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The Book of Numbers. By John Horton Conway and Richard K. Guy. Springer-Verlag, 1996, 320, $29.95.

Reviewed by **Andrew Bremner**

Am I the only person ever to have claimed a pineapple as deduction against income tax? The arrival of Conway and Guy's *Book of Numbers* may well mean that others follow suit when classroom teachers discover the pedagogical virtues of using the fruit to demonstrate the occurrence in nature of the Fibonacci numbers. What a delight it is to turn the pages of this book, being simultaneously entertained and enlightened by these masters of arithmetic mathesis. The planning of such a book displays enormous conceit, and bringing it to realisation is a remarkable achievement. Readers must enjoy discovering for themselves quite how successful the authors have been. With "numbers" as the theme of the book, the volume could well have been arbitrarily large; yet in practice it is quite a slim tome of just ten chapters (short of fourteen, one notes, the infinity of Borges).

It is in fact admirably restrained in size; it is also beautifully produced by Springer, and comfortably priced to boot. What more could one ask? There is a clear readership amongst the devoted followers of Martin Gardner, and *Scientific American*'s Mathematical Games columns. Readers familiar with the two volume *Winning Ways* by Berlekamp, Conway, and Guy **[1]**, will undoubtedly recognize inimitable matters of style common to both works, including many whimsically delightful illustrations. But the material of the *Book of Numbers* is intentionally far more accessible to the lay mathematician than that of *Winning Ways*, and can be perused with pleasure and profit by pappous schoolboy and pompous professor. There is truly something here for everyone.

The first chapter starts with the language of number, from the hypothesis that "Hickory, Dickory, Dock" is a corruption of some rustic counting sequence for "Eight, Nine, Ten", to the syntax of number names in different languages. As illustration, a table of cardinals in Welsh is given, where we learn for instance that "eighteen" occurs as "two times nine" and "fifty" as "half a hundred". (Those of us who have slogged through classical Hawaiian will recall the intrigue at "seventy" translating as "forty with thirty remainder" despite the regular formations of the other multiples of ten.) There follow individual ruminations based on the numbers between one and a hundred. So we find the familiar, the French "quinzaine" denoting the same period of time as the English "fortnight"; and the less familiar —"punch", a drink with five ingredients, from the Hindi word for five.

This whole section begs a parlour game with the aim of finding some literary reference to the integers between one and a hundred (maybe the Bible should be disbarred for this purpose. John Buchan provides an easy start; I suspect a search of *Tristram Shandy* would also prove fruitful.) The visual equivalent exists on celluloid in Greenaway's "Drowning by Numbers": the viewer becomes aware that the numbers from one to one hundred are occurring in sequence in the film, with an abrupt ending at the appearance of 100.

We progress to historical number systems (and discover how Caesar dealt with fractions: for instance, the five spots on a die is the Roman quincunx, the symbol for five uncia, or five twelfths. Of course, there are still today twelve ounces in both the troy and the apothecary's pound). Later in the book is a consequent discussion of tablet number 7289 from the Yale Babylonian Collection, an astonishing base 60 computation of the square root of 2. Its accuracy is such that the value is correct to six decimal places. Also shown is a photograph of the tablet Plimpton 322 which in Babylonian cuneiform appears to be a table of Pythagorean triangles (here duly completed by the authors, in Babylonian of course).

The second chapter includes Patterns Providing Pretty Proofs, with some geometrically inspired formulations of series summations. The Ackermann numbers are described, being the sequence $1, 4, 7625597484987, \ldots$ whose fourth term is so staggeringly immense that the cerebellum quivers merely at the thought of trying to comprehend it. Yet this is only a springboard for the authors' "chained-arrow" numbers; these in turn are of such magnitude as guaranteed to leave your brainbox smoking.

The following chapter concentrates on sequences, and in particular methods for determining the rule of formation of a sequence of integers. Little is assumed, so that binomial coefficients and Pascal's triangle are introduced from first principles. But there is an interesting discussion of Difference Fans and of Number Walls, of which I choose the latter to illustrate here with an example. If standard differencing techniques do not reveal the rule behind a sequence, then the sequence cannot be of polynomial type. To detect sequences of exponential type (linear recurrence relations) one forms a Number Wall: below a row of ones place the terms of the sequence, and then compute further rows in the wall using the rule that for each cross of five bricks,

$$(\text{centre-brick})^2 = (\text{north-brick})(\text{south-brick}) + (\text{east-brick})(\text{west-brick})$$

That is, for the cross

$$\begin{array}{ccc} & N & \\ W & C & E \\ & S & \end{array}$$

$C^2 = NS + EW$. So for the Fibonacci sequence, for example, we develop the wall

$$\begin{array}{ccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & \cdots \\ & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots \\ & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{array}$$

If the sequence is genuinely exponential, then ultimately a row of zeroes will appear, and the number of rows between the ones and zeroes gives the length of the recurrence relation, which is then easy to compute. What is so intriguing about this construction is the surely non-obvious fact that every brick will be computed as an *integer*. To prove all the properties of the Number Wall will clearly afford excellent exercise. In more substantial vein, I apply the Number Wall technique to the Shallit sequence (see [**2**]): a_{n+1} is the least integer such that

$$\frac{a_{n+1}}{a_n} > \frac{a_n}{a_{n-1}},$$

with $a_0 = 8$, $a_1 = 55$. The sequence is thus $8, 55, 379, 2612, 18002, \ldots$. We construct the wall

1	1	1	1	1	1	1	1	1	⋯
8	55	379	2612	18002	124071	855106	5893451	40618081	⋯
	−7	−19	−214	−1448	−5171	−87785	−82185	−4520276	⋯
		−3	7	55	−809	8515	−66185	501931	⋯
			−1	−6	−36	−216	−1296	−7776	⋯
				0	0	0	0	0	⋯

which indicates a four term recurrence relation. Indeed, we appear to find

$$a_n = 6a_{n-1} + 7a_{n-2} - 5a_{n-3} - 6a_{n-4}.$$

But beware, O constructor murorum! It is anti-intuitive that the sequence $\{a_n\}$ be exponential, and in fact with sufficient computer-aided effort it has been verified that the linear recurrence is valid only in the range $4 \le n \le 11055$. The term a_{11056} differs by 1 from that computed from the recurrence! It is instructive to estimate the size of a_{11056}, and contemplate the fact that the bottom row of the above wall starts out with 11051 zeroes before becoming non-zero.

This is heady stuff, but we are propelled forward into Famous Families of Numbers, where the Great Arithmetician of Ulm, Johann Faulhaber, is finally given full credit for his discovery and use in the *Academiae Algebrae* (1631) of what later became known as the Bernoulli numbers. It was almost eighty years later that Bernoulli himself embarked on his extensive study of these numbers (with, it should be remarked, full due to Faulhaber). So we find Bell and Catalan, Ramanujan and Stirling, and of course Fibonacci. There is an extended discussion of the occurrence in nature of the Fibonacci numbers, from the edible sorosis to leaf phyllotaxis. A mature pineapple will display eight spirals of bracts in one direction, and thirteen in the other, clockwise and anticlockwise (a supermarket sortie found no counterexample to this rule; the lady who observed my protracted handling of every single pineapple advanced with the advice: "You should *smell* them, you know"). Similarly a sunflower head will usually display 34 and 55 spirals of seeds. The authors provide what seems a very plausible explanation for this phenomenon ("Say, bud, where do you think you are going?"). D'Arcy Thompson's admirable *On Growth and Form* was a favourite tome of the childhood bookshelf, but is sadly remiss in failing even to mention the Fibonacci sequence.

In the Primacy of Primes, we meet *inter alia* Conway's famous Prime Producing Machine, and we are given the current status on the factorization of the Mersenne and Fermat primes. The latter, $F_n = 2^{2^n} + 1$, is now known to be composite for $5 \le n \le 23$; factorizations are given for $5 \le n \le 11$, and partial factorizations for $n = 12, 13$. This is certainly very much up-to-date, with F_{10} and F_{11} having only just fallen (however the authors do miss two very recently announced Mersenne primes).

Fruitfulness of Fractions is a rag-bag collection of Farey fractions, decimal expansions of prime reciprocals, shuffles, and Pythagorean triangles. Continued fractions, which themselves might merit an entire volume, are restricted to a mention in terms of the astronomical Metonic cycle, used to determine the Jewish calendar and the date of Easter. (What a shame Conway was not prevailed upon to add a page or two at this stage, even if not *strictly* relevant. Master of much, he is supreme in the quirks of the calendar, and I have savoured for many years his perspicacious observation that any Swede born on February 29 in the year 1696 would not have celebrated a first birthday for another 48 years!)

"Algebraic Numbers" leads to the ruler-and-compass construction of the regular polygons, and beautifully elegant constructions with appropriate diagrams are given for the 3-, 5-, 7-, 9-, 13-, and 17-gons ("But..." I hear you murmur; yes, indeed, the luxury of an angle-trisector is used for the 7-, 9-, and 13-gons.) Some problems are given whose solution will involve a specific algebraic number, for instance, that of finding a hexagon of largest area given that no two vertices are more than one unit of distance apart. Ron Graham solves the problem with a hexagon of area A, with A satisfying an irreducible polynomial of degree 10 ($A = 0.674981\ldots$; the area of a regular hexagon of unit side is $3\sqrt{3}/8 = 0.649519\ldots$). There is even a somewhat contrived problem whose solution involves the root of a polynomial of degree 71. It should perhaps be stressed that throughout the book, all these deeper results are simply quoted, and each chapter has a sizeable bibliography referring the reader to research papers where necessary, and to appropriate literature in general.

The remaining chapters include sections on imaginary and transcendental numbers, with mention of the connection between Euler's prime-producing polynomial $n^2 - n + 41$ and the fact that $e^{\pi\sqrt{163}}$ is an integer. ("But..." I hear you mutter again: you must simply go away and compute, but make sure to give yourself at least 31 significant figures.) There is also a nice discussion of Gregory numbers, where the name of Lewis Carroll arises.

Finally, in the unique style of the authors, there is a chapter on the infinite and the infinitesimal. We are shown how to add and multiply, just as if infinities occurred every day in our cheque-books. Surrealism comes into play, as it were, but it's only a game. Sorry. You just have to get hold of a copy of this book; trying to summarize adequately the 320 pages of Conway & Guy is as demanding an exercise as extracting the plums from a particularly rich Christmas pudding.

So the book has multifarious virtues: what are its faults? Being ever greedy, one can lament what the authors have chosen to omit, as much as rejoice in what they include. Here again is another parlour game, to alphabetize missing topics that perhaps merit mention: amicable numbers and Alcuin's sequence, Beatty sequences, congruent numbers... There is a slightly irritating and curious inconsistency with the chosen type-faces. Springer have chosen a rather spidery font for some of the tables, which can render the content impactless. For instance, Table 6.1 lists those repeating decimals that occur in fractions with denominator a given prime. One of the features that should leap to the eye is the equal length of the entries for a given prime: 027, 054, 081, ... for the prime 37. As it stands, however, entries such as that at 53 look anything but equal in length. Yet other tables, such as Table 1.4, decimal expansions of "some of our favourite numbers", are set perfectly. Which is not in the index; at least, "perfect" is not in the index. A little unfortunate, since this was the very first item that I looked up on receiving the book (perfect numbers are indeed mentioned on pages 136–137). The index also mispaginates at least one item. Tut. However, if such be the sum total of faults, then there is not much cause for curmudgeonly grumble.

It is clear that this eclectic review can only begin to convey the pleasure that this book has provided. Throughout, the authors communicate their enthusiasm and exuberance with great éclat. There is joy at being in the care of mathematicians who delight in the sheer *friendliness* of numbers. The lucid explanations and insights can be startling and impressive. Gounod set to music in his curiously admonitory duet "L'Arithmétique":

> ...
> Cultiver cet art salutaire
> C'est apprendre à garder son bien,
> Car, mes amis, sur cette terre,
> Sachez, qu'on a souvent affaire
> A des gens qui comptent trop bien.

How fortunate that these authors who count so much better than most of us have imparted their wisdom to the printed medium. In Japan at New Year, the takarabune is the treasure-laden ship of the Gods of Good Fortune. Here is our ship, with Conway and Guy the Bringers of Happiness.

REFERENCES

1. E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways: for your mathematical plays*, Academic Press, 1982 (2 vols.).
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> There is an old Armenian saying, "He who lacks a sense of the past is condemned to live in the narrow darkness of his own generation." Mathematics without history is mathematics stripped of its greatness: for, like the other arts—and mathematics is one of the supreme arts of civilization—it derives its grandeur from the fact of being a human creation
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