.81
$\tilde{\Delta}_h y,\ \Delta_h p$						
ν	Mesh	$\lvert y_h - R_h y\rvert_0$	$\lvert u_h - R_h u\rvert_0$	$\lvert y_h - R_h z\rvert_0$	ρ	CPU s
10^{-3}	256^2	0.24(-4)	0.12(-3)	0.39(-1)	0.04	0.82
10^{-3}	1024^2	0.15(-5)	0.80(-5)	0.39(-1)	0.03	12.34
10^{-6}	256^2	0.71(-5)	0.48(-2)	0.42(-4)	0.03	1.00
10^{-6}	1024^2	0.43(-6)	0.30(-3)	0.39(-4)	0.04	15.82

Bang-bang control

The case $\nu = 0$ and box constraints $u_L = -1$ and $u_H = 1$, and a non attainable target function given by $z(x_1, x_2) = \sin(4\pi x_1)$

	$\Delta_h y$ $\Delta_h p$		$\tilde{\Delta}_h y$, $\tilde{\Delta}_h p$	
Mesh	ρ	CPU s	ρ	CPU s
128×128	0.45	0.35	0.40	0.25
256×256	0.45	1.39	0.52	1.01
512×512	0.45	5.10	0.50	3.95
1024×1024	0.45	21.29	0.45	15.98
2048×2048	0.45	87.28	0.45	64.07





A state-constrained elliptic optimal control problem

$$
\left\{
\begin{array}{rcll}
\min_{u \in L^2(Q)} J(y,u) & := & \frac{1}{2}||y-z||^2_{L^2(\Omega)} + \frac{\nu}{2}||u||^2_{L^2(\Omega)} & \\
\Delta y & = & u + f & \text{in } \Omega \\
y & = & 0 & \text{on } \partial\Omega \\
y_L \quad \leq \quad y & \leq & y_H & \text{in } \Omega
\end{array}
\right.
$$

The solution approach through Lagrange multipliers associated with the state constraints leads to difficulties:

- The Lagrange multipliers associated with the state constraints are regular Borel measures.
- Methods relying on Lagrange multipliers must be adapted.

Remedy: Lavrentiev-type or Moreau-Yosida regularizations.

Regularized state-constrained optimal control problem

$$\left\{\begin{array}{rclll} \min_{u\in L^2(Q)} J(y,u) & := & \frac{1}{2}||y-z||^2_{L^2(\Omega)} + \frac{\nu}{2}||u||^2_{L^2(\Omega)} & & \\ \Delta y & = & u+f & \text{in } \Omega & \\ y & = & 0 & \text{on } \partial\Omega & \\ y_L \quad \leq \quad y-\lambda u & \leq & y_H & \text{in } \Omega, & \lambda>0 \end{array}\right.$$

Set $v = y - \lambda u$. It becomes a 'control-constrained' optimal control problem

$$\left\{\begin{array}{rcll} \min_{v\in L^2(Q)} J(y,v) & := & \frac{1}{2}||y-z||^2_{L^2(\Omega)} + \frac{\nu}{2\lambda^2}||y-v||^2_{L^2(\Omega)} & \\ \Delta y - y/\lambda + v/\lambda & = & f & \text{in } \Omega \\ y & = & 0 & \text{on } \partial\Omega \\ y_L \quad \leq v & \leq & y_H & \text{in } \Omega \end{array}\right.$$

- The associated Lagrange multipliers are $L^2(\Omega)$.

Optimality system

The objective functional $J(y, v)$ is strictly convex and lower semicontinuous. One can prove existence and uniqueness of the optimal solution.

This solution is characterized by the following optimality system

$$\begin{aligned} \Delta y - y/\lambda + v/\lambda &= f \\ \Delta p - p/\lambda + (y - z) + k\,(y - v) &= 0 \\ (p/\lambda - k\,(y - v), t - v) &\geq 0 \quad \text{for all } t \in V_{ad} \end{aligned}$$

where $k = \nu/\lambda^2$ and

$$V_{ad} = \{v \in L^2(\Omega) \,|\, y_L(x) \leq v(x) \leq y_H(x) \text{ a.e. in } \Omega\},$$

Smoothing

- Compute the Jacobian for the y, p system

$$J_{ij}^{-1} = \frac{1}{\det J_{ij}} \begin{pmatrix} -(c_{ij}^{p} + \frac{h^2}{\lambda}) & 0 \\ -h^2(1+k) & -(c_{ij}^{y} + \frac{h^2}{\lambda}) \end{pmatrix},$$

Smoothing

- Compute the Jacobian for the y, p system

$$J_{ij}^{-1} = \frac{1}{\det J_{ij}} \begin{pmatrix} -(c_{ij}^{p} + \frac{h^2}{\lambda}) & 0 \\ -h^2(1+k) & -(c_{ij}^{y} + \frac{h^2}{\lambda}) \end{pmatrix},$$

- Define a local Newton update for y_{ij} and p_{ij} at i,j

$$\begin{pmatrix} y_{ij}(v_{ij}) \\ p_{ij}(v_{ij}) \end{pmatrix} = \begin{pmatrix} y_{ij} \\ p_{ij} \end{pmatrix} + J_{ij}^{-1} \begin{pmatrix} r_{ij}^{y}(v_{ij}) \\ r_{ij}^{p}(v_{ij}) \end{pmatrix},$$

Smoothing

- Compute the Jacobian for the y, p system

$$J_{ij}^{-1} = \frac{1}{\det J_{ij}} \begin{pmatrix} -(c_{ij}^{p} + \frac{h^2}{\lambda}) & 0 \\ -h^2(1+k) & -(c_{ij}^{y} + \frac{h^2}{\lambda}) \end{pmatrix},$$

- Define a local Newton update for y_{ij} and p_{ij} at i, j

$$\begin{pmatrix} y_{ij}(v_{ij}) \\ p_{ij}(v_{ij}) \end{pmatrix} = \begin{pmatrix} y_{ij} \\ p_{ij} \end{pmatrix} + J_{ij}^{-1} \begin{pmatrix} r_{ij}^{y}(v_{ij}) \\ r_{ij}^{p}(v_{ij}) \end{pmatrix},$$

- Find v_{ij}^* such that $\frac{p_{ij}(v_{ij}^*)}{\lambda} - k\,(y_{ij}(v_{ij}^*) - v_{ij}^*) = 0$. Set

$$v_{ij} = \begin{cases} y_{H\,ij} & \text{if} \quad v_{ij}^* \geq y_{H\,ij} \\ v_{ij}^* & \text{if} \quad y_{L\,ij} < v_{ij}^* < y_{H\,ij} \\ y_{L\,ij} & \text{if} \quad v_{ij}^* \leq y_{L\,ij}. \end{cases}$$

Smoothing

- Compute the Jacobian for the y, p system

$$J_{ij}^{-1} = \frac{1}{\det J_{ij}} \begin{pmatrix} -(c_{ij}^p + \frac{h^2}{\lambda}) & 0 \\ -h^2(1+k) & -(c_{ij}^y + \frac{h^2}{\lambda}) \end{pmatrix},$$

- Define a local Newton update for y_{ij} and p_{ij} at i, j

$$\begin{pmatrix} y_{ij}(v_{ij}) \\ p_{ij}(v_{ij}) \end{pmatrix} = \begin{pmatrix} y_{ij} \\ p_{ij} \end{pmatrix} + J_{ij}^{-1} \begin{pmatrix} r_{ij}^y(v_{ij}) \\ r_{ij}^p(v_{ij}) \end{pmatrix},$$

- Find v_{ij}^* such that $\frac{p_{ij}(v_{ij}^*)}{\lambda} - k\,(y_{ij}(v_{ij}^*) - v_{ij}^*) = 0$. Set

$$v_{ij} = \begin{cases} y_{H\,ij} & \text{if} \quad v_{ij}^* \geq y_{H\,ij} \\ v_{ij}^* & \text{if} \quad y_{L\,ij} < v_{ij}^* < y_{H\,ij} \\ y_{L\,ij} & \text{if} \quad v_{ij}^* \leq y_{L\,ij}. \end{cases}$$

- Set $p_{ij} = p_{ij}(v_{ij})$ and $y_{ij} = y_{ij}(v_{ij})$

Preliminary numerical results

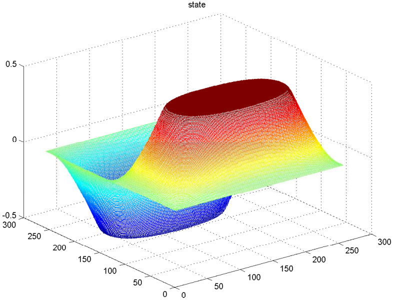
Consider $z(x_1, x_2) = \sin(2\pi x_1)\sin(\pi x_2)$ and $y_L(x) = -1/2$ and $y_L(x) = 1/2$. Mesh 513×513
Convergence factors choosing $\nu = \lambda^2$

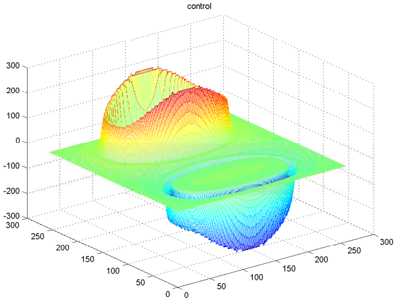
	λ	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
2nd-order	ρ	0.03	0.04	0.10	0.10	0.09
4th-order	ρ	0.09	0.07	0.12	0.08	0.08

Further results for ρ with $\nu = 10^{-6}$ and $\lambda = 10^{-3}$

Mesh	257×257	513×513	1025×1025
2nd-order	0.12	0.10	0.10
4th-order	0.15	0.12	0.09

Solution


state
0.5
0
-0.5
300
250
200
150
100
50
0
0
50
100
150
200
250
300


control
300
200
100
0
-100
-200
-300
300
250
200
150
100
50
0
0
50
100
150
200
250
300