# Spherical varieties and arc spaces

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## 1 Lecture 1

This is a joint work with Anne Moreau. Let us begin with a few notations. We consider $G$ a reductive connected algebraic group over $\mathbb{C}$, with Borel subgroup $B$.

Let $X$ be a spherical $G$ variety, assumed to be projective and smooth. Recall that *spherical* means that $X$ admits an open $B$-orbit. Then it also admits an open $G$-orbit. Let $X_0$ be the open $G$-orbit in $X$, which can be written as a quotient $X_0 = G/H$, for some *spherical subgroup* $H$ of $G$.

Spherical varieties can be described by some combinatorial data, which involve both the group $G$ and the space $X$. More precisely, this description splits in two parts. First one needs to understand spherical subgroups, and second to understand equivariant embeddings of $X_0$ in $X$. Both steps of the description are combinatorial and it turns out that the second part is easier than the first one. Equivariant embeddings are described by the Luna-Vust theory via *colored fans*. The spherical subgroups can be described up to conjugacy by some more involved combinatorial data.

With this picture in mind, there should be an interpretation of the *betti numbers* of $X$, $b_i(X) = \dim_{\mathbb{C}} H^i(X, \mathbb{C})$, in terms of combinatorics.

Problem. How to compute $b_i(X)$ using combinatorics involving $X$ and $G$?

This problem has a solution and it is the purpose of these lectures to explain it. Let us first look at some examples.

**Example 1.** Assume that $X$ is homogeneous, that is $X = X_0$, in which case $H$ is parabolic since $X$ is projective. For simplicity, take $P = B$. We want to compute

$$B(G/B, t) = \sum_{i \geq 0} b_{2i}(G/B) t^i. \tag{1.1}$$

Recall that the homogeneous space $G/B$ is spherical since the group $W = B\backslash G/B$ is finite. The Betti polynomial $B(G/B, t)$ can be expressed in terms of the length of elements in $W$:

$$B(G/B, t) = \sum_{g \in W} t^{\ell(g)}. \tag{1.2}$$

When $G = SL(2)$, $G/B = \mathbb{P}^1$ and $B(G/B, t) = 1 + t$.

**Example 2.** Assume that $G = T = B$ and that $H$ is the trivial group $\{e\}$. In other words, $X$ is a smooth projective toric variety. Let $M = X(T)$ be the character lattice, $N = X^*(T)$ the dual lattice and $\mathcal{F}$ the fan of cones corresponding to $X$. It spans $N_{\mathbb{R}}$ because $X$ is projective.

There is a one-to-one correspondance between $G$-orbits in $X$ and cones $\sigma$ in $\mathcal{F}$. These cones have different dimensions but satisfy

$$\dim(\sigma) = \operatorname{codim}(\mathcal{O}_\sigma), \tag{1.3}$$

where $\mathcal{O}_\sigma$ denotes the $G$-orbit in $X$ corresponding to the cone $\sigma \in \mathcal{F}$.

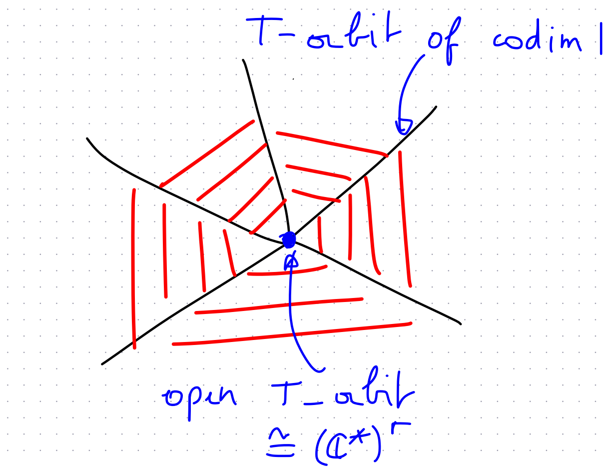


Table 1: Fan of a toric variety

These orbits are themselves tori of smaller dimensions. Therefore,

$$B(X,t) = \sum_\sigma B(\mathcal{O}_\sigma, t) = \sum_\sigma (t-1)^{codim \sigma}. \tag{1.4}$$

Note that $B(X,p) = |X(\mathbb{F}_p)|$ for prime integer $p$.

For instance, $\mathbb{P}^1$ has three orbits under $\mathbb{C}^*$. Two of 0 dimension and one of dimension 1. The previous formula gives $B(\mathbb{P}^1, t) = 1 + 1 + (t-1) = 1 + t$.

**Example 3.** Let us now consider the case of *horospherical varieties* which includes both Example 1 and Example 2. Recall that the spherical homogeneous space $X = G/H$ is called *horospherical* if $H$ contains the unipotent radical $U$ of some Borel subgroup $B$. Any embedding $X_0 \hookrightarrow X$ of a horospherical homogeneous space $X_0 = G/H$ is called *horospherical*. To make the things more concrete, let us consider $G = SL(2)$ and $U \subset H \subset B$. The dimension of $H$ can be either 1 or 2. The case $H = B$ has already been studied. So let us assume that $\dim H = 1$. The subgroup $H$ can be the unipotent group $U$ itself or some finite extension:

$$U_k = \{\begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix}, \ \lambda^k = 1\}, \qquad U_k/U \cong \mu_k, \tag{1.5}$$

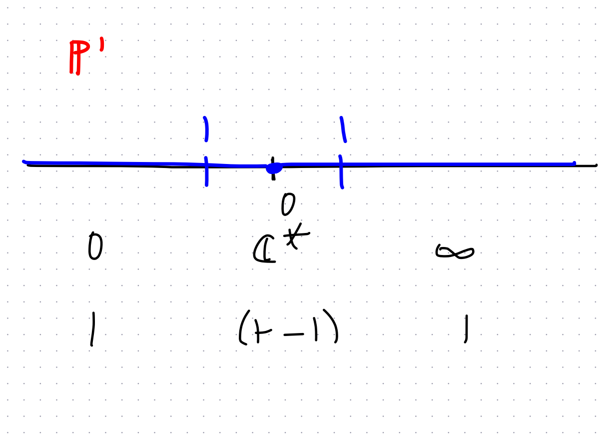


Table 2: Fan of $\mathbb{P}^1$

where $\mu_k$ is the cyclic group of $k$-roots of unity.

If $G/H$ is horospherical, the normalizer $P = N_G(H)$ is parabolic. The commutator $[P, P]$ is a subset of $H$, therefore $T = P/H$ is a torus of dimension 1. The following map

$$G/H \hookrightarrow G/P \tag{1.6}$$

is a fibration of the projective homogeneous space $G/P$ with toric fibers. Any horospherical homogeneous space can be determined by the choice of a parabolic group $P$ and of a subgroup $H$ which seats in between $U$ and $B$. Consider an embedding of $G/H$ into a smooth projective spherical varietie $X$:

$$G/H \hookrightarrow X. \tag{1.7}$$

A naive way to proceed would be two use the decomposition of $X$ as a disjoint union of its $G$-orbits through the formula (which we used in example 2):

$$B(X, t) = \sum_x B(\mathcal{O}_x, t). \tag{1.8}$$

The problem here is that in full generality it is not obvious to compute the Betti polynomials of the $G$-orbits. We will explain another way of computing them using arc spaces.

Let us go back to our example where $H = U_k$ for some $k \geqslant 1$. There are little choices to embed $G/H$ in a smooth projective spherical variety. When $k = 1$, there are two such embeddings. It is either $\mathbb{P}^2$ or $\mathbb{F}^1$ (which is a blowup of a point in $\mathbb{P}^2$). If $k \geqslant 2$, there is only one choice for $X$, namely $\mathbb{F}_k$, the blowing-up of $\mathbb{P}^2$ at $k$ general points. As for the combinatorial data needed to describe such embeddings via the Luna-Vust theory, it depends on $(\mathbb{C}^2 \backslash 0)/\mu_k$ where $\mu_k$ is the group of $k$-roots of unity.

Let us know write down the formula which allows to compute Betti polynomials. On the $\mathbb{Q}$-span of the lattice $N$, there is a nice negative piecewise linear (on each cone of the fan of $X$) function $\kappa$:

$$\kappa : N_{\mathbb{Q}} \mapsto \mathbb{Q}. \tag{1.9}$$

The formula for the Betti polynomial reads

$$B(X,t) = B(G/H,t) \sum_{n \in N} t^{-\kappa(n)}. \tag{1.10}$$

In the case $k = 1$, we have

$$\begin{aligned} B(\mathbb{P}^2,t) &= B(\mathbb{C}^2 \backslash 0,t)(1 + (t^{-1} + t^{-2} + ...) + (t^{-2} + t^{-4} + ...)) \\ &= (t^2 - 1)(1 + \frac{1}{t-1} + \frac{1}{t^2-1}) = 1 + t + t^2. \end{aligned}$$

and

$$\begin{aligned} B(\mathbb{F}_1,t) &= B(\mathbb{C}^2 \backslash 0,t)(1 + (t^{-1} + t^{-2} + ...) + (t^{-1} + t^{-2} + ...)) \\ &= (t^2 - 1)(1 + \frac{1}{t-1} + \frac{1}{t-1}) = 1 + 2t + t^2. \end{aligned}$$

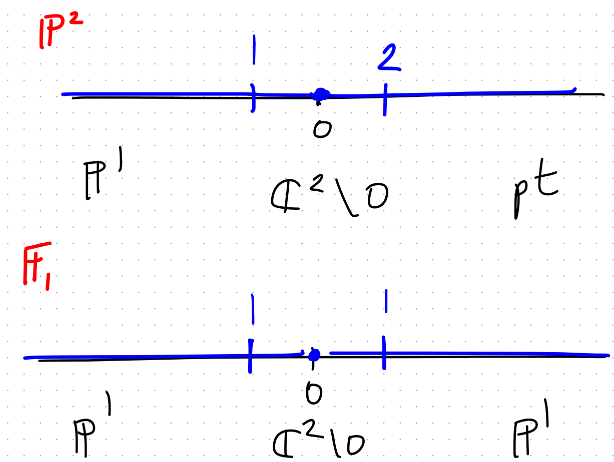


Table 3: Colored fans of $\mathbb{P}^2$ and $\mathbb{F}_1$

This formula works for all horospherical embeddings and involve infinite series in $t^{-1}$. For instance, we can revisit the example of a toric spherical variety $T \hookrightarrow X$ with equation (1.10) in mind. Recall that $T$-orbits in $X$ are parameterized by cones $\sigma$ and that

$$B(X,t) = \sum_{\sigma} (t-1)^{\mathrm{codim}\sigma} = (t-1)^d \sum_{\sigma} \frac{1}{(t-1)^{\mathrm{dim}\sigma}} \in B(T,t)\mathbb{Q}((t^{-1})). \tag{1.11}$$

Now, we move on more complicated spherical varieties where we have to consider *spherical roots*. Spherical roots are in some sense the analogue of simple roots of reductive algebraic groups in the theory of spherical varieties. One can associate to spherical roots some finite group, called the *small Weyl group*, generated by reflections. The fundamental domain for this action is called the *valuation cone*. The Luna-Vust theory puts restrictions on how spherical and simple roots should interact each other. These sophisticated constraints did not appear in the horospherical case precisely because a horospherical variety does not have spherical roots. We now consider a fourth example of a spherical variety which admits spherical roots.

**Example 4.** $G = SL(2)$, $H = T$. In this case, $G/H \cong \mathbb{P}^1 \times \mathbb{P}^1 \backslash (\Delta \cong \mathbb{P}^1)$. The torus $T$ is the stabilizer of $(0, \infty)$. $X_0 = G/H$ and $X = \mathbb{P}^1 \times \mathbb{P}^1$. The coordinate ring of the homogeneous space $G/H$ is filtered

$$\mathbb{C}[G/H] = \bigcup_{i \geqslant 0} R^i, \tag{1.12}$$

where

$$R^i = \left\{ \frac{f(x_0, x_1, y_0, y_1)}{\begin{vmatrix} x_0 & y_0 \\ x_1 & y_1 \end{vmatrix}^i} \mid f \text{ is bihomogeneous of degree } i \right\}. \tag{1.13}$$

It is clear that $\dim R^i = (i+1)^2$. The associated graded ring has blocks of odd dimensions:

$$grR = \bigoplus_{i \geqslant 0} R^i / R^{i-1}. \tag{1.14}$$

As a $G$-module, it splits into the direct sum of all even simple modules of $G$, $V(2i)$.

The graded ring $\mathbb{C}[SL(2)/T]$ is isomorphic to the coordinate ring of $G/U_2$, as a $G$-module, but not as a ring.

$$\mathbb{C}[SL(2)/U_2] \cong grR. \tag{1.15}$$

In some sense, the horospherical variety $G/U_2$ is a degeneration of $G/T$. We call $U_2$ a *horospherical satellite* of $H$. We have the following formula for the Betti polynomial of $X$:

$$\begin{aligned} B(X, t) &= B(G/T, t) + B(G/U_2, t) \sum_{n > 0} t^{-n} \\ &= (t^2 + t) + (t^2 - 1)(t^{-1} + t^{-2} + \cdots) \\ &= (t^2 + t) + (t^2 - 1)\frac{1}{t - 1} = t^2 + 2t + 1. \end{aligned}$$

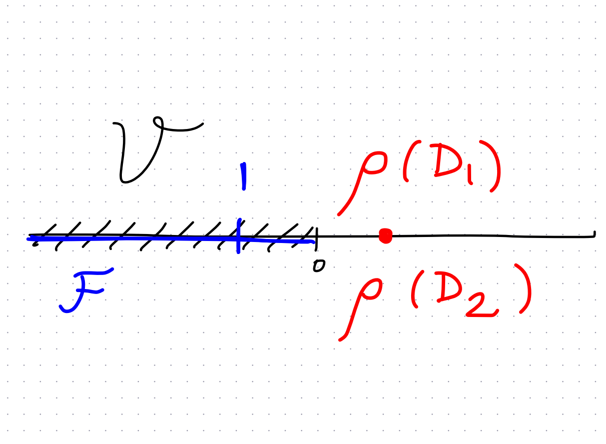


Table 4: Colored fans of $X = \mathbb{P}^1 \times \mathbb{P}^1$

# 2 Lecture 2

Let $X$ be a (spherical) projective algebraic variety over $\mathbb{C}$. Set

$$E(X; u, v) = \sum_{0 \leqslant p,q \leqslant \dim X} (-1)^{p+q} h^{p,q}(X) u^p v^q,$$

where $h^{p,q}(X) = \dim h^p(\Omega^q)$, $0 \leqslant p, q \leqslant \dim X = d$. Since $X$ is spherical, we have $E(X; u, v) = B(X, t) = \sum_i b_{2i}(X) t^i$, with $t = uv$, since $h^{p,q}(X) \neq 0$ implies that $p = q$.

One can embed $G/H$ into some projective space $\mathbb{P}(V)$ for some finite dimensional vector space $V$, and consider the closure $\overline{G/H}$. But in general this closure is singular. For such embeddings, the string theory is necessary to consider analog of the formula for Betti polynomials.

## 2.1 Stringy $E$-functions

One needs additional assumptions.

Assume that $X = \overline{G/H}$ is normal and $\mathbb{Q}$-Gorenstein, that is, $\ell K_X$ is Cartier for some positive integer $\ell \in \mathbb{N}$.

To define the *stringy E-function* $E_{st}$, one needs a resolution of singularities.

Let $\rho\colon Y \to X$ be a resolution of singularities with $Y$ smooth projective and such that the irreducible components $D_1 \cup \ldots D_r$ of $\mathrm{exc}(\rho)$ are smooth projective simple normal crossing divisors. This choice is always possible.

Then

$$K_Y = \rho^*(K_X) + \sum_{i=1}^r a_i D_i, \qquad a_i > -1, \quad \ell a_i \in \mathbb{N} \text{ for some } \ell.$$

Set $I = \{1, \ldots, r\}$. For $J \subset I$, set $D_J = Y$ if $J = \varnothing$, and $D_J = \bigcap_{j \in J} D_j$ otherwise.

**Definition 1.**

$$E_{st}(X;u,v) = \sum_{J\subset I} E(D_J;u,v) \prod_{j\in J} \left( \frac{uv-1}{(uv)^{a_i+1}-1} - 1 \right).$$

**Theorem 1** (B., 1998)**.** *$E_{st}(X;u,v)$ does not depend on the resolution $\rho$.*

**Corollary 2.** *If $X$ is smooth then $E_{st}(X;u,v) = E(X;u,v)$.*

The independence of the resolution comes from integration over the space of arcs of $Y$: $\mathbb{C}((s)) \supset \mathbb{C}[[s]] \supset \mathbb{C}$. The motivic integral looks $\int\limits_{Y(\mathbb{C}[[s]])} \mu \in R$ where $R$ is some ring and $\mu$ some measure. Then $\int\limits_{Y(\mathbb{C}[[s]])} \mu = \int\limits_{Y_{reg}(\mathbb{C}[[s]])}$ by the properties of the motivic integrals... The independence property can be viewed as the Jacobi transform for motivic integrals.

What foregoes actually holds for non spherical varieties.

## 2.2 Luna-Vust theory (1983).

Recall that $X$ is spherical. Then $X$ is an embedding of some spherical homogeneous space $X_0 = G/H$.

Considering the fields $\mathbb{C}(X)^{(B)} \subset \mathbb{C}(X)$, we define the set of weights of elements of $\mathbb{C}(X)^{(B)}$. Let $M \cong \mathbb{Z}^r$ be the weight lattice, that is, the set of such weights. The open $B$-orbit $B.x \subset X_0$ is isomorphic to $(\mathbb{C}^*)^r \times \mathbb{C}^{d-r}$, with $r \leqslant d = \dim X$. The number $r$ is the *rank* of $G/H$. Set $N = \mathrm{Hom}(M,\mathbb{Z})$, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$.

The *valuation cone* $\mathcal{V}$ is contained in $N_{\mathbb{Q}}$. It is cosimplicial and we have

$$\mathcal{V} = \{x \in N_{\mathbb{Q}} | \langle s_i \mid x\rangle \leqslant 0 \; \forall i\},$$

where the $s_1, \ldots, s_k$, $k \leqslant r$, are the *spherical roots.*

Let $\mathcal{O} = \mathbb{C}[[t]]$ and $\mathcal{K} = \mathbb{C}((t))$. The set $G(\mathcal{O})$ acts on $(G/H)(\mathcal{K})$ and

$$G(\mathcal{O}) \setminus (G/H)(\mathcal{K}) \cong \mathcal{V} \cap N.$$

**Example 5.** Let $X$ be the set of $n\times n$-size matrices, and $G = GL(n) \times GL(n)$. Then $G$ acts on $X$ by $(g_1,g_2).A = g_1 A g_2^{-1}$. Take $B = \{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \times \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$. We have $X_0 \cong GL(n) \subset M(n\times n, \mathbb{C}) = X$ and

$$\begin{aligned} G(\mathcal{O}) \setminus (G/H)(\mathcal{K}) &\cong GL(n)(\mathcal{O}) \setminus GL(n)(\mathcal{K}) / GL(n)(\mathcal{O}) \\ &\cong \left\{ \begin{pmatrix} t^{a_1} & & & * \\ & t^{a_2} & & \\ & & \ddots & \\ * & & & t^{a_n} \end{pmatrix}, \quad a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \right\}. \end{aligned}$$

## 2.3 Satellites of spherical subgroups

Let $v \in \mathcal{V} \cap N$ be a primitive point. Then consider the embedding $X_0 = G/H \hookrightarrow X_v = X_0 \cup D_v$, with $D_v$ a $G$-invariant divisor, corresponding to the uncolored ray $\mathbb{Q}_+ v$. It is an *elementary embedding*, that is, it consists in only two $G$-orbits, with one of codimension one.

**Example 6.** The spherical embedding $(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta) \cup \Delta$ is elementary.

The group $G$ acts on the normal bundle $N_{D_v}$ of $D_v$ in $X_v$. Then $N_{D_v}$ is spherical and it is an elementary embedding of some spherical homogeneous space $G/H_v$,

$$N_{D_v} = G/H_v \cup D_v,$$

with $\dim H_v = \dim H$.

**Theorem 3.** *The subgroup $H_v$ only depends on the minimal face of $\mathcal{V}$ containing $v$.*

Hence, there are only, up to $G$-conjugacy, $2^k$ such subgroups $H_v$ which we can the *satellites* of $H$.

**Example 7.** Return to the example $X = M(n \times n, \mathbb{C})$ and $G = GL(n) \times GL(n)$. The spherical roots $s_1, \ldots, s_{n-1}$ are in bijection with the simple roots $\alpha_1, \ldots, \alpha_{n-1}$. In the case, the *small Weyl group* is $S_n \times S_n$. It acts on $N_{\mathbb{Q}}$, and $\mathcal{V}$ is a fundamental domain for it.

Let $I \subset \{s_1, \ldots, s_{n-1}\}$. Consider the two opposite parabolic subgroups $P_I^+$ and $P_I^-$ "containing" the root spaces associated to the $\alpha_i$'s, $i \in I$. and set $L_I = P_I^+ \cap P_I^-$. The faces of $\mathcal{V}$ are in bijection with subsets $I$, and

$$H_I = \{(g_1, g_2) \in P_I^+ \times P_I^- \mid L(g_1) = L(g_2)\},$$

with $L\colon P_I^\pm \to L_I$ the canonical projection. We have $\dim H_I = n^2 = \dim H$. Note that the space

$$H_\varnothing \cong \{ \begin{pmatrix} a & & 0 \\ & \ddots & \\ * & & a \end{pmatrix} \times \begin{pmatrix} a & & * \\ & \ddots & \\ 0 & & a \end{pmatrix} \mid a \in \mathbb{C}^* \}$$

is horospherical. On the opposite side, $H_{\{1,\ldots,n-1\}} \cong H$.

**Theorem 4.**

$$E_{st}(X; u, v) = \sum_{I \subset \{1,\ldots,k\}} E(G/H_I; u, v) \sum_{n \in \mathcal{V}_I^0} (uv)^{-\kappa(n)}.$$

*Here $X$ is a $\mathbb{Q}$-Gorenstein projective spherical variety, with colored fan $\mathcal{F}$ in $N_{\mathbb{Q}} \supset \mathcal{V}$, and $\kappa\colon N_{\mathbb{Q}} \to \mathbb{Q}$ is a piecewise linear function, defined by some conditions depending on the canonical divisor $K_X = \sum_{D_i\ G\text{-inv}} (-D_i) + \sum_{\substack{D_i\ B\text{-inv},\\ \text{not } G\text{-inv}}} c_i D_i$. Note that $c_i \leqslant -1$ for all $i$.*

Remark. It is not so satisfying to summate over the different $\mathcal{V}_I$. Somehow, we would wish a more global formula.

Write the formula for $E_{st}(X;u,v)$ as

$$E_{st}(X;u,v) = E(G/H;u,v) \sum_{I\subset\{1,\dots,k\}} \frac{E(G/H_I;u,v)}{E(G/H;u,v)} \sum_{n\in\mathcal{V}_I^0} (uv)^{-\kappa(n)}$$

in order to factorise by the $E$-function of the open $G$-orbit. Then it is natural to study the ratio $\dfrac{E(G/H_I;u,v)}{E(G/H;u,v)}$. We conjecture that it is a polynomial in $t^{-1}$ with integer coefficients. We proved it only in a few cases.

For example, assume that $r = 1$, then $\mathcal{V}$ is a half-line and we observe that

$$\frac{E(G/H_I;u,v)}{E(G/H;u,v)} = 1 \pm t^{-a}, \qquad a \in \mathbb{N}.$$

Question. Can we interpret the sign and the integer $a$, for instance in term of the small Weyl group?