# R-matrices, affine quantum groups and applications

From a mini-course given by David Hernandez at Erwin Schrödinger Institute in January 2017.

**Abstract**

R-matrices are solutions of the quantum Yang-Baxter equation. At the origin of the theory of quantum groups, they can be interpreted as intertwining operators in representation theory. After reviewing standard constructions from quantum affine algebras, we will present recent development in the theory. Maulik-Okounkov gave a general geometric construction of stable basis and R-matrices. In another direction, monoidal categorifications of cluster algebras have been established using R-matrices to categorify Fomin-Zelevinsky mutations relations. We also discuss recent advances on transfer matrices derived from R-matrices, which give new informations on corresponding quantum integrable systems as well as on the ODE/IM correspondence seen in the context of affine opers.

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## 1 Algebraic solution (Drinfeld)

**Définition 1.1.** Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ and $z$ be an indeterminate. The Yang-Baxter equation (with spectral parameters) is as follows:

$$\mathcal{R}_{12}(z)\mathcal{R}_{13}(zw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(zw)\mathcal{R}_{12}(z),$$

where $\mathcal{R}(z) \in (\mathcal{A}\otimes\mathcal{A})(z)$, $\mathcal{R}_{12}(z) = \mathcal{R}(z)\otimes 1$, $\mathcal{R}_{23}(z) = 1\otimes\mathcal{R}(z)$ and $\mathcal{R}_{13}(z) = (P\otimes\mathrm{Id})\mathcal{R}_{23}(z)$; $P$ is a "twist".

**Exemple 1.2.** A fundamental (historical) example (Yang R-matrix): $\mathcal{A} = \mathrm{End}(\mathbb{C}^2)$, $q \in \mathbb{C}^*$,

$$\mathcal{R}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q^{-1}(z-1)}{z-q^{-2}} & \frac{1-q^{-2}}{z-q^{-2}} & 0 \\ 0 & \frac{z(1-q^{-2})}{z-q^{-2}} & \frac{(z-1)q^{-1}}{z-q^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

.

It is attached to the 6-vertex model (ice) of XXZ-spin chain model.

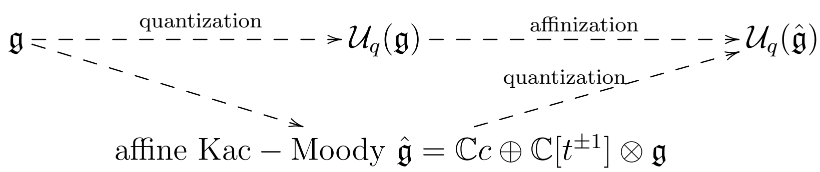
If one writes $z = e^u$, $q = e^{\frac{h}{2}}$, then

$$\begin{pmatrix} \frac{q^{-1}(z-1)}{z-q^{-2}} & \frac{1-q^{-2}}{z-q^{-2}} \\ \frac{z(1-q^{-2}}{z-q^{-2}} & \frac{(z-1)q^{-1}}{z-q^{-1}} \end{pmatrix} \underset{h,u\to 0}{\sim} \frac{1}{u+h}\begin{pmatrix} u & h \\ h & u \end{pmatrix}.$$

Algebraic construction: $V$ is a 2 dimensional representation of $\mathcal{U}_q(\hat{\mathrm{sl}_2})$ (affine quantum group).

We can generalize by replacing $\mathrm{sl}_2$ by $\mathfrak{g}$ a general simple finite dimensional Lie algebra and $V$ by a finite dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$. This gives rise to many $\mathcal{R}$-matrices.

By Drinfeld-Jimbo, we have the following:




**Theorem 1.3.** *(Drinfeld)"Commutative diagram" gives* 2 *presentation of* $\mathcal{U}_q(\hat{\mathfrak{g}})$.

**Remark 1.4.** $\mathcal{U}_q(\hat{\mathfrak{g}})$ is a Hopf algebra.

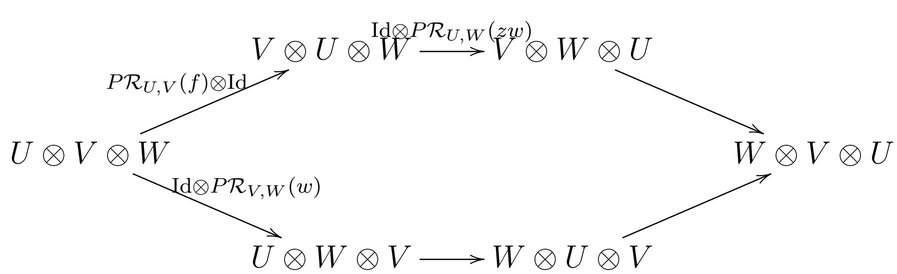
**Theorem 1.5.** $\mathcal{U}_q(\hat{\mathfrak{g}})$ *has a universal non trivial* $\mathcal{R}$*-matrix:* $\mathcal{R}(z) \in (\mathcal{U}_q(\hat{\mathfrak{g}})\hat{\otimes}\mathcal{U}_q(\hat{\mathfrak{g}})[[z]]$, *where* $\hat{\otimes}$ *is some completion of the tensor product for some filtration, such that from two finite dimensional representations* $V, W$ *of* $\mathcal{U}_q(\hat{\mathfrak{g}})$, *we get* $P\mathcal{R}_{V,W}(z) : V \otimes W[[z]] \to W \otimes V[[z]]$, *where* $R_{V,W}(z)$ *is the image of* $R(z)$ *in* $\mathrm{End}(V \otimes W)[[z]]$*: we get a well defined morphism of* $\mathcal{U}_q(\hat{\mathfrak{g}})$*-modules.*

**Proposition 1.6.** *If* $V, W$ *are simple and finite dimensional, there exists* $f_{V,W} \in \mathbb{C}((z))$ *such that* $f_{V,W}(z)R_{V,W}(z)$ *is rational in* $\mathrm{End}((V \otimes W)) \otimes \mathbb{C}(z)$. *Let* $d \in \mathbb{Z}$ *be the order of* 1 *as a pole of* $R_{V,W}(z)f_{V,W}(z)$. *Then* $\lim_{z\to 1}[(z-1)^d P f_{V,W}(z)R_{V,W}(z)] := \mathcal{R}^{norm}_{V,W} : V \otimes W \to W \otimes V$ *is a non-zero* $\mathcal{U}_q(\hat{\mathfrak{g}})$*-module morphism.*

**Remark 1.7.** In general we do not know the orders of poles and the zeros of rational points.

**Exemple 1.8.** see example above, $\mathfrak{g} = \mathrm{sl}_2$, $V = V(1)$, $W = W(q^2)$, evaluation representation of $\mathcal{U}_q(\hat{\mathrm{sl}_2})$ (morphism of algebra $\mathcal{U}_q(\hat{\mathrm{sl}_2}) \to \mathcal{U}_q(\hat{\mathrm{sl}_2})$). Then $d = 1$, i.e 1 is a pole of order 1 and $\mathcal{R}^{norm}_{V,W}$ has rank 1 (not invertible).

**Remark 1.9.** If $U, V, W$ are finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$:




By the Yang-Baxter equation, this diagram is commutative.

# 2 Geometric construction (theory stable envelop, Maulik-Okounkov

We consider the couple $(X, \omega)$, where $X$ is a quasi-projective, non singular algebraic variety and $\omega$ is a holomorphic simplectic form. We have an action of $(\mathbb{C}^*)^{R+n} \simeq T \supset A \simeq (\mathbb{C}^*)^R$.

Hypothesis:

- $\mathbb{C}\omega \subset H^0(\Omega^2\omega)$ is stable for the induced action of $T$. We have a corresponding character $h : T \to \mathbb{C}^*$ and $A \subset \mathrm{Ker} h$
- there exists a proper $T$-equivariant affine map $\mathcal{F} : X \to X_0$ satisfying an additional technical formality condition.

**Exemple 2.1.**

- $A = (\mathbb{C}^*)^{n+1}$, $T = A \times \mathbb{C}^*$, $X = T^*\mathbb{P}^n$ (action on coordinates of $\mathbb{P}^n$). An additional functor acts on the fibers.
- Nakajima quiver varieties.

Let $X^A$ be the set of fixed points for the action of $A$, and $N(X^A)$ be the normal bundle $X^A$. It has a structure of a direct sum of 1 dimensional representations of $A$.

**Définition 2.2.** Let $\Delta$ be the set of characters of $A$ corresponding to $A$-simple modules occurring in $N(X^A)$.

One has $\Delta \subset \mathcal{X}^*_{\mathbb{R}} = C(A) \otimes_{\mathbb{Z}} \mathbb{R}$, where $C(A)$ is the group of characters of $A$. One has $\mathcal{X}_{\mathbb{R}} \simeq \mathbb{R}^n \subset \mathrm{Lie}(A)$.

For $\alpha \in \Delta$, we have the hyperplane $\alpha^\perp = \{v \in \mathcal{X}_{\mathbb{R}} | \alpha(v) = 0\}$.

We have the decomposition $\mathcal{X}_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta} \alpha^\perp = \bigsqcup_i \mathcal{C}_i$, where the $\mathcal{C}_i$ are the open chambers.

For a connected component $Z \subset X^A$ and a chamber $\mathcal{C}$, we set

$$\mathrm{leaf}_{\mathcal{C}}(Z) = \{x \in X | \lim_{z \to 0} (\sigma(z).x) \text{ exists and is in } Z\}$$

for some $\sigma \in \mathcal{C}$ (this does not depend on the choice of $\sigma$).

For each chamber, we define a partial ordering $\preceq_{\mathcal{C}}$ on the connected components of $X^A$ by saying $Z' \preceq_{\mathcal{C}} Z$ if $\overline{\mathrm{leaf}_{\mathcal{C}} Z} \cap Z' \neq \emptyset$. The slope of a connected component $Z$ is then $\mathrm{Slope}_{\mathcal{C}}(Z) = \bigsqcup_{Z' \preceq Z} \mathrm{leaf}_{\mathcal{C}}(Z')$.

**Exemple 2.3.** $X = T^*\mathbb{P}^1$, $\Delta = \{\alpha, -\alpha\}$, $\alpha(a, b) = a^{u_0} b^{-u_1}$ where $u_0, u_1$ are characters for the action on $A$.

One has $\mathcal{X}_{\mathbb{R}} = \mathbb{R}^2 = \mathcal{C}_+ \sqcup \alpha^\perp \sqcup \mathcal{C}_-$, where $\mathcal{C}_\epsilon = \{(x, y) \in \mathbb{R}^2 | \epsilon u_0 x > u_1 y\}$ for $\epsilon \in \{-, +\}$.

Let $p_0 = [1 : 0]$, $p_1 = [0 : 1]$. One has $\mathbb{P}^1 \subset T^*\mathbb{P}^1$.

One has $\mathrm{leaf}_{\mathcal{C}_+}(\{p_0\}) = T^*_{p_0}(\mathbb{P}^1)$, $\mathrm{leaf}_{\mathcal{C}_-}(\{p_1\}) = T^*_{p_1}(\mathbb{P}^1)$, $\mathrm{leaf}_{\mathcal{C}_+}(\{p_1\}) = \mathbb{P}^1 \setminus \{p_1\}$, $\mathrm{leaf}_{\mathcal{C}_+}(\{p_0\}) = \mathbb{P}^1 \setminus \{p_0\}$.

Let $H^\bullet_T(X)$, $H^\bullet_T(X^A)$ be the $T$-equivariant cohomology.

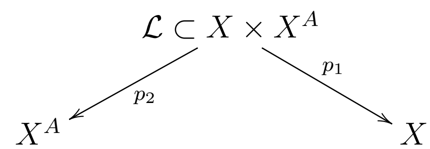
**Theorem 2.4.** *(Maulik-Okounkov) There exists a unique* $\mathrm{Stab}_{\mathcal{C}} : H^\bullet_T(X^A) \to H^\bullet_T(X)$ *morphism of* $H^\bullet_T(\{pt\})$*-modules such that for* $\gamma \in H^\bullet_T(Z)$*, with* $Z$ *a connected component of* $X^A$ *and* $\mu = \mathrm{Stab}_{\mathcal{C}}(\gamma)$*, we have :*

1. $\mathrm{Supp}(\mu) \subset \mathrm{Slope}_{\mathcal{C}}(Z)$

2. $\mu_{|Z} = \pm e(N_-(Z)) \cup \gamma$ *where* $N_-(Z)$ *is the "negative part of* $\mu(Z)$ *with respect to* $\mathcal{C}$*" (the sign is decided with the choice of a "polarization")*

3. *for* $Z' \succ_{\mathcal{C}} Z$, $\deg_A(\mu_{|Z'}) < \mathrm{codim}_Z(Z')$.

Idea of construction: based on the construction of a Lagrangian correspondence such that $p_1$ is proper.

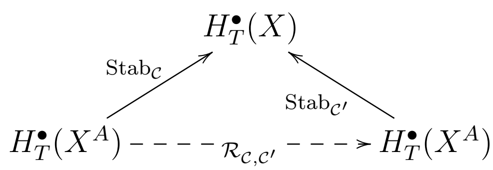



Then $\mathrm{Stab}_{\mathcal{C}} = (p_1)_* \circ (p_2)^*$.

**Exemple 2.5.** $X = T^*(\mathbb{P}^1)$, $u_0, u_1, h$ characters, $H_T^\bullet(\{pt\}) = \mathbb{C}[u_0, u_1, h]$, $H_T^\bullet(X) = \mathbb{C}[u_0, u_1, h] \oplus \mathbb{C}[u_0, u_1, h]c$, where $c$ is the first Cherr character of $\mathcal{O}(-1)$.

$H_T^\bullet(X^A) = \mathbb{C}[u_0, u_1, h][p_0] \oplus \mathbb{C}[u_0, u_1, h][p_1]$,
$\mathrm{Stab}_{\mathcal{C}_+}([p_0]) = u_1 - c$, $\mathrm{Stab}_{\mathcal{C}_-}([p_0]) = u_1 - c - h$
$\mathrm{Stab}_{\mathcal{C}_+}([p_1]) = u_0 - c - h$, $\mathrm{Stab}_{\mathcal{C}_-}([p_1]) = u_0 - c$.

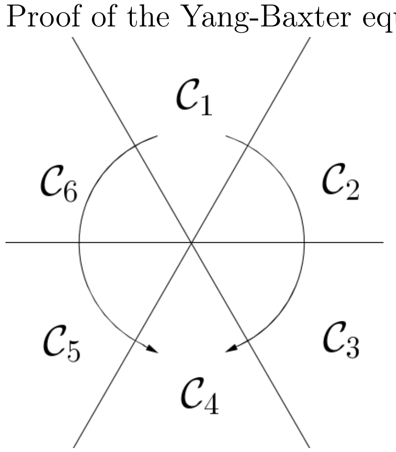
**How do we get $R$-matrices :** $\mathcal{C}, \mathcal{C}'$ chambers. Up to a localization, $\mathrm{Stab}_{\mathcal{C}'}$ is invertible and we set: $\mathcal{R}_{\mathcal{C}',\mathcal{C}} = \mathrm{Stab}_{\mathcal{C}'}^{-1} \circ \mathrm{Stab}_{\mathcal{C}}$, where $\mathrm{Stab}_{\mathcal{C}} : H_T^\bullet(X^A) \to H_T^\bullet(X^A)$.




**Exemple 2.6.** In the basis of fixed points: $\mathrm{Stab}_{\mathcal{C}_+} = \begin{pmatrix} -u & -h \\ 0 & u-h \end{pmatrix}$, $\mathrm{Stab}_{\mathcal{C}_-} = \begin{pmatrix} -u-h & 0 \\ -h & u \end{pmatrix}$, with $u = u_1 - u_0$, $u + h \neq 0$ and $\mathcal{R}_{\mathcal{C}_-,\mathcal{C}_+}(u) = \frac{1}{u+h}\begin{pmatrix} u & h \\ h & u \end{pmatrix}$, which is the Yang $R$-matrix.

More generally, $X$ : Nakajima quiver variety, $Q$ quiver with $n$ vertices give more $R$-matrices.

Proof of the Yang-Baxter equation (idea):




$\mathcal{R}_{\mathcal{C}_4,\mathcal{C}_3}\mathcal{R}_{\mathcal{C}_3,\mathcal{C}_2}\mathcal{R}_{\mathcal{C}_2\mathcal{C}_1} = \mathcal{R}_{\mathcal{C}_4,\mathcal{C}_5}\mathcal{R}_{\mathcal{C}_5,\mathcal{C}_6}\mathcal{R}_{\mathcal{C}_6\mathcal{C}_1}$.

**Remark 2.7.** RTT construction: $R$ matrices give quantum groups.

For $Q$ a quiver, using Nakajima varieties $\mathcal{U}_q(W)$, $Y_Q$ (Maulik-Okounkov Yangian) is a Hopf algebra.

# 3 $\mathcal{R}$ matrices and categorification

Let $\mathfrak{g}$ be a simple finite dimensional algebra and $\mathfrak{n} \subset \mathfrak{g}$ be a nilpotent sub-algebra. Then $\mathcal{U}_q(\mathfrak{n}) \subset \mathcal{U}(\mathfrak{g})$. There is a realization of $\mathcal{U}_q(\mathfrak{n})$ in term of quiver-Hecke algebra (Khovanov-Lauda, Rouquier): it may be seen as an analog of affine Hecke algebras of type $A$: $\mathcal{U}_q(\mathfrak{n}) \simeq K(\oplus_{v\in\mathbb{N}^n} R_v - \text{Mod})$, where $Q$ is a quiver, $n$ is the number of vertices, $v \in \mathbb{N}^n$ and $\oplus_{v\in\mathbb{N}^n} R_v - \text{Mod}$ is a graded finite dimensional module with a convolution product .

There is a bijection between dual canonical basis and the classes of simple modules (Varagnolo-Vasserot).

Moreover: $\mathcal{U}_q(\mathfrak{n})$ has a quantum cluster algebra structure (Bernstein-Zelevinsky).

It has a distinguished set of generators (cluster variables) defined inductively by mutation relations (Fomin-Zelevinsky)

We have the following: $\chi * \chi = \Pi_i \chi_i + \Pi_j \chi_j$, where $\chi$, $\chi*$, $\chi_i$, $\chi_j$ are cluster variables.

**Conjecture 3.1.** Cluster variables belong to the dual canonical basis.

**Notion of categorification of cluster algebras:** cluster algebras correspond to certain classes of simple modules.

Let $\mathcal{A}$ be a cluster algebra and $\mathcal{M}$ be a monoidal category.

Then $K(\mathcal{M}) \simeq \mathcal{A}$ (ring isomorphism). Under this isomorphism,

{ class of simple modules} correspond to {cluster variables}

The sequences below correspond to mutation relations

$$0 \to \bigotimes_j [\chi_j] \to [\chi^*] \otimes [\chi] \to \bigotimes_i [\chi_i] \to 0$$

.

**Exemple 3.2.**
- Hernandez-Leclerc: using finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$
- Nakajima, Kimura-Qin: using perverse sheaves on quiver varieties
- Kang-Kashiwara-Kim-Oh: proved the conjecture using KLR algebras.

**Theorem 3.3.** *(Kang-Kashiwara-Kim-Oh)*
$\bigoplus_{v\in\mathbb{N}^n} R_v - Mod$ *is a categorification of the cluster algebra* $\mathcal{U}_q(\mathfrak{n})$.

This theorem proves a conjecture of Fomin-Zelevinsky (see also Qin).

Idea of the proof: a crucial point is the construction of normalized $\mathcal{R}$-matrices $\mathcal{R}^{norm}_{V,W}$ : $V \circ W \to W \circ V$ (no notion of universal $R$-matrices).

Then we realize mutation relations as exact sequences:

$$0 \to \text{Ker} \to V \circ W \overset{\mathcal{R}^{norm}_{V,W}}{\to} \text{Im} \to 0.$$

$\mathcal{R}$-matrices give categorification of cluster algebras.

If $Q$ is a quiver, $\mathcal{A}_Q$ (quantum) is a cluster algebra generated (as a ring) by cluster variables.

Let $\mathcal{M}$ be a monoidal category such that $\mathcal{A}_Q \simeq K_0(\mathcal{M})$ (ring isomorphism).

cluster variable $\subset$ class of simple modules.

**Exemple 3.4.** • (categorical setting) $\mathfrak{g} = \mathrm{sl}_3$, $\mathcal{U}_q(\hat{sl}_3)$. For $a \in \mathbb{C}^*$, let $\mathrm{ev}_a : \mathcal{U}_q(\hat{\mathrm{sl}}_3) \to \mathcal{U}_q(\mathrm{sl}_3)$ be the evaluation morphism.

Let $V_1$, $V_2$ be fundamental representations of $\mathcal{U}_q(\mathrm{sl}_3)$. For all $a \in \mathbb{C}^*$, we have $V_1(a)$, $V_2(a)$, 3 dimensional simple $\mathcal{U}_q(\hat{\mathrm{sl}}_3)$ representations.

$$0 \subset W \subset V_1(1) \otimes V_1(q^2) \overset{\mathcal{R}^{norm}_{V_1(q)\otimes V_1(q^2)}}{\to} V_2(q) \to 0,$$

where $W = \ker(R^{norm})$ is simple and $V_2(q) \subset V_1(q^2) \otimes V_1(1)$.

Let $\mathcal{C}$ be the category of all finite dimensional representations and $\mathcal{C}_0 \subset \mathcal{C}$ be the category of all finite dimensional $\mathcal{U}_q(\hat{\mathrm{sl}}_3)$-representation $V$ satisfying :

$$[V] \in \mathbb{C}[[V_1(1)], [V_1(q^2)], [V_2(q)]],$$

where $[V]$ is the class of $V$ in $K(\mathcal{C})$ and $[V_1(1)], [V_1(q^2)]$ and $[V_2(q)]]$ are classes in $K(\mathcal{C}_0)$.

By construction, $\mathcal{C}_0$ is monoidal.

• $\mathrm{SL}_3 \supset N = \{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\}$, $\mathbb{C}[N] = \mathbb{C}[x, y, z]$.

One has $\mathbb{C}[N] \supset B = \{x^a y^b (xy - z)^c, a, b, c \in \mathbb{N}\} \cup \{y^a z^b (xy - z)^c, a, b, c \in \mathbb{N}\}$. The set $B$ is the canonical basis.

We have a ring isomorphism:

$$\begin{aligned} \mathbb{C}[N] &\overset{\psi}{\simeq} K(\mathcal{C}_0) \\ x &\leftrightarrow [V_1(1)] \\ y &\leftrightarrow [V_1(q^2)] \\ z &\leftrightarrow [V_2(q)] \\ xy - z &\leftrightarrow [V_1(1) \otimes V_1(q^2)] - [V_2(q)] = W \end{aligned}.$$

The map $\psi$ defines a bijection between $B$ and the classes of simple modules.

**Cluster algebra structure** $Q : 3 \to 1 \to 2$

$\mathcal{A}_Q \subset \mathbb{Q}(x_1, x_2, x_3)$. The variables $x_1$, $x_2$ and $x_3$ are cluster variables of $\psi$. The mutation relations are $X_1^* = x_1^{-1}(x_2 + x_3)$, $x_1^* x_1 = x_2 + x_3$.

One has $\mathbb{C} \otimes \mathcal{A}_Q \simeq \mathbb{C}[N] \simeq K(\mathcal{C}_0)$, $x_1 \leftrightarrow x$, $x_2 \leftrightarrow z$, $x_3 \leftrightarrow xy - z$ and $x_1^* \leftrightarrow y$.

Cluster variables $\subset$ simple modules.

# 4 Quantum integrable system

**Transform-matrix construction** Let $V$ be a finite dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$, $t_V(z) = (\mathrm{Tr}_V \otimes \mathrm{Id})\mathcal{R}(z) \in \mathcal{U}_q(\hat{\mathfrak{g}})[[z]] = \sum_{m \geq 0} t_V[m] z^m$ ( the $t_V[m] \in \mathcal{U}_q(\hat{\mathfrak{g}})$ are transformation matrices).

By Yang-Baxter equation, $t_V(z) t_W(w) = t_W(w) t_V(z)$, the $t_V[m]$, for $V$ finite dimensional and $m \in \mathbb{N}$ generate a commutative sub-algebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (apply $\mathrm{Tr}_{V \otimes W} \otimes \mathrm{Id}$).

We have actions of $K(\mathcal{C})$ on some spaces: the quantum integrable systems.

$$[V] \to t_V(z) \to \mathrm{End}(W)[[z]].$$

1. "XXZ"-type: $W$ finite dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$, action of $K(\mathcal{C})$ on $W \otimes \mathbb{C}[[z]]$
2. Quantum KdV-models: $W$ a Fork space over Virasoro algebra.

Spectrum ? Eigenvalues of the operators $t_V(z)$ on $W$ ? In the case 1: conjecture of Frenkel-Reshetikhin when $V, W$ finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$

**Theorem 4.1.** *(Frenkel-Hernandez)*
*The eigenvalues of $t_V(z)$ on $W$ can be expressed as a generalized Baxter relation in term of polynomials.*

**Exemple 4.2.** $\mathfrak{g} = \mathrm{sl}_2$, $V$: 2 dimensional simple module, $W$ tensor product of simple modules. Eigenvalues: $\lambda_j(Z) = D(z)\frac{P_j(zq^{-2})}{P_j(z)} + A(z)\frac{P_j(zq^2)}{P_j(z)}$ (the Baxter TQ relations)
$D(z)$, $A(z)$: universal functions, $P_j(z)$ polynomial.

**Remark 4.3.** In general, more than two terms.

Zeros of the $P_j(z)$ ? Hint from Langlands duality ODE/IM (integral models) correspondence (Dorey-Tateo).
Spectral determinants of Schroedinger operators (differential operators) correspond to eigenvalues of KdV systems.

**Conjectural generalization** (Feigin-Frenkel)
Affine $\mathcal{G}$ opers (without monodromy) are in correspondence with eigen values of quantum KdV systems.
Masoero-Razmondo-Valeri: on oper side they established $Q\tilde{Q}$ relations, which implies Bethe- Ansatz equations.
Let $\mathcal{C}$ be the finite dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and $\mathcal{O}$ be the category $\mathcal{O}$ of representation of $\mathcal{U}_q(\hat{\mathfrak{b}})$, where $\hat{\mathfrak{b}}$ is a Borel sub-algebra ($\mathcal{C} \subset \mathcal{O}$)

**Theorem 4.4.** *(Frenkel-Hernandez) The $Q\tilde{Q}$-system hold in $K(\mathcal{O})$.*
*In particular, the roots of the $P_j$ satisfy the Bethe Ansatz equations generically. The genericity conditions have been removed by Feigin-Jimbo-Miwa-Mukhin.*