Powers of translation quivers

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Contents

I propose we leave math to the machines and go play outside.
Calvin

Introduction and motivation

Let m, $n \in \mathbb{N}$ and let Π be a regular $nm+2$-gon. The goal of this thesis is to understand the structure of the quiver $((\Gamma^1_{A_{nm-1}})^m, (\tau_1)^m)$ where $\Gamma^1_{A_{nm-1}}$ is obtained from a geometric construction on the set of diagonals of Π, $(\Gamma^1_{A_{nm-1}})^m$ is its mth power and $(\tau_1)^m$ is an automorphism of the vertices of $(\Gamma^1_{A_{nm-1}})^m$. [BMA] were the first to study the structure of this quiver. They have shown that the pair $((\Gamma^1_{A_{nm-1}})^m, (\tau_1)^m)$ is a stable translation quiver and that it contains as connected component the translation quiver $\Gamma^m_{A_{n-1}}$ whose vertices are the diagonals in Π which divide Π into two polygons whose numbers of sides are congruent to 2 modulo m, called m-diagonals, and whose arrows are obtained by a simple geometric rule. The main result in [BMA] is that the mesh category of $(\Gamma^m_{A_{n-1}}, \tau_m)$ ($\tau_m := (\tau_1)^m|_{\Gamma^m_{A_{n-1}}}$) is equivalent to the m-cluster category of type A_{n-1},

$$\mathcal{C}^m_{A_{n-1}} := D^b(kA_{n-1})/\tau^{-1} \circ [m],$$

where k is an algebraically closed field, τ is the Auslander-Reiten translation of $D^b(kA_{n-1})$ and [1] is the shift. Following [BMA], we want to express each connected component of $((\Gamma^1_{A_{nm-1}})^m, (\tau_1)^m)$ in function of the Auslander-Reiten quiver of a quotient of $D^b(kA_{n'})$ for some n'. In particular, we will show that $n' = n-1$ or n.

The structure of this thesis will be as follows: In Chapter 1, Section 1.1 we introduce some basic concepts of finite-dimensional representation theory. In particular, we define the path algebra kQ for Q a finite quiver and the associated category $\operatorname{mod} kQ$ of left kQ-modules. Then in Section 1.2 we present one of the most important theorems of the finite-dimensional representation theory, Gabriel's Theorem, which states that there are finitely many irreducible objects in $\operatorname{mod} kQ$ if and only if the underlying graph of Q is a finite union of simply-laced Dynkin graphs. In Chapter 2, we define the m-cluster category of type Q, $\mathcal{C}^m_Q := D^b(kQ)/\tau^{-1} \circ [m]$ where $D^b(kQ)$ denotes the bounded derived category of kQ-modules, for Q a quiver with underlying graph a finite union of simply-laced Dynkin graphs. The m-cluster category was introduced in Keller [KEL] and has also been studied by Baur-Marsh [BMA], Thomas [THO], Zhu [ZHU] and others. In Chapter 3, Section 3.1, we introduce the mesh category associated to the quiver $\Gamma^m_{A_n-1}$ whose vertices are m-diagonals of an $nm+2$-gon. This construction has been studied by [BMA] and [THO] and has its origin in the construction of [CCS] for the case $m = 1$. In Section 3.2, we define the mth power, (Γ^m, τ^m), of a translation quiver (Γ, τ) whose vertices are the same as in (Γ, τ) but whose arrows are given by sectional paths in Γ of length m, and state some useful results linked to it due to [BMA]. In Section 3.3 we prove that the stable translation quiver $((\Gamma^1_{A_{nm-1}})^m, (\tau_1)^m)$ has the translation quiver $(\Gamma^m_{A_n-1}, \tau_m)$ as connected component following [BMA]. Finally in Chapter 4, Section 4.1 we introduce the result on the component $\Gamma^m_{A_{n-1}}$ of $(\Gamma^1_{A_{nm-1}})^m$ following [BMA] which proves that the m-cluster category $\mathcal{C}^m_{A_{n-1}}$ is equivalent to the mesh category of $\Gamma^m_{A_{n-1}}$. Then in Section 4.2 we show that the other components of $(\Gamma^1_{A_{nm-1}})^m$ can always be described in relation to some quotient of $D^b(kA_{n'})$ ($n' = n$ or $n-1$). In particular, we prove that for m odd, the connected components of $((\Gamma^1_{A_{nm-1}})^m, (\tau_1)^m)$ are isomorphic to the disjoint union of the Auslander-Reiten quivers of

$$D^b(kA_{n-1})/\tau^{-1} \circ [m]$$

and $\frac{m-1}{2}$ copies of

$$D^b(kA_n)/\tau^{-\alpha} \circ [m+1]$$

where $\alpha = \frac{m-1}{2}n - \frac{m-3}{2}$.

Due to time constraints, we will not rewrite all the proofs needed in this thesis and thus we will from time to time refer the reader to the original literature. However, we have included the proofs of the results which are most important for our purpose. We strongly recommend the books [HD] and [ASS] and the articles [KEL] and [BMA] for additional information on the different topics that we treat here.

Acknowlegements

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Chapter 1

Quiver representations

1.1 General principles

Convention 1.1.1. *We assume some basic knowledge on module theory and we refer the reader to [ASS] for terminology, such as hereditary, section, retraction, radical, socle or top. Furthermore, throughout this thesis k will always denote an algebraically closed field.*

Definition 1.1.2. *A* finite quiver $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ *is given by:*

- *a finite set of vertices, Q_0, labeled by $\{1, 2, \cdots, n\}$;*
- *a finite set of arrows, Q_1,*

such that for all arrow a in Q_1, a starts at the vertex $s(a)$ (or sa to simplify) and terminates at the vertex $t(a)$ (or ta).

Remark. The definition of *infinite quiver* is the natural extension of the above definition. In this thesis, a quiver will always be a finite quiver except when mentioned explicitly.

From a quiver Q we can define a k-algebra kQ, called *path algebra*. Within the representation theory of finite-dimensional algebras, the study of these path algebras has given one of the first main theorems in this field, Gabriel's Theorem (see Theorem 1.2.4). In this thesis, we need these path algebras in order to define the m-cluster category (see Section 2.2). But first we introduce this algebra and state some basic results related to it.

Definition 1.1.3. *Let Q be a quiver. A path x in Q is*

1. *either a trivial path e_i which starts and terminates at vertex $i \in Q_0$;*
2. *or a non-trivial path a which is a sequence $a_1 a_2 \cdots a_m$ $(m \geq 1)$ of arrows in Q_1 which satisfies*

$$ta_{j+1} = sa_j \text{ for } 1 \leq j \leq m-1.$$

For a non-trivial path, $a_1a_2\cdots a_m$, we say that the path starts at sa_m and ends at ta_1. We use the notation sx and tx to denote the starting and terminating vertex of a path x.

Definition 1.1.4. *Let Q be a quiver.*

1. *Q is* acyclic *if for all i in Q_0 there exists no non-trivial path starting and terminating at i.*
2. *The* underlying graph *of Q is the quiver obtained from Q by forgetting the orientation of all the arrows in Q_1. The non-oriented arrows are called* edges*.*
3. *Q is* connected *if every two vertices in Q_0 can be linked by a (non-oriented) path in its underlying graph.*

Definition 1.1.5. *Let Q be a quiver. The* path algebra *kQ is the k-algebra with basis the paths in Q, and with product defined as:*

$$xy = \begin{cases} \text{obvious composite path (if } sx = ty) \\ 0 \text{ (else).} \end{cases}$$

for any paths x, y in Q. The multiplication is associative.

Definition 1.1.6. *Let Q be a quiver. The* path category *$k\langle Q\rangle$ is defined on the same model as the path algebra kQ, that is:*

1. *the objects are k-linear combinations of the vertices of Q;*
2. *the morphisms are k-linear combinations of the paths in Q.*

We give some of the interesting properties of this k-algebra.

Lemma 1.1.7. *Let Q be a quiver and let $A := kQ$ be its associated path algebra.*

1. *The e_i are orthogonal idempotents: $e_ie_j = 0$ if $i \neq j$ and $e_i^2 = e_i$.*
2. *A has an identity given by $1 = \sum_{i=1}^n e_i$.*
3. *The space Ae_i, respectively e_jA, has as basis the paths starting at i, respectively terminating at j. The space e_jAe_i has as basis the paths starting at i and terminating at j.*
4. *$A = \bigoplus_{i=1}^n Ae_i$, so that each Ae_i is a projective left A-module.*
5. *If X is a left A-module, then $Hom_A(Ae_i, X) \cong e_iX$.*
6. *If $0 \neq f \in Ae_i$ and $0 \neq g \in e_iA$, then $fg \neq 0$.*

7. *The e_i are primitive idempotents, i.e Ae_i is an indecomposable module.*

8. *If $e_i \in Ae_jA$, then $i = j$.*

9. *The e_i are inequivalent, i.e. $Ae_i \ncong Ae_j$ for $i \neq j$.*

Proof. See [CB], page 4. □

One of the main interest in studying the path algebra kQ is to understand its indecomposable modules. This can be done by understanding the structure of its associated quiver Q. In particular, we are interested in the category $Rep_k(Q)$ of representations of Q which is defined as follow:

Definition 1.1.8.

- *A* representation V *of* Q *is a collection* $\{V_i \,|\, i \in Q_0\}$ *of* k*-vector spaces together with a collection* $\{V_a : V_{sa} \to V_{ta} \,|\, a \in Q_1\}$ *of* k*-linear maps.*

- *A* morphism of representations $\theta : V \to V'$ *is a collection of* k*-linear maps* $\theta_i : V_i \to V_i'$ *for all* $i \in Q_0$*, such that the diagram*

$$\begin{array}{ccc} V_{sa} & \xrightarrow{V_a} & V_{ta} \\ \downarrow{\scriptstyle \theta_{sa}} & & \downarrow{\scriptstyle \theta_{ta}} \\ V_{sa'} & \xrightarrow[V_{a'}]{} & V_{ta'} \end{array}$$

commutes for each arrow a *in* Q_1*. Moreover* θ *is an* isomorphism *if* θ_i *is invertible for every* i *in* Q_0*.*

Then for a quiver Q*, the* category of representations of Q, $Rep_k(Q)$ *is defined as the category whose objects are representations of* Q *and whose morphisms are defined as above.*

Example 1. Let $S(i)$ be the representation such that $\forall\, j \in Q_0$

$$S(i)_j = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and $\forall\, a \in Q_1$

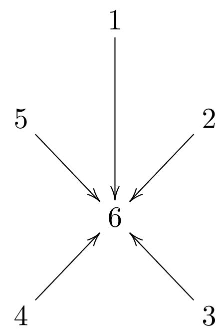
$$S(i)_a = 0.$$

For example, we consider the quiver $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$ and the associated representation $S(2)$.

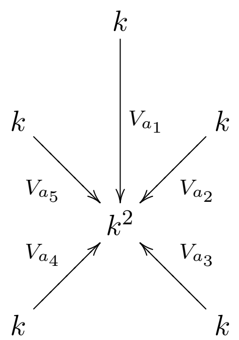
$$0 \xrightarrow{0} k \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0\,.$$

Note that the representation $S(i)$ can be defined on any quiver Q, for any vertex $i \in Q_0$.

Example 2. Let's consider the star quiver Q_{star}.



A representation of Q_{star} is a collection of six vector spaces $V_1, V_2, \cdots, V_6$ and of five linear maps $V_{a_i} : V_i \to V_6$. A special representation of the star quiver is the representation $V_{\nu,\lambda}$, for $\nu, \lambda \in k$:



with V_{a_1}, V_{a_2}, V_{a_3}, V_{a_4}, V_{a_5} given by

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\\nu\end{pmatrix}, \begin{pmatrix}1\\\lambda\end{pmatrix}.$$

Note that the family $\{V_{\nu,\lambda} | \nu, \lambda \in k\}$ is a 2-dimensional family of k-vector spaces.

A well-known result in the theory of representations of quivers states that the category $Rep_k(Q)$ is equivalent to the category of left kQ-modules, $\operatorname{Mod} kQ$. More precisely:

Lemma 1.1.9. *Let Q be a connected and acyclic quiver. There exists an equivalence of categories $\operatorname{Mod} kQ \cong Rep_k(Q)$ that restricts to an equivalence $\operatorname{mod} kQ \cong rep_k(Q)$, where $\operatorname{mod} kQ$ denotes the category of finitely generated left kQ-modules and $rep_k(Q)$ the category of finite-dimensional k-linear representations of Q.*

Proof. We only give the construction. If $\mathcal{X}$ is a left kQ-module, define a representation X with

$$X_i = e_i \mathcal{X} \quad \forall\, i \in Q_0,$$
$$X_a(x) = ax = e_{ta}ax \in X_{ta} \quad \forall\, a \in Q_1 \text{ and } \forall\, x \in X_{sa}.$$

If X is a representation of Q, define a kQ-module $\mathcal{X}$ via

$$\mathcal{X} = \oplus_{i=1}^{n} X_i.$$

Let $X_i \xrightarrow{\epsilon_i} \mathcal{X} \xrightarrow{\pi_i} X_i$ be the canonical maps $\forall\, i \in Q_0$. Then for all $x \in \mathcal{X}$, we define the left action of kQ via:

$$\begin{array}{rcl} a_1 a_2 \cdots a_m x & = & \epsilon_{ta_1} X_{a_1} X_{a_2} \cdots X_{a_m} \pi_{sa_m}(x) \\ e_i x & = & \epsilon_i \pi_i(x). \end{array}$$

It is straightforward, but tedious, to check that these are inverses and that morphisms behave, etc. □

Example 3. Let $A = kQ$ be a path algebra. Then $S(i)$ is the representation corresponding to the simple module associated to vertex i.

Example 4. Let $A = kQ$ be a path algebra. We denote the representation associated to the left projective module Ae_i by $P(i)$. For example for the quiver

$$1 \longrightarrow 2 \longleftarrow 3\ ,$$

we have:

$$\begin{aligned} P(1) &= (\ k \xrightarrow{1} k \xleftarrow{0} 0\ ) \\ P(2) &= (\ 0 \xrightarrow{0} k \xleftarrow{0} 0\ ) \\ P(3) &= (\ 0 \xrightarrow{0} k \xleftarrow{1} k\ ). \end{aligned}$$

Example 5. Let $A = kQ$ be a path algebra and $e_i A$ the right injective module associated to vertex i. Then $Hom_k(e_i A, k)$ is a left injective module. Its associated representation is denoted by $I(i)$. For example for the quiver

$$1 \longrightarrow 2 \longleftarrow 3\ ,$$

we have:

$$\begin{aligned} I(1) &= (\ k \xrightarrow{0} 0 \xleftarrow{0} 0\ ) \\ I(2) &= (\ k \xrightarrow{1} k \xleftarrow{1} k\ ) \\ I(3) &= (\ 0 \xrightarrow{0} 0 \xleftarrow{0} k\ ). \end{aligned}$$

Remark. We have now a powerful tool for understanding the modules of any algebra isomorphic to a path algebra. In fact we can easily prove that for any hereditary connected finite-dimensional algebra A, there exists a quiver Q_A such that $A \cong kQ_A$ (see Theorem 3.7, Chapter 2 in [ASS]).

Definition 1.1.10. *If V and W are two representations of the same quiver Q, we define their* direct sum $V \oplus W$ *by*

$$(V \oplus W)_i := V_i \oplus W_i$$

for all $i \in Q_0$, and

$$(V \oplus W)_a := \begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix} : V_{sa} \oplus W_{sa} \to V_{ta} \oplus W_{ta}$$

for all $a \in Q_1$.

We say that V is a trivial representation *if $V_i = 0$ for all $i \in Q_0$. If V is isomorphic to a direct sum $W \oplus Z$, where W and Z are non-trivial representations, then V is called* decomposable*. Otherwise V is called* indecomposable.

It is easily proved that every representation has a unique decomposition into indecomposable representations (up to isomorphism and permutation of components). Hence the classification problem in the category $Rep_k(Q)$ reduces to classifying the indecomposable representations of Q. By Lemma 1.1.9, this is equivalent to classifying the indecomposable left kQ-modules. In particular, much attention has been devoted to the question of when there are finitely many indecomposable representations or, equivalently, finitely many indecomposable kQ-modules.

1.2 Path algebras of finite representation type

The aim of this section is to introduce Gabriel's Theorem for the classification of path algebras of *finite representation type* (or *finite type* for short), that is path algebras which have only a finite number of indecomposable modules up to isomorphism. But first we give a simple construction of the *Weyl group* W_Σ associated to Σ where Σ is a finite connected graph (with no orientation of the edges) without loops (i.e no arrows starting and terminating at the same vertex) and with vertices given by $\Sigma_0 = \{1, \cdots, n\}$. We define W_Σ to be the group of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ generated by *reflections* $\sigma_1, \cdots, \sigma_n$ defined as follows. For $\underline{x} = (x_1, \cdots, x_n)$, we have

$$\sigma_i(x_1, \cdots, x_i, \cdots, x_n) := (x_1, \cdots, -x_i + \sum_{i-j} x_j, \cdots, x_n)$$

where the sum runs over all edges having i as endpoint.

Remark. Typically, the Weyl group appears in the representation theory of Lie Algebras. In this framework a more precise definition of the group is required. However we do not need these specifications. For more details on this subject, we refer the reader to [HU].

Example 6. If Σ is the graph:

$$1 \text{ ——— } 2 \text{ ——— } 3$$

we have

$$\begin{aligned} \sigma_1(x_1, x_2, x_3) &= (-x_1 + x_2, x_2, x_3) \\ \sigma_2(x_1, x_2, x_3) &= (x_1, -x_2 + x_1 + x_3, x_3) \\ \sigma_3(x_1, x_2, x_3) &= (x_1, x_2, -x_3 + x_2). \end{aligned}$$

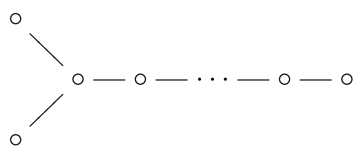
Associated with the Weyl group are the well-known quivers of simply-laced Dynkin type.

Definition 1.2.1. *Let Q be a quiver. Then Q is a* quiver of simply-laced Dynkin type *if its underlying graph is an union of* Dynkin graphs of type A, D or E*:*

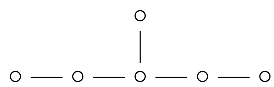
Dynkin graph

A_n

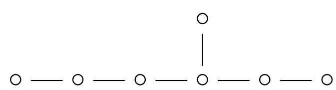
D_n



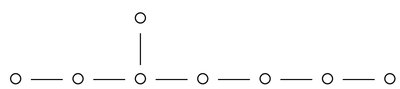
E_6



E_7



E_8



Here is the connection between the Weyl group and the Dynkin graphs.

Theorem 1.2.2. *Let Σ be a finite connected graph without loops and W_Σ its associated Weyl group. Then Σ is a union of Dynkin graphs if and only if W_Σ is a finite group.*

Proof. C.f. [REI] □

Definition 1.2.3. *Let* Σ *be a Dynkin graph with* n *vertices,* $w \in W_\Sigma$ *and* $\{\underline{f_i}\,|i = 1, \cdots, n\}$ *a set of elements of* ImW_Σ *which forms a basis of* $\mathbb{R}^n$*. Then the elements of* $\mathbb{R}^n$ *of the form* $w(\underline{f_i})$ *are called* roots*. A root* $\underline{x}$ *is said to be* positive *if* $x_i \geq 0$ *for all* i*.*

Remark.

1. Notice that for $\underline{x}$ a positive root there exists at least one $i \in \{1, \cdots, n\}$ such that $x_i > 0$.

2. The set of positive roots forms a basis of $\mathbb{R}^n$.

3. It is known that for each root $\underline{x}$ either $\underline{x}$ or $-\underline{x}$ is positive.

Example 7. In Example 6, the set of positive roots is

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 1, 1)\}.$$

Theorem 1.2.4 (Gabriel's theorem)**.**
Let Q *be an acyclic connected quiver.*

1. *A path algebra* kQ *has a finite representation type if and only if the underlying graph of* Q *is a Dynkin graph of type* A_n*,* D_n*,* E_6*,* E_7 *or* E_8*.*

2. *Assume that the underlying graph of* Q *is a Dynkin graph as above with vertices* $\{1, \cdots, n\}$*, and let* V *be a representation of* Q *over* k*. Then the assignment*

$$V \mapsto (dim_k V_1, \cdots, dim_k V_n) \in \mathbb{R}^n$$

provides a one-to-one correspondence between isomorphism classes of indecomposable representations of Q *(or indecomposable* kQ*-modules) and positive roots of the Dynkin graph.*

Proof. C.f. [REI] □

Example 8. Let consider the quiver $1 \longrightarrow 2 \longrightarrow 3$ with underlying graph and set of positive roots given in Example 6 and 7. Then the positive roots $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ are associated to the indecomposable representations:

$$k \xrightarrow{0} 0 \longrightarrow 0$$

$$0 \xrightarrow{0} k \xrightarrow{0} 0$$

$$0 \longrightarrow 0 \xrightarrow{0} k$$

$$k \xrightarrow{1} k \xrightarrow{0} 0$$

$$0 \xrightarrow{0} k \xrightarrow{1} k$$

$$k \xrightarrow{1} k \xrightarrow{1} k.$$

Chapter 2

The m-cluster category

In this chapter, we study first the so-called Auslander-Reiten quiver and then we give a formal description of the m-cluster category.

2.1 The Auslander-Reiten quiver

Definition 2.1.1.

1. *A* translation quiver *is a pair* (Q, τ) *where*

   (a) $Q = (Q_0, Q_1)$ *is a locally finite quiver;*

   (b) $\tau : Q_0' \to Q_0$ *is an injective map defined on a subset* $Q_0' \subseteq Q_0$ *such that for any* $x \in Q_0$, $y \in Q_0'$, *the number of arrows from* x *to* y *is equal to the number of arrows from* $\tau(y)$ *to* x*:*

   $$\sharp\{\text{arrows } x \to y\} = \sharp\{\text{arrows } \tau y \to x\}.$$

   τ *is often called the* translation of Q.

   The vertices in $Q_0 \backslash Q_0'$ *are called* projective. *If* $Q_0' = Q_0$ *and* τ *is bijective, the couple* (Q, τ) *is called* stable translation quiver.

2. *A* morphism of translation quivers *is a morphism of quivers which is compatible with the translation* τ, *that is: let* (Q_a, τ_a) *and* (Q_b, τ_b) *be two translation quivers. Then* $f : Q_a \to Q_b$ *is a morphism of translation quiver if* f *is a morphism of quivers and*

   $$\tau_b(f(i)) = f(\tau_a(i)) \text{ for all } i \in (Q_a)_0.$$

   The translation quivers together with these morphisms form the category of translation quivers.

3. *A translation quiver is said to be* connected *if* $Q_0 \neq \emptyset$ *and if starting at some vertex in* Q *one can reach any other vertices by a sequence of arrows and/or* τ. *Notice that this does not imply that the underlying graph is connected.*

An example of a translation quiver is the Auslander-Reiten quiver of a finite-dimensional algebra.

Definition 2.1.2. *The* Auslander-Reiten quiver Q^{AR} *of a finite-dimensional algebra A over a field k is defined as follows:*

1. *the vertices are the isomorphism classes of indecomposable finite-dimensional left A-modules;*

2. *the number of arrows between vertices x and y is given by the dimension of the space of irreducible morphisms (see Definition A.2.4) from a representative of the isomorphism class of x to one of y.*

Lemma 2.1.3. *Let τ be the translation defined in Definition A.1.5. Then the Auslander-Reiten quiver Q^{AR} of an algebra A is a translation quiver, the translation τ^{AR} being defined for all vertices $[M]$, by $\tau^{AR}[M] = [\tau M]$.*

Proof. See [ASS], Chapter 4, Lemma 4.8. □

Remark.

1. The construction of the Auslander-Reiten translation with some of the useful proprieties associated to it is given in Annexe A.1.

2. It is clear that Q^{AR} is finite (or, equivalently, has finitely many vertices) if and only if A is of finite representation type.

3. Using *almost split sequences* (see Definition A.2.9) we can get a better understanding of the structure of the Auslander-Reiten quiver, in particular of the irreducible morphisms (see Definition A.2.4) between two classes of indecomposable modules and of the Auslander-Reiten translation. Below we state some of the major results taken from [ASS]. The reader is referred to the Annexe A.2 and to [ASS] for more details on this subject.

Theorem 2.1.4. 1. *For any indecomposable non-projective A-module M, there exists an almost split sequence*

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$$

in $\operatorname{mod} A$.

2. *For any indecomposable non-injective A-module N, there exists an almost split sequence*

$$0 \longrightarrow N \longrightarrow F \longrightarrow \tau^{-1} N \longrightarrow 0$$

in $\operatorname{mod} A$.

Proof. See [ASS], Chapter 4, Theorem 3.1. □

Remark. From Theorem 2.1.4 and Proposition A.2.2, we can conclude that for any non-projective (respectively non-injective) A-module M (respectively N) and τM (respectively $\tau^{-1}N$), if there exists an almost split sequence

$$0 \longrightarrow \tau M \longrightarrow X \longrightarrow Y \longrightarrow 0$$

$$(\, 0 \longrightarrow Z \longrightarrow W \longrightarrow \tau^{-1}N \longrightarrow 0 \,),$$

then it is isomorphic to

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$$

$$(\, 0 \longrightarrow N \longrightarrow F \longrightarrow \tau^{-1}N \longrightarrow 0 \,).$$

Theorem 2.1.4 implies that there exists a right (respectively left) minimal almost split morphism ending (respectively starting) at any indecomposable non-projective (respectively non-injective) module. On the other side, the following proposition shows the existence of such a homomorphism ending (respectively starting) at indecomposable projective (respectively injective) modules.

Proposition 2.1.5.

1. *Let P be an indecomposable projective module in* $\operatorname{mod} A$. *An A-module homomorphism $g : M \to P$ is right minimal almost split if and only if g is a monomorphism with image equal to* $\operatorname{rad} P$.

2. *Let I be an indecomposable injective module. An A-module homomorphism $f : I \to M$ is left minimal almost split if and only if f is an epimorphism with kernel equal to* $\operatorname{soc} I$.

Proof. We only prove (1), (2) can be proved similarly. It suffices, by the duality of Proposition A.2.2, to show that the inclusion homomorphism $g : \operatorname{rad} P \to P$ is right minimal almost split. Because g is a monomorphism, g is right minimal. Clearly, g is not a retraction. Let thus $v : V \to P$ be a homomorphism that is not a retraction. As P is projective, the module $\operatorname{rad} P$ is the unique maximal submodule of P. Because v is not an epimorphism $v(V) \subset \operatorname{rad} P$, that is v factors through g. □

Example 9. Let Q be the quiver: $1 \longleftarrow 2$. We denote its associated path algebra kQ by A. We consider the short exact sequence

$$0 \longrightarrow S(1) \xrightarrow{f} P(2) \xrightarrow{g} S(2) \longrightarrow 0$$

in $\operatorname{mod} A$, where

$$S(1) = (\, k \xleftarrow{0} 0 \,),$$

$$P(2) = (\ k \xleftarrow{1} k\ ),$$

$$S(2) = (\ 0 \xleftarrow{0} k\ ).$$

Then f is the embedding of $S(1)$ as the radical of $P(2)$ and g the canonical homomorphism of $P(2)$ onto its top. Because $P(2)$ is both injective and projective, it follows from Proposition 2.1.5 that f is right almost split and g is left almost split.

Proposition 2.1.6.

1. *Let M be an indecomposable non-projective module in* $\mathrm{mod}\, A$. *There exists an irreducible morphism $f : X \to M$ if and only if there exists an irreducible morphism $f' : \tau M \to X$.*

2. *Let N be an indecomposable non-injective module in* $\mathrm{mod}\, A$. *There exists an irreducible morphism $g : N \to Y$ if and only if there exists an irreducible morphism $g' : Y \to \tau^{-1}N$.*

Proof. See [ASS], Chapter 4, Proposition 3.8. □

Corollary 2.1.7.

1. *Let S be a simple projective non-injective module in* $\mathrm{mod}\, A$.
If $f : S \to M$ is irreducible, then M is projective.

2. *Let S be a simple injective non-projective module in* $\mathrm{mod}\, A$.
If $g : N \to S$ is irreducible, then N is injective.

Proof. See [ASS], Chapter 4, Corollary 3.9. □

2.2 The m-cluster category

Definition 2.2.1. *Let $\mathcal{A}$ be an abelian category. We define $D^b(\mathcal{A})$ as the* derived category of bounded complexes over $\mathcal{A}$. *We denote the* subcategory of indecomposable objects of $D^b(\mathcal{A})$ *by ind $D^b(\mathcal{A})$.*

Remark.

1. The category $D^b(\mathcal{A})$ is the triangulated category obtained from $K^b(\mathcal{A})$ by localizing with respect to the set of quasi-isomorphisms.

2. In our case $\mathcal{A}$ will always be the category $\mathrm{mod}\, kQ$. We will denote $D^b(\mathrm{mod}\, kQ)$ by $D^b(kQ)$ for short.

3. It is known that $D^d(kQ)$ is triangulated, Krull-Schmidt and satisfies other properties that we won't use. The interested reader is referred to Happel [HD].

Happel [HD] has proved the following useful lemma:

Lemma 2.2.2. *Let $\dot{X}$ be an indecomposable object in $D^b(kQ)$. Then $\dot{X}$ is isomorphic to a* stalk complex *with indecomposable* stalk*, that is $\dot{X}$ has the form:*

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

where M is an indecomposable module in $\operatorname{mod} kQ$.

Proof. See [HD], page 49. □

Definition 2.2.3. *Let $\mathcal{A}$ be* $\operatorname{mod} kQ$. *We can define the* Auslander-Reiten quiver Q^{AR} *on $D^b(kQ)$ (more precisely on ind $D^b(\operatorname{mod} kQ)$) as follows:*

1. *the vertices are the isomorphism classes of objects in ind $D^b(kQ)$;*
2. *the number of arrows between vertices x and y is given by the dimension of the space of irreducible morphisms (see Definition A.2.4) from a representative of the isomorphism class of x to one of y.*

Lemma 2.2.4. *Let τ_D be defined as in Definition A.1.7. Then the Auslander-Reiten quiver Q^{AR} is a stable translation quiver, the translation τ^{AR} being defined for all vertices $[\dot{M}]$, by $\tau^{AR}[\dot{M}] = [\tau_D(\dot{M})]$.*

Proof. See [HD] page 41. □

For the case where Q is a quiver of simply-laced Dynkin type, i.e its underlying graph G is an union of Dynkin graphs, Happel [HD] has shown that *ind* $D^b(kQ)$ is equivalent to another category, the mesh category of $\mathbb{Z}G$. In the following we give a formal definition of this new category.

Definition 2.2.5. *Let Q be a quiver of simply-laced Dynkin type. Then $\mathbb{Z}Q$ is the quiver associated to Q defined as follows:*

1. *the vertices are the couples (n, i) where $n \in \mathbb{Z}$ and $i \in Q_0$;*
2. *for every arrow $a \in Q_1$ such that $a : i \to j$, there are two arrows*

$$(n, i) \to (n, j) \text{ and } (n, j) \to (n + 1, i) \in (\mathbb{Z}Q)_1.$$

Lemma 2.2.6. *Let τ be the translation on $\mathbb{Z}Q$ such that for all $n \in \mathbb{Z}$ and all $i \in Q_0$:*

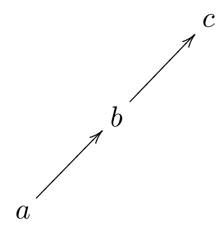
$$\tau((n, i)) = (n - 1, i).$$

Then $(\mathbb{Z}Q, \tau)$ is a stable translation quiver.

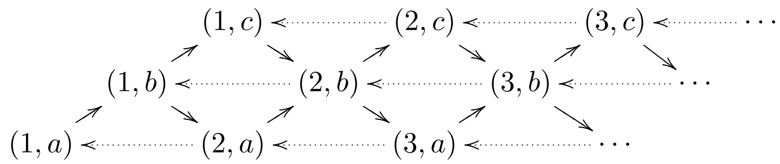
Proof. See [HD] p.53. □

Remark. It can be shown that for two quivers Q and Q' such that Q and Q' have the same underlying Dynkin graph of type A, D or E, the associated quivers $\mathbb{Z}Q$ and $\mathbb{Z}Q'$ are equivalents (see Keller [KEL]). Hence, in our case, $\mathbb{Z}Q$ is independent of the orientation of the arrows of Q and we can denote it by $\mathbb{Z}G$.

Example 10. Let $\vec{A}_3$ be the quiver:



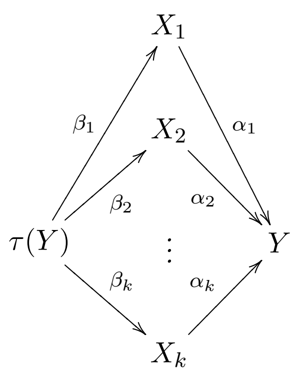
Then the associated translation quiver $(\mathbb{Z}\vec{A}_3, \tau)$ is given by:



where τ is indicated by the doted arrows.

We define now the mesh category associated to a stable translation quiver:

Definition 2.2.7. *Let (Q, τ) be a stable translation quiver with no multiple arrows. Let $Y \in Q_0$ and let $X_1, \cdots, X_k$ be all the vertices with arrows to Y, denoted by $\alpha_i : X_i \to Y$. Let $\beta_i : \tau(Y) \to X_i$ be the corresponding arrows from $\tau(Y)$ to X_i ($i = 1, \cdots, k$). Then the* mesh ending at Y *is defined to be the subquiver consisting of the vertices Y, $\tau(Y)$, $X_1, \cdots, X_k$ and the arrows $\alpha_1, \alpha_2, \cdots, \alpha_k$ and $\beta_1, \beta_2, \cdots, \beta_k$.*



Let $k\langle Q\rangle$ be the path category defined in Definition 1.1.6. The mesh relation at Y *is defined as:*

$$m_Y := \sum_{i=1}^{k} \beta_i \alpha_i \in Hom_{k\langle Q\rangle}(\tau(Y), Y).$$

Let I_m be the ideal in $k\langle Q\rangle$ generated by the mesh relations m_Y where Y runs over all vertices of Q. Then the mesh category of Q *is defined to be the quotient*

$$k\langle Q\rangle / I_m.$$

Remark.

1. In particular the *mesh category of* $\mathbb{Z}Q$ is well-defined.
2. As $\mathbb{Z}Q$ doesn't depend on the orientation of its arrows, its mesh category is also independent of the arrows orientation.

The following lemma is due to Happel [HD]:

Lemma 2.2.8. *Let Q be a quiver of simply-laced Dynkin type with underlying graph G. Then $ind\, D^b(kQ)$ is equivalent to the mesh category of $\mathbb{Z}G$.*

Proof. See [HD], page 52. □

Remark. Let Q be a quiver of simply-laced Dynkin type with underlying graph G and $(\mathbb{Z}G, \tau)$ be defined as above. It can be shown that τ corresponds to the Auslander-Reiten translation τ^{AR} of $ind\, D^b(kQ)$. The *shift* $[m]$ which sends a vertex in a copy of the module category of kQ to the corresponding vertex in the mth next copy of the module category is as well consistent with $\mathbb{Z}G$. Therefore the Auslander-Reiten quiver of $ind\, D^b(kQ)$ is isomorphic to $\mathbb{Z}G$.

Example 11. In Example 10, $(\mathbb{Z}\vec{A}_3, \tau)$ is isomorphic to the Auslander-Reiten quiver of $ind\, D^b(k\vec{A}_3)$.

Definition 2.2.9. *Let Q be a quiver of a simply-laced Dynkin type with underlying graph G. Then*

$$\mathcal{C}_Q^m = (ind\, D^b(kQ))/\tau^{-1} \circ [m]$$

is called the m-cluster category of type G.

Remark.

1. Keller [KEL] was the first to introduce the m-cluster category for Q a quiver of a simply-laced Dynkin type. Actually he observed that triangulated orbit categories, such as $ind\, D^b(kQ)/[m+1]$, provide easily constructed examples of triangulated categories, the m-cluster category (see Theorem, Section 4). Furthermore this category has the following properties:

(a) it has the Calabi-Yau property (see Corollary, Section 8.3 in [KEL]);

(b) it is Krull-Schmidt;

(c) it has almost split triangles (see Section 3, page 3 in [KEL]).

Thomas [THO], Marsh [BMA] and Zhu [ZHU] have been studying the m-cluster categories, too.

2. Let ρ_m denote the automorphism of $\mathbb{Z}G$ induced by the autoequivalence $\tau^{-1} \circ [m]$. The Auslander-Reiten quiver of $\mathcal{C}_Q^m$ is isomorphic to the quotient $\mathbb{Z}G/\rho_m$ and $ind\,\mathcal{C}_Q^m$ is equivalent to the mesh category of $\mathbb{Z}G/\rho_m$.

Example 12. Consider the path algebra kQ with Q given by:

$$c \xleftarrow{\beta} b \xleftarrow{\alpha} a \ .$$

We want to understand the Auslander-Reiten quiver of $\mathcal{C}_Q^m$. First we look for the indecomposable modules of $D^b(kQ)$. By Lemma 2.2.2, it suffices to find the indecomposable modules of $\operatorname{mod} kQ$. We have a complete list of the indecomposable projective or injective kQ-modules, given as representations (see Example 8):

$$\begin{aligned} P(1) &= (\ k \longleftarrow 0 \longleftarrow 0\ ) = S(1) \\ P(2) &= (\ k \xleftarrow{1} k \longleftarrow 0\ ) \\ P(3) &= (\ k \xleftarrow{1} k \longleftarrow k\ ) = I(1) \\ I(2) &= (\ 0 \longleftarrow k \xleftarrow{1} k\ ) \\ I(3) &= (\ 0 \longleftarrow 0 \longleftarrow k\ ) = S(3), \end{aligned}$$

where $P(i)$ is the representation associated to the projective module Ae_i (See Example 4) and $I(i)$ is associated to the injective modules $Hom_k(e_iA, k)$. Notice the $P(1)$ and $I(3)$ are also simple modules, denoted by $S(1)$ and $S(3)$ respectively. We have also a simple module

$$S(2) = (\ 0 \longleftarrow k \longleftarrow 0\ ),$$

which is neither projective nor injective. Further we have the relations:

$$\begin{aligned} P(1) &= \operatorname{rad} P(2) \\ P(2) &= \operatorname{rad} P(3) \\ I(3) &= I(2)/S(2) \\ I(2) &= I(1)/S(1) = P(3)/S(1). \end{aligned}$$

We want to find the irreducible morphisms and the Auslander-Reiten translation τ_D in $ind\, D^b(kQ)$. In Annexe $A.1$, we have seen that

$$\tau_D(P(i)) = [-1]I(i).$$

Moreover by Lemma A.1.8 we know that for X a non-projective module τ the Auslander-Reiten translation on $\operatorname{mod} kQ$ and τ_D are equivalent. Hence if we restrict our search to the irreducible morphisms and the translation of non-projective representations, by Lemma 2.2.2, it suffices to find the irreducible morphisms and DTr-Auslander-Reiten translation of the Auslander-Reiten quiver of $\operatorname{mod} kQ$:

1. Starting at $P(1)$. From Corollary 2.1.7 the target of an irreducible morphism starting at $P(1)$ has to be projective. By Proposition 2.1.5, if there exists an irreducible morphism $f : P(1) \to X$, then $P(1)$ has to be part of the summand of $\operatorname{rad} X$. Here the only two other projective modules are $P(2)$ and P(3). Hence the inclusion $i : P(1) \to P(2)$ ($P(1) = \operatorname{rad} P(2)$) is the only irreducible morphism starting at $P(1)$ and by Proposition 2.1.5 it is actually the only right minimal almost split morphism ending at $P(2)$. Thus we get the almost split sequence

   $$0 \longrightarrow P(1) \xrightarrow{i} P(2) \longrightarrow Coker\, i \longrightarrow 0\,.$$

   It is easily seen that $Coker\, i = P(2)/P(1) = S(2)$. This gives us two new important data:

   (a) There exists an irreducible morphism from $P(2)$ to $S(2)$.
   (b) By Theorem 2.1.4 and Proposition A.2.2, $\tau(S(2)) = P(1)$.

2. Starting at $P(2)$. We have already found an irreducible morphism $P(2) \to S(2)$. Moreover $P(2) = \operatorname{rad} P(3)$, hence the inclusion is an irreducible morphism.

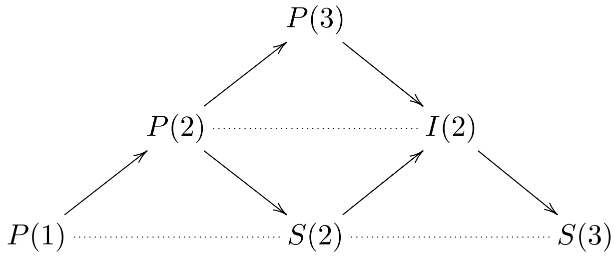
3. Starting at $P(3)$. We have that $P(3)$ is non-simple, projective and injective. We have the relations: $\operatorname{soc} P(3) = S(1)$ and $P(3)/S(1) = I(2)$. Therefore $P(3) \to I(2)$ is an irreducible morphism. Notice that $P(2) \to P(3)$ is the unique (up to isomorphism) irreducible morphism ending at $P(3)$ and that $P(3) \to I(2)$ is the unique (up to isomorphism) irreducible morphism starting from it. From Proposition 2.1.6 it follows that $\tau(I(2)) = P(2)$.

4. Starting at $I(2)$. We have $\operatorname{soc} I(2) = S(2)$. Hence there is an irreducible morphism $I(2) \to I(2)/S(2) = S(3)$ which is by Theorem 2.1.4 left minimal right exact, so that we have an almost split sequence:

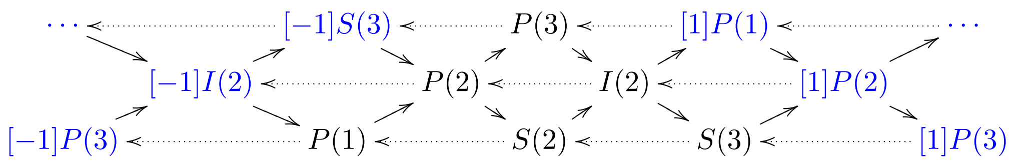
   $$0 \longrightarrow S(2) \longrightarrow I(2) \longrightarrow S(3) \longrightarrow 0\,.$$

   Therefore, there exists an irreducible morphism $I(2) \to S(3)$ and $\tau\ (S(3)) = S(2)$.

Putting all this information together we get the Auslander-Reiten quiver of mod kQ:



and by extension the Auslander-Reiten quiver of $ind\,D^b(kQ)$ is:



Chapter 3

Triangulations and quivers

3.1 Triangulations of polygons

In this section we want to introduce the mesh category associated to diagonals, called m-diagonals, of a polygon Π. This construction has been studied by [BMA], [TZA] and [THO] and has at its origin the construction of [CCS] for the case $m = 1$. The structure of this section is mainly based on [BMA].

But first we have to recall some basic concepts:

Definition 3.1.1. *1. Let X be a finite set and $\triangle \subseteq \mathcal{P}(X)$ be a collection of subsets. If $\triangle$ is closed under taking subsets (that is $A \in \triangle$ and $B \subseteq A$ imply $B \in \triangle$), then we call $\triangle$* an abstract simplicial complex *on the ground set X. The* vertices *of $\triangle$ are the singletons of $\triangle$ ($\{i\} \in \triangle$). The* faces *are the elements of $\triangle$ and the* facets *are the maximal among those (that is $A \in \triangle$ such that $A \subseteq B$ and $B \in \triangle$ imply $A = B$).*

2. Let $\triangle$ be an abstract simplicial complex as above. Then the dimension *of a face A is equal to $|A| - 1$ where $|A|$ is the cardinality of A. $\triangle$ is said to be of* pure dimension d *if all its facets have dimension d.*

It is possible to construct such an abstract simplicial complex on the m-diagonals of a polygon Π following [TZA]:

Definition 3.1.2. *Let Π be an $(nm + 2)$-gon, (m, $n \in \mathbb{N}$, with vertices numbered clockwise from 1 to $nm + 2$). In the following we will regard all operations on Π modulo $nm + 2$.*

1. *A* diagonal *D is a curve inside Π, considered up to isotopy in Π, such that D connects two non-consecutive vertices of Π. A diagonal is denoted by the pair (i, j) (or simply ij if $1 \leq i, j \leq 9$) of vertices that it joins. Thus (i, j) is the same as (j, i).*
2. *D is called an* m-diagonal *in Π if it is a diagonal and D divides Π into an $(mj + 2)$-gon and an $(m(n - j) + 2)$-gon where j can take value in the set $\{1, \cdots, \lceil \frac{n-1}{2} \rceil\}$.*

3. *Let X be the set of all the m-diagonals in Π. Then the abstract simplicial complex $\triangle^m_{A_{n-1}}$ on the ground set X, defined as in Definition* 3.1.1*, has the following properties:*

   (a) *the vertices are the m-diagonals of Π;*

   (b) *the faces are the sets of m-diagonals which pairwise don't cross. They are also called m-*divisible dissections *(of Π);*

   (c) *the facets are the maximal faces, that is the maximal sets of m-diagonals which pairwise don't cross.*

Remark. 1. It can be easily shown that each facet contains exactly $n-1$ elements. Hence the abstract simplicial complex $\triangle^m_{A_{n-1}}$ is pure of dimension $n-2$.

2. The above definition is a generalization of the concept of triangulations of a disc (case $m = 1$).

Definition 3.1.3. *Let $\triangle^m_{A_{n-1}}$ be defined as above. We associate to $\triangle^m_{A_{n-1}}$ a quiver $\Gamma^m_{A_{n-1}}$ given by:*

1. *a set of vertices where the vertices are the m-diagonals in Π, that is the vertices of $\triangle^m_{A_{n-1}}$.*

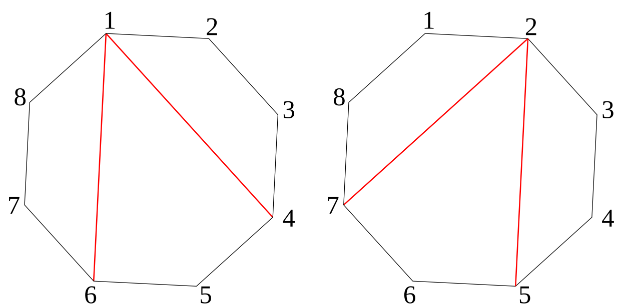
2. *a set of arrows such that there is an arrow from D to D' (where D and D' are m-diagonals) if the following conditions are met:*

   (a) *D and D' have a common vertex i in Π.*

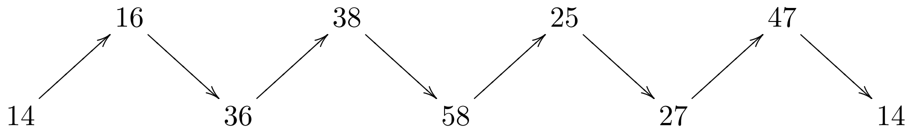
   (b) *Let j, j' be the other endpoints of D and D' respectively. Then D, D' and the sides joining j to j' form an $(m+2)$-gon in Π.*

   (c) *D can be rotated clockwise to D' inside Π about the common endpoint i.*

Example 13. Let $n = 3$, $m = 2$, that is we consider a octagon with vertices numbered from 1 to 8.



Then the diagonals 14, 16, 25, 27, 36, 38, 47, 58 are the complete set of 2-diagonals. The subsets $\{14, 16\}$ or $\{14, 58\}$ are facets of the associated abstract simplicial complex $\triangle^2_{A_2}$ and the associated quiver $\Gamma^2_{A_2}$ is given by a cycle:



Definition 3.1.4. *Consider* $\triangle^m_{A_{n-1}}$ *and its associated quiver* $\Gamma^m_{A_{n-1}}$*. Let* D *and* D' *be two* m*-diagonals.*

1. *If there is an arrow from the* D *to* D' *in* $\Gamma^m_{A_{n-1}}$*, then we say that* D *is related to* D' *by an* m-pivoting elementary move*. We denote by* P_i^m *(or* P_i*) the* m*-pivoting elementary move with* pivot i *in* Π *where* i *is the common endpoint of* D *and* D'*.*

2. *An* m-pivoting path *from* D *to* D' *is a sequence of* m*-pivoting elementary moves starting at* D*. Then* D *and* D' *are said to be* related by a sequence of m-pivoting elementary moves*.*

3. *If* D *is related to* D' *by two consecutive* m*-pivoting elementary moves with distinct pivots, we can define the* mesh relation $P_{j'}P_i = P_{i'}P_j$*, where* i*,* j *(respectively* i'*,* j'*) are the vertices of* D *(respectively* D'*) such that* $P_{i'}P_j(D) = P_{j'}P_i(D) = D'$*. More generally, a* mesh relation *is an equality between two pivoting paths which differ only in two consecutive* m*-pivoting elementary moves by such a change.*

Remark. In the definition of the mesh relation, the sides of Π can be one of the two edges of the m-pivoting path between D and D' with the following convention:

- If one of the two edges in the m-pivoting path is a slide, the corresponding term in the mesh relation is replaced by zero.

Associated to $\Gamma^m_{A_{n-1}}$ we can now define the automorphism of quivers

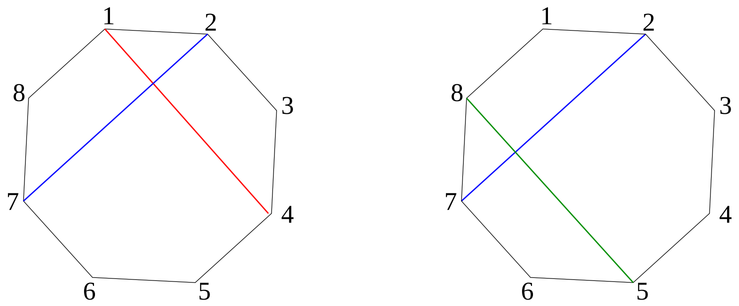
$$\tau_m : \Gamma^m_{A_{n-1}} \to \Gamma^m_{A_{n-1}} \text{ given by } \tau_m(D) = D'$$

where D' is the m-diagonal obtained from D by an anticlockwise rotation through $\frac{2m}{nm+2}\pi$ about the center of the given polygon Π.

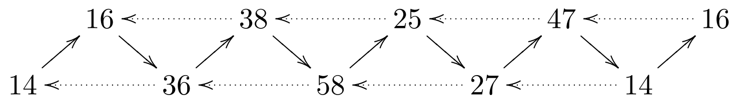
Proposition 3.1.5. *The pair* $(\Gamma^m_{A_{n-1}}, \tau_m)$ *is a connected stable translation quiver.*

Proof. See [BMA], Proposition 2.2. □

Example 14. In our previous example, $\tau_2(14) = 27$ and $\tau_2(27) = 58$:



Thus we get the translation quiver $(\Gamma^2_{A_2}, \tau_2)$:



We can now define the mesh category associated to $\Gamma^m_{A_{n-1}}$:

Definition 3.1.6. *Let $k < \Gamma^m_{A_{n-1}} >$ be the path category associated to $\Gamma^m_{A_{n-1}}$ as in Definition 1.1.6. Furthermore let J_m be the ideal in $k < \Gamma^m_{A_{n-1}} >$ generated by the mesh relations ($< P_{i'}P_j - P_{j'}P_i >$) running over all m-pivoting path from (i, j) to (i', j'). Then the* mesh category of $\Gamma^m_{A_{n-1}}$ *is defined to be the quotient*

$$k < \Gamma^m_{A_{n-1}} > /J_m.$$

Remark. The definition of $k < \Gamma^m_{A_{n-1}} > /J_m$ corresponds to the construction of the mesh category $k\langle Q\rangle/I_m$ defined in Section 2.2.

3.2 The ℓth power of a translation quiver

In this section we define a new translation quiver which, under certain conditions, has $\Gamma^m_{A_{n-1}}$ as connected component.

Definition 3.2.1. *Let (Γ, τ) be a translation quiver. We define Γ^ℓ, the ℓ*th power of Γ, *as follows:*

1. *the vertices are the same as the vertices of Γ;*
2. *the arrows are the sectional paths of length ℓ, where a path*

$$x = x_0 \to x_1 \to \cdots \to x_{\ell-1} \to x_\ell = y$$

is said to be sectional *if $\tau x_{i+1} \neq x_{i-1}$ for $i = 1, \cdots, \ell - 1$.*

Associated to Γ^ℓ, we define the ℓth power of the translation, τ^ℓ, where

$$\tau^\ell = \tau \circ \tau \circ \cdots \circ \tau \,(\ell \text{ times}).$$

Example 15. For all N, $\ell \in \mathbb{N}$, we can consider $(\Gamma^1_{A_{N-3}})^\ell$ and the associated ℓth power of the translation τ^ℓ. For example let $N = 11$ and $\ell = 3$, then $(\Gamma^1_{A_8}, \tau)$ is given by the first translation quiver in Figure 3.1 and the associated 3rd power of $(\Gamma^1_{A_8}, \tau)$ consists of the other two connected components in Figure 3.1.

Definition 3.2.2. *Consider the category of translation quivers with morphisms the isomorphisms. Then taking the ℓth power can be extended to a functor, $\mathcal{P}^\ell$, such that:*

1. *for every object, (Q, τ), $\mathcal{P}^\ell(Q, \tau) = (Q^\ell, \tau^\ell)$.*
2. *for every isomorphism of translation quivers,*

$$\varphi : (Q, \tau) \to (Q', \tau'),$$

we have

$$\mathcal{P}^\ell(\varphi) := \varphi^{(\ell)} : (Q^\ell, \tau^\ell) \to (Q'^\ell, \tau'^\ell)$$

is the isomorphism of translation quivers given by

$$\varphi^{(\ell)}(i) = \varphi(i) \text{ for all } i \in Q_0.$$

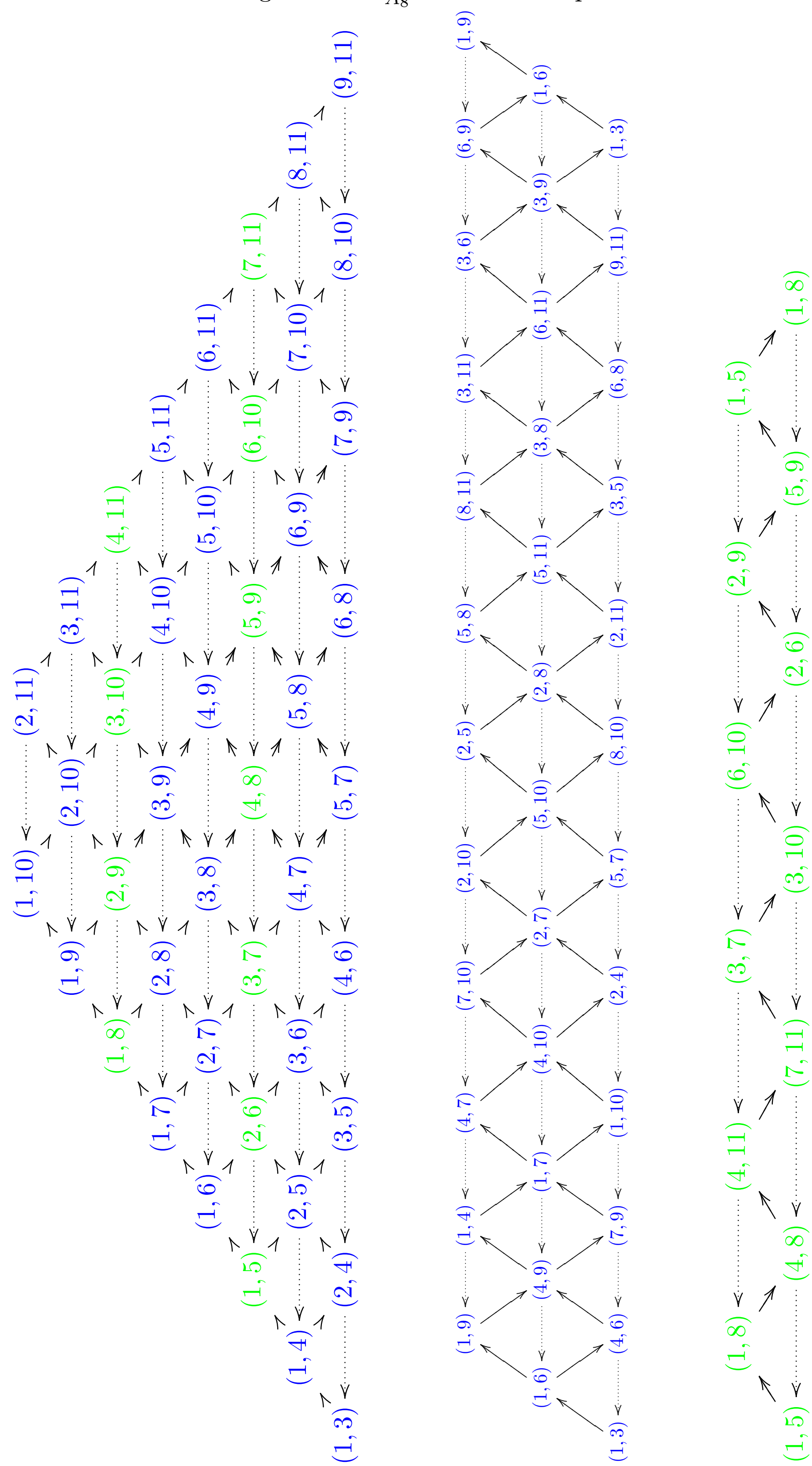
Remark. It is easily checked that if $x = x_0 \to x_1 \to \cdots \to x_\ell = y$ is a sectional path (that is $\tau(x_{i+2}) \neq x_i$) then

$$\varphi^{(\ell)}(x) = \varphi^{(\ell)}(x_0) \to \varphi^{(\ell)}(x_1) \to \cdots \to \varphi^{(\ell)}(x_\ell) = \varphi^{(\ell)}(y)$$

is as well a sectional path (using the propriety $\tau'(\varphi(x)) = \varphi(\tau(x))$).

In the following we will prove that, under certain conditions, the couple (Γ^ℓ, τ^ℓ) is a (stable) translation quiver. But before we need to recall and/or introduce the notion of a projective vertex, of a stable translation quiver and of a hereditary translation quiver.

Figure 3.1: $\Gamma^1_{A_8}$ and its third power



Definition 3.2.3. *Let* (Γ, τ) *be a translation quiver, where* $\tau : Q_0' \to Q_0$ *is an injective map defined on a subset* $Q_0' \subseteq Q_0$.

1. *The vertices in* $Q_0 \backslash Q_0'$ *are called* projective *vertices.*
2. (Γ, τ) *is said to be* stable *if* $Q_0' = Q_0$ *and* τ *is a bijection.*
3. (Γ, τ) *is said to be* hereditary *if:*
   (a) *for any non-projective vertex* z*, there is an arrow from some vertex* z' *to* z*;*
   (b) *there is no oriented cyclic path of length at least one containing projective vertices;*
   (c) *If* y *is a projective vertex and there is an arrow from* x *to* y*, then* x *is projective.*

The following result is due to [BMA]:

Theorem 3.2.4. *Let* (Γ, τ) *be a translation quiver such that if* y *is a projective vertex and if there is an arrow* $x \to y$*, then* x *is projective. Then* (Γ^ℓ, τ^ℓ) *is a translation quiver.*

Proof. We first prove the following statement by induction on ℓ:

- *If there is a sectional path*

$$x = x_0 \to x_1 \to \cdots \to x_\ell = y$$

in Γ *and* $\tau^\ell y$ *is defined, then* $\tau^i x_i$ *is defined for* $i = 1, \cdots, \ell - 1$ *and there is a sectional path*

$$\tau^\ell y = \tau^\ell x_\ell \to \tau^{\ell-1} x_{\ell-1} \to \cdots \to \tau x_1 \to x = x_0.$$

Furthermore, if the multiplicities of arrows between consecutive vertices in the first path are $k_1, k_2, \cdots, k_\ell$*, the multiplicities of arrows between consecutive vertices in the second path are* $k_\ell, k_{\ell-1}, \cdots, k_1$.

Clearly it is true for $\ell = 1$ since (Γ, τ) is a translation quiver.
Suppose that is true for $\ell - 1$ and that

$$x = x_0 \to x_1 \to \cdots \to x_\ell = y$$

is a sectional path in Γ. Furthermore suppose that $\tau^\ell y$ is defined. Hence $\tau^{\ell-1} y$ is also defined and by induction on the sectional path

$$x_1 \to \cdots \to x_\ell = y,$$

we obtain that $\tau^{i-1} x_i$ is defined for $i = 1, 2, \cdots, \ell$ and that there is a sectional path

$$\tau^{\ell-1} x_\ell \to \tau^{\ell-2} x_{\ell-1} \to \cdots \to x_1$$

in Γ, with multiplicities of the arrows $k_\ell, k_{\ell-1}, \cdots, k_2$.
Under (A), we prove that $\tau^i x_i$ exists for $i = 1, 2, \cdots, \ell$ and that it is non-projective. Under (B) we show that the multiplicities of the arrows are consistent.

(A) As $\tau^\ell x_\ell$ is defined, $\tau^{\ell-1} x_\ell$ is non-projective. Moreover as there is an arrow $\tau^{\ell-1} x_\ell \to \tau^{\ell-2} x_{\ell-1}$, by our assumption $\tau^{\ell-2} x_{\ell-1}$ is also non-projective. Hence by induction we can show that $\tau^{i-1} x_i$ exists and is non-projective for $i = 1, 2, \cdots, \ell$. Finally this proves that there exists $\tau^i x_i$ for all i.

(B) For $i = 2, 3, \cdots, \ell$, there are k_i arrows from $\tau^{i-1} x_i$ to $\tau^{i-2} x_{i-1}$,

$$\tau^{i-1} x_i \xrightarrow{k_i} \tau^{i-2} x_{i-1} \ .$$

Thus there are k_i arrows

$$\tau(\tau^{i-2}) x_{i-1} \xrightarrow{k_i} \tau(\tau^{i-1}) x_i \ ,$$

that is

$$\tau^{i-1} x_{i-1} \xrightarrow{k_i} \tau^i x_i \ .$$

For $i = 1$ as there are k_1 arrows from x_0 to x_1, there are k_1 arrows from τx_1 to x_0.

It remains to show that the new path is sectional, that is $\tau(\tau^i x_i) \neq \tau^{i+2} x_{i+2}$ for all i. But if this isn't the case then $x_i = \tau x_{i+2}$ (for some i), contradicting the fact that $x_0 \to x_1 \to \cdots \to x_m$ is sectional. Hence we have proved the statement.

The above statement implies that the number of sectional paths with a sequence of vertices $x_0, x_1, \cdots, x_\ell$ is less than or equal to the number of sectional paths with a sequence of vertices $\tau^\ell x_\ell, \tau^{\ell-1} x_{\ell-1}, \cdots, \tau x_1, x_0$. We want now to show that there is a bijection between the set of sectional paths of length ℓ from x to y and the set of sectional paths of length ℓ from $\tau^\ell y$ to x.

Consider the sectional path $x = x_0 \to x_1 \to \cdots \to x_\ell = y$ and suppose that

$$x = x'_0 \to x'_1 \to \cdots \to x_{\ell'} = y$$

is another sectional path from x to y, that is there exists i, $0 < i < \ell$, such that $x_i \neq x'_i$. It follows that $\tau^i x_i \neq \tau^i x'_i$ and thus the sectional path from $\tau^\ell y$ to x provided by the above statement is also on a different sequence of vertices. By applying the statement to every sectional path of length ℓ from x to y, we obtain an injection from the set of sectional paths of length ℓ from x to y to the set of sectional paths of length ℓ from $\tau^\ell y$ to x. A similar argument shows that there is an injection from the set of sectional paths of length ℓ from $\tau^\ell y$ to x to the set of sectional paths of length ℓ from x to y. This proves the existence of the bijection. Hence the number of sectional paths of length ℓ from x to y equal the number of sectional paths of length ℓ from $\tau^\ell y$ to x and finally (Γ^ℓ, τ^ℓ) is a translation quiver. □

Corollary 3.2.5.

1. *Let* (Γ, τ) *be a hereditary translation quiver. Then* (Γ^ℓ, τ^ℓ) *is a translation quiver.*

2. *Let* (Γ, τ) *be a stable translation quiver. Then* (Γ^ℓ, τ^ℓ) *is a stable translation quiver.*

Proof. 1. It is immediate from Theorem 3.2.3 and the definition of a hereditary translation quiver.

2. If (Γ, τ) is stable, then it has no projective vertex. So (Γ^ℓ, τ^ℓ) is a translation quiver by Theorem 3.2.3. Since τ is defined on all vertices of Γ, so is τ^ℓ.

□

Remark. The ℓth power of a hereditary translation quiver need not to be hereditary.

3.3 The case $(\Gamma^1_{A_{nm-1}})^m$, an introduction

In the following we will study the mth power of $\Gamma^1_{A_{nm-1}}$. As this quiver is not connected (see Example 15 where $n = 3$, $m = 3$ and $N = 11$), we will thus study its different components. But first we present the following general proposition on $(\Gamma^1_{A_{N-3}})^m$ $(N \geq 4)$ due to [BMA]:

Proposition 3.3.1. *One of the connected components of the quiver* $(\Gamma^1_{A_{N-3}})^m$ *is a translation quiver of* m*-diagonals if and only if* $N = nm + 2$ *for some* $n \in \mathbb{N}$*.*

Proof. Let Π denote the associated N-gon. If $N \neq nm + 2$ then, by definition, Π doesn't contain any m-diagonal. Hence $(\Gamma^1_{A_{N-3}})^m$ can't contain a translation quiver of m-diagonals.

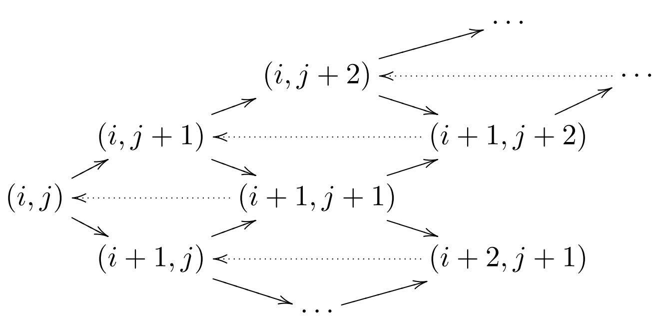
Now assume $N = nm+2$ for some n. We denote $(\Gamma^1_{A_{N-3}})^m$ $(= (\Gamma^1_{A_{nm-1}})^m)$ by Γ and $\Gamma^m_{A_{n-1}}$ by $\mathcal{Q}$. First as the vertices of Γ are the diagonals of Π and as the vertices of $\mathcal{Q}$ are the m-diagonals of Π, we see that the vertices of $\mathcal{Q}$ are a subset of the vertices of Γ. Secondly we claim that the arrows between those vertices are the same for $\mathcal{Q}$ and for Γ^m. In other words, we claim that there is a sectional path of length m between D and D' if and only if D can be rotated clockwise to D' about a common endpoint and if D, D' and the sides joining j to j' form an $(m + 2)$-gon in Π. Let start with D and D' in

Γ such that there exists a sectional path of length m from D to D' i.e there exists an arrow $D \to D'$ in Γ. We denote D (respectively D') by the couple of its endpoints (i, j) (respectively (i', j')). Without loss of generality $i < j$ (respectively $i' < j'$). By assumption there exists a sectional path

$$(i, j) = D \to D_1 \to D_2 \to \cdots \to D_{m-1} \to D_m = D' = (i', j').$$

We want to describe the diagonals D_k for $k = 1, \cdots, m$ in function of i and j.

The first arrow $D \to D_1$ is either $(i, j) \to (i, j+1)$ or $(i, j) \to (i+1, j)$ (where every vertex is considered $\mod N$). In the first case, one then has either the arrow $(i, j+1) \to (i, j+2)$ or $(i, j+1) \to (i+1, j+1)$. In the second case, one gets either the arrow $(i+1, j) \to (i+1, j+1)$ or the arrow $(i+1, j) \to (i+2, j)$. This leads to the general form of the quiver $\Gamma^1_{A_{nm-1}}$:



where $\dashleftarrow$ represents the translation τ given by $\tau(i+1, j+1) = (i, j)$. Thus the sectional path has to be either of the form

$$D = (i, j) \to (i, j+1) \to (i, j+2) \to \cdots \to (i, j+m) = D_m = D'$$

or of the form

$$D = (i, j) \to (i+1, j) \to (i+2, j) \to \cdots \to (i+m, j) = D_m = D'$$

where all vertices are taken $\mod N$. In particular the arrow $D \to D'$ corresponds in the first case to the rotation about the common endpoint i of D and D' and in the second case to the rotation about the common endpoint j of D and D'. In each case D, D' and the sides joining their endpoints form an $(m+2)$-gon in Π, so there is an arrow from D to D' in $\mathcal{Q}$. Since it is clear that every arrow in $\mathcal{Q}$ arises in this way, we see that the arrows between the vertices of $\mathcal{Q}$ and the arrows of the corresponding connected component of Γ are the same.

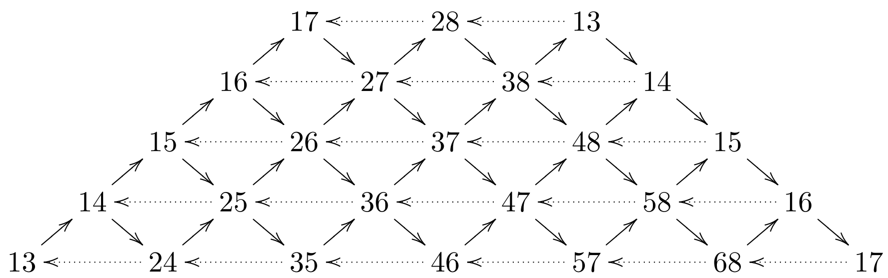
□

Thus we get the following proposition due to [BMA]:

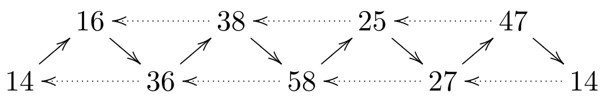
Proposition 3.3.2. $\Gamma^m_{A_{n-1}}$ *is a connected component of* $(\Gamma^1_{A_{nm-1}})^m$.

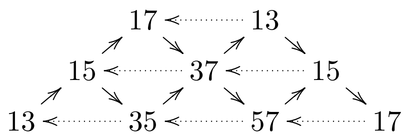
Proof. We know by Proposition 3.1.6 that $\mathcal{Q} = \Gamma^m_{A_{n-1}}$ is a connected stable translation quiver. If there is an arrow $D \to D'$ in Γ^m where D is an m-diagonal then D' is an m-diagonal. Similarly, $\tau_1^m(D)$ is also an m-diagonal. $\square$

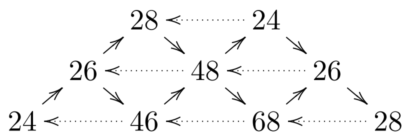
Example 16. Let $n = 3$ and $m = 2$. Then $(\Gamma^1_{A_5}, \tau)$ is given by:



And the associated 2nd power of $(\Gamma^1_{A_5}, \tau)$ has three connected components:







As proved in Proposition 3.3.1, the first component corresponds to $\Gamma^2_{A_2}$ given in Example 13.

In Chapter 4, we will first present the result of [BMA], which formulates an equivalence between the mesh category of $\Gamma^m_{A_{n-1}}$ and the m-cluster category $\mathcal{C}^m_{A_{n-1}}$. We will finally present the new result of this thesis, which is the understanding of the remaining components of $(\Gamma^1_{A_{nm-1}})^m$ relating them to $ind\, D^b(kQ)$.

Chapter 4

The components of $(\Gamma^1_{A_{nm-1}})^m$

The goal of this chapter is first to introduce the result on the component $\Gamma^m_{A_{n-1}}$ of $(\Gamma^1_{A_{nm-1}})^m$ from [BMA] which proves that the m-cluster category $\mathcal{C}^m_{A_{n-1}}$ is equivalent to the mesh category of $\Gamma^m_{A_{n-1}}$. Secondly we will show that the other components of $(\Gamma^1_{A_{nm-1}})^m$ can as well be described in relation with some quotient of $ind\, D^b(kQ)$. In order to get a better understanding of the different components of $(\Gamma^1_{A_{nm-1}})^m$ we have implemented an algorithm in the programming language Python which calculates these components in function of m and n. The reader is referred to Appendix B for the presentation of this algorithm.

4.1 The component $\Gamma^m_{A_{n-1}}$

In Section 2.2 we have defined the m-cluster category of type G

$$\mathcal{C}^m_Q = (ind\, D^b(kQ))/\tau^{-1} \circ [m]$$

for Q a quiver of simply-laced Dynkin type with underlying graph G. In this chapter we will focus on the case where $G = A_{n-1}$, $n \geq 2$. Let ρ_m denote the automorphism of the quiver $\mathbb{Z}A_{n-1}$ induced by the autoequivalence $\tau^{-1} \circ [m]$. From the previous chapter we know that the Auslander-Reiten quiver of $\mathcal{C}^m_{A_{n-1}}$ is isomorphic to the quotient $\mathbb{Z}A_{n-1}/\rho_m$ and that $ind\,\mathcal{C}^m_{A_{n-1}}$ is equivalent to the mesh category of $\mathbb{Z}A_{n-1}/\rho_m$. Now we want to show that $ind\,\mathcal{C}^m_{A_{n-1}}$ is equivalent to the mesh category of $\Gamma^m_{A_{n-1}}$. In fact it is enough to show that, as translation quiver, $\mathbb{Z}A_{n-1}/\rho_m$ is isomorphic to $\Gamma^m_{A_{n-1}}$.

We first need to define a new set, $\Phi^m_{\geq -1}$:

Definition 4.1.1. *Let Φ a root system of type A with the set of positive roots denoted by Φ^+ and with simple roots $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then*

1. *for every* $\alpha \in \Phi^+$, *we define the* m-coloured positive root *to be* α^k *for* $k \in \{1, 2, \cdots, m\}$. α^k *is said to have color* k.

2. *We define the* set of m-coloured almost positive roots, $\Phi^m_{\geq -1}$, *with elements the union of* m*-coloured positive roots and negative simple roots* $-\alpha_i$ *(*α_i *a positive simple root). We consider that the negative simple roots have colour* 1 *for convenience (*$-\alpha_i$ *can thus also be denoted* $-\alpha_i^1$*).*

The main result of this section will be established in three steps due respectively to [FR], [THO], [ZHU] and [BMA]:

1. We show that there exists a bijection D from $\Phi^m_{\geq -1}$ to the set of m-diagonals of Π following [FR].

2. We show that there exists a bijection V from $\Phi^m_{\geq -1}$ to the vertices of the Auslander-Reiten quiver of $\mathcal{C}^m_{A_{n-1}}$ following [THO] and [ZHU].

3. Finally we show that the composition of the inverse of D with V is an isomorphism of stable translation quivers following [BMA].

1. Let Π be the $(nm+2)$-gon associated to $\Gamma^m_{A_{n-1}}$ with vertices numbered clockwise from 1 to nm+2. We denote by R_m the anticlockwise rotation of Π which takes vertex i to vertex $i-1$ for $i \geq 2$ and vertex 1 to vertex $nm+2$. By extension R_m is defined on the diagonals (i, j) $(i < j)$ such that $R_m((i, j)) = (i-1, j-1) \mod mn + 2$. We now construct a one-to-one correspondence between $\Phi^m_{\geq -1}$ and the m-diagonals of Π:

(a) For $1 \leq i \leq \frac{n}{2}$, the negative simple root $-\alpha_{2i-1}$ corresponds to the diagonal $((i-1)m+1, (n-i)m+2)$. For $1 \leq i \leq \frac{n-1}{2}$, the negative simple root $-\alpha_{2i}$ corresponds to the diagonal $(im+1, (n-i)m+2)$ (see Figure ?). Notice that the m-diagonals associated to $-\alpha_i$ form a facet of $\triangle^m_{A_{n-1}}$, called the *m-snake facet.*

(b) For $1 \geq i \geq j \geq n$, we can show that there are exactly m m-diagonals intersecting the diagonals labeled $-\alpha_i$, $-\alpha_{i+1}$, $\cdots$, $-\alpha_j$ and no other diagonals labeled with negative simple roots. These diagonals are of the form

$$D,\, R_m(D),\, R_m^2(D)[= R_m(R_m(D))],\, \cdots,\, R_m^{m-1}(D)$$

for some diagonal D. Then for $1 \leq k \leq m$, α^k corresponds to $R_m^{k-1}(D)$ where α denotes the positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. Generally, for an m-coloured almost positive root β^k, we denote the corresponding diagonal $D(\beta^k)$.

Let $I = I_+ \cup I_-$ be the decomposition of the graph of type A_{n-1} with I_+ the even-numbered vertices of A_{n-1} and I_- the odd-numbered ones.

Notice that for this decomposition there are no arrows between vertices in I_+ or between vertices in I_-. Fomin-Reading [FR] define a bijection $R^{FR}_m : \Phi^m_{\geq -1} \to \Phi^m_{\geq -1}$ as follows: for $\beta^k \in \Phi^m_{\geq -1}$

$$R^{FR}_m(\beta^k) = \begin{cases} \beta^{k+1} & \text{if } \beta \in \Phi^+ \text{ and } k < m, \\ ((\tau_- \tau_+(\beta))^1 & \text{otherwise.} \end{cases}$$

Here [FR] use the involution $\tau_\pm : \Phi_{\geq -1} \to \Phi_{\geq -1}$ given by

$$\tau_\epsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i, \text{ for } i \in I_{-\epsilon} \\ (\Pi_{i \in I_\epsilon} \sigma_i)(\alpha) & \text{otherwise.} \end{cases}$$

The set $\{\sigma_i : i \in I\}$ is the set of simple reflections corresponding to the simple roots α_i introduced in Section 1.2. [FR] have proved:

Lemma 4.1.2. *For all $\beta^k \in \Phi^m_{\geq -1}$, we have:*

$$D(R^{FR}_m(\beta^k)) = R_m(D(\beta^k)).$$

Proof. See the discussion in [FR] Section 4.1. □

2. Let $I = I_+ \cup I_-$ be the decomposition of A_{n-1} as above. Moreover we define the quiver A^{alt}_{n-1} as follow:

- A^{alt}_{n-1} has A_{n-1} as underlying graph;
- the orientation of the arrows is obtained by orienting every arrow from a vertex in I_+ to a vertex in I_-.

Given a representation V of A^{alt}_{n-1}, its dimension is by definition

$$dim(V) = \sum_{i=1}^{n} dim_k(V(i))\alpha_i.$$

By Gabriel's Theorem (Theorem 2.1.4), dim is a bijection from indecomposable representations of A^{alt}_{n-1} to Φ^+. We write $V(\alpha)$ for the indecomposable representation associated to $\alpha \in \Phi^+$, regarded as an indecomposable objects in $D^b(kA^{\text{alt}}_{n-1})$. Then it is clear from the definition that the indecomposable objects in $\mathcal{C}^m_{A_{n-1}}$ are given by

(a) the objects $[k-1]V(\alpha)$ for $k = 1, 2, \cdots, m$ and $\alpha \in \Phi^+$.

(b) the objects $[-1]I(i)$ where $I(i)$ is the injective representation corresponding to vertex $i \in Q_0$ (see Example 3). This comes from the observation that $\tau P(i) = [-1]I(i)$ where $P(i)$ is the projective representation corresponding to vertex $i \in I$.

By defining $V(\alpha^k)$ to be $[k-1]V(\alpha)$ for $k = 1, 2, \cdots, m$ and $V(-\alpha_i)$ to be $[-1]I(i)$, we get the wanted bijection between the objects of $ind\,\mathcal{C}^m_{A_{n-1}}$ and the m-coloured almost positive roots of $\Phi^m_{\geq -1}$. We have:

Lemma 4.1.3. *For all $\alpha^k \in \Phi^m_{\geq -1}$,*

$$V(R^{FR}_m(\alpha^k)) \cong [1]V(\alpha^k)$$

where $[1]$ *denotes the autoequivalence of $\mathcal{C}^m_{A_{n-1}}$ induced by the shift on $D^b(kA^{alt}_{n-1})$.*

Proof. See [THO], Lemma 2 and [ZHU], Section 3.8. □

3. Composing the inverse of the map D (m-coloured roots to m-diagonals) of **1.** with the map V (m-coloured roots to indecomposable modules) of **2.** we obtain a bijection ψ from the set of m-diagonals of Π to $ind\,\mathcal{C}^m_{A_{n-1}}$.

Lemma 4.1.4. *For every m-diagonal (i, j) of Π, we have*

$$\psi(R_m((i, j))) \cong [1]\psi((i, j)),$$

and therefore

$$\psi(\tau_m((i, j))) \cong \tau_m(\psi((i, j))).$$

Proof. The first statement follows immediately from Lemma 4.1.2 and 4.1.3. We can deduce from this

$$\psi(\tau_m(D)) = \psi(R^m_m(D)) = [m]\psi(D)$$

and thus obtain the second statement since $[m]$ coincides with τ_m on every indecomposable object of $\mathcal{C}^m_{A_{n-1}}$ by the definition of this category. □

The following results are due to [BMA].

Proposition 4.1.5. *The map ψ from m-diagonals in Π to indecomposable objects in $\mathcal{C}^m_{A_{n-1}}$ is an isomorphism of quivers.*

Proof. See [BMA], Proposition 5.4. □

Proposition 4.1.6. *There is an isomorphism ψ of translation quivers between the stable translation quiver $\Gamma^m_{A_{n-1}}$ of m-diagonals and the Auslander-Reiten quiver of the m-cluster category $\mathcal{C}^m_{A_{n-1}}$.*

Proof. See [BMA], Proposition 5.5. □

We therefore have our main result.

Theorem 4.1.7. *The m-cluster category $\mathcal{C}^m_{A_{n-1}}$ is equivalent to the additive category generated by the mesh category of the stable translation quiver $\Gamma^m_{A_{n-1}}$ of m-diagonals.*

Proof. Clear from Proposition 4.1.6.

4.2 The other components

From the previous section (Proposition 4.1.6) we can conclude

$$\Gamma^1_{A_{nm-1}} \cong \mathbb{Z}A_{nm-1}/\rho_1,$$

where $\mathbb{Z}A_{nm-1}$ is the quiver defined in Definition 2.2.5 and ρ_s ($s \in \mathbb{Z}$) is the automorphism on $\mathbb{Z}A_{nm-1}$ corresponding to the automorphism $\tau^{-1} \circ [s]$ on $ind\, D^b(kA_{nm-1})$. We define a new quiver $\tilde{\Gamma}^1_{A_{nm-1}}$ associated to $\Gamma^1_{A_{nm-1}}$:

Definition 4.2.1. *Consider the stable translation quiver $(\Gamma^1_{A_{nm-1}}, \tau_1)$. Then we set $\tilde{\Gamma}^1_{A_{nm-1}}$ to be the quiver given by:*

1. *The vertices $\{[(i,j),k] \mid (i,j) \in (\Gamma^1_{A_{nm-1}})_0$ and $k \in \mathbb{Z}\}$ where the vertex $[(i,j),k]$ is called the kth* copy of *(i,j).*

2. *The arrows:*

   (a) *$[(i,j),k]$ to $[(i,j'),k]$ if there exists an arrow in $\Gamma^1_{A_{nm-1}}$ from the vertex (i,j) to the vertex (i,j');*

   (b) *$[(i,j),k]$ to $[(i',j),k]$ if there exists an arrow in $\Gamma^1_{A_{nm-1}}$ from the vertex (i,j) to the vertex (i',j);*

   (c) *$[(i,j),k]$ to $[(i',i),k+1]$ if there exists an arrow in $\Gamma^1_{A_{nm-1}}$ from the vertex (i,j) to the vertex (i',i);*

   (d) *$[(i,j),k]$ to $[(j,j'),k]$ if there exists an arrow in $\Gamma^1_{A_{nm-1}}$ from the vertex (i,j) to the vertex (j,j').*

Furthermore we associate to $\tilde{\Gamma}^1_{A_{nm-1}}$ a translation $\tilde{\tau}_1$ where

$$\tilde{\tau}_1([(i,j),k]) = [(i',j'),k']$$

if one of the following conditions is met:

1. *$\tau_1(i,j) = (i',j')$, $k = k'$ and $i \neq 1$;*

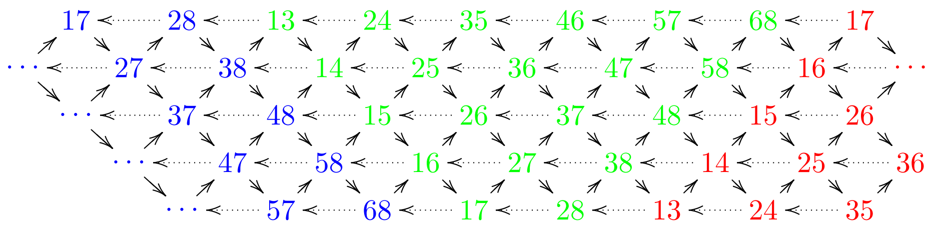
2. *$\tau_1(i,j) = (i',j')$, $k = k'+1$ and $i = 1$.*

Lemma 4.2.2. *The pair $(\tilde{\Gamma}^1_{A_{nm-1}}, \tilde{\tau}_1)$ is a stable translation quiver.*

Proof. Clear from the construction. □

Convention 4.2.3. *In the following we will designate $\tilde{\Gamma}^1_{A_{nm-1}}$ by $\tilde{\Gamma}_{A_{nm-1}}$ and $\tilde{\tau}_1$ by τ in order to simplify the notation.*

Example 17. This rather technical definition corresponds to the following construction. Consider the stable translation quiver $(\Gamma^1_{A_5}, \tau_1)$. Then it can be represented by the infinite quiver:



where the vertices having the same name (for example 13, 13) are the same vertex in $\Gamma^1_{A_5}$. Then $\tilde{\Gamma}_{A_5}$ corresponds to the above quiver with:

1. the blue part corresponding to the vertex set $\{[(i,j),0] \mid (i,j) \in \Gamma^1_{A_5}\}$;

2. the green part corresponding to the vertex set $\{[(i,j),1] \mid (i,j) \in \Gamma^1_{A_5}\}$;

3. the red part corresponding to the vertex set $\{[(i,j),2] \mid (i,j) \in \Gamma^1_{A_5}\}$.

Proposition 4.2.4. *Let Q be a quiver with underlying graph A_{nm-1}. Then*

$$\mathbb{Z}A_{nm-1} \cong \tilde{\Gamma}_{A_{nm-1}}$$

that is the category ind $D^b(kQ)$ is equivalent to the mesh category associated to $\tilde{\Gamma}_{A_{nm-1}}$ defined in Definition 3.1.6.

Proof. Clear from Lemma 2.2.8 and Proposition 4.1.6. □

Convention 4.2.5. *We fix a numeration of the rows of $\tilde{\Gamma}_{A_{nm-1}}$ from bottom to top. Since there are exactly $nm-1$ diagonals starting at any given vertex of Π, $\tilde{\Gamma}_{A_{nm-1}}$ has $nm-1$ rows. Clearly if $[(i,j),k]$ is in row s for k even, then $[(i,j),k+1]$ is in row $nm-s$. Furthermore for k even, the kth copy of $\Gamma^1_{A_{nm-1}}$ will always designate the copy of $\Gamma^1_{A_{nm-1}}$ with vertex $[(1,3),k]$ in row 1.*

In the following, we will study the mth power of $(\tilde{\Gamma}_{A_{nm-1}}, \tau)$. In particular we will show that the mth power of $\tilde{\Gamma}_{A_{nm-1}}$ is isomorphic to a disjoint union of one copy of $\mathbb{Z}A_{n-1}$ and of several copies of $\mathbb{Z}A_n$. But first we prove some useful proprieties of $(\tilde{\Gamma}_{A_{nm-1}}, \tau)$:

Proposition 4.2.6. *Let $(\tilde{\Gamma}_{A_{nm-1}}, \tau)$ be defined as above and $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ be one of the connected components of $(\tilde{\Gamma}_{A_{nm-1}})^m$. Then:*

1. *if $[(i,j),k]$ and $[(i',j'),k']$ are both vertices in $\mathcal{Q}_0$, then one of the two sets of conditions is met:*

$$(a) \begin{cases} i \equiv i' & (\mathrm{mod}\, m) \\ j \equiv j' & (\mathrm{mod}\, m) \\ k' \equiv k & (\mathrm{mod}\, 2) \end{cases}$$

$$(b) \begin{cases} j \equiv i' & (\mathrm{mod}\, m) \\ i \equiv j' & (\mathrm{mod}\, m) \\ k' \equiv k+1 & (\mathrm{mod}\, 2). \end{cases}$$

2. *Each connected component has vertices in exactly one of the first m rows.*

Proof. 1. Recall that in the proof of Proposition 3.3.1 we proved that there exists an arrow $D\ :\ (i,j)\ \rightarrow\ (i',j')$ in $(\Gamma^1_{A_{nm-1}})^m$ if and only if $(i',j')\ =\ (i,j+m)$ or $(i+m,j)$. Using the same argument one can show that there is an arrow $D\ :\ [(i,j),k]\ \rightarrow\ [(i',j'),k']$ in $\mathcal{Q}$ if and only if one of the following conditions is met:

   (a) $k' = k$ and $(i',j')\ =\ (i,j+m)$ or $(i+m,j)$

   (b) $k' = k+1$ and $(i',j') = (j+m,i)\,(\mathrm{mod}\, nm+2)$.

   Since any pair of vertices of $\mathcal{Q}$ are joined by a sequence of arrows, the statement follows.

2. Clear from 1.

□

Lemma 4.2.7. *Let $(\tilde{\Gamma}_{A_{nm-1}},\tau)$ be defined as above, $\mathcal{Q} = (\mathcal{Q}_0,\ \mathcal{Q}_1)$ be one of the connected components of $(\tilde{\Gamma}_{A_{nm-1}})^m$ and $[(i,j),k] \in \mathcal{Q}_0$ a vertex situated in row $s\ \in\ \{1,\ 2,\ \cdots,\ m\}$ of $\tilde{\Gamma}_{A_{nm-1}}$. Then $\mathcal{Q}$ has only vertices in rows $s+\alpha m$ of $\tilde{\Gamma}_{A_{nm-1}}$ where $\alpha = \{0,\ 1,\ 2,\ \cdots \lfloor \frac{nm-1-s}{m} \rfloor\}$.*

Proof. In the proof of Proposition 4.2.6 we have shown that there is an arrow $D\ :\ [(i,j),k]\ \rightarrow\ [(i',j'),k']$ in $\mathcal{Q}$ if and only if one of the following conditions is met:

1. $k' = k$ and $(i',j')\ =\ (i,j+m)$ or $(i+m,j)$

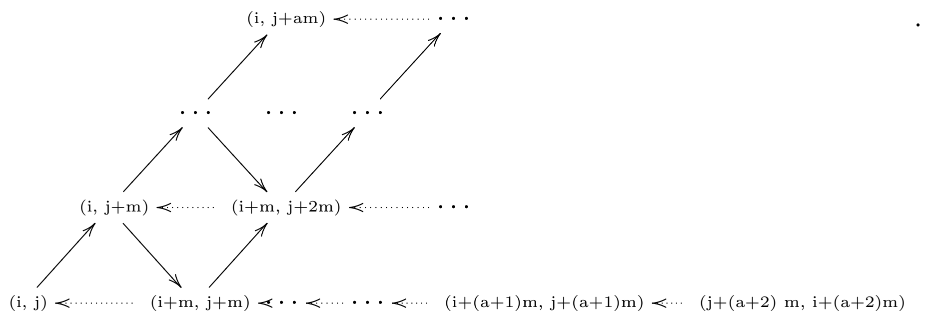
2. $k' = k+1$ and $(i',j') = (j+m,i)\,(\mathrm{mod}\, nm+2)$.

Without loss of generality, assume that k is even. In order to simplify the notation we denote the vertex $[(i',j'),k]$ by (i',j') and we set $a = \lfloor \frac{nm-1-s}{m} \rfloor$. Consider first the case where $n = 2$ and $s = m$. Then for all m, $\mathcal{Q}$ has the form:

$$(i, j) \leftarrow (i+m, j+m) \leftarrow \cdots \leftarrow \cdots \leftarrow (i+(a+1)m, j+(a+1)m) \leftarrow (j+(a+2)\,m, i+(a+2)m)$$

i.e it is situated in only one row.

For the case, where $n \neq 2$, then starting at vertex (i,j) we get that $\mathcal{Q}$ has for all m the general form:



It is straightforward to see that (i, j), $(i+m, j+m)$, $\cdots$ are in row s of $\tilde{\Gamma}_{A_{nm-1}}$ and that $(i, j+m)$, $(i+m, j+2m)$, $\cdots$ are in row $s+m$ of $\tilde{\Gamma}_{A_{nm-1}}$, etc. □

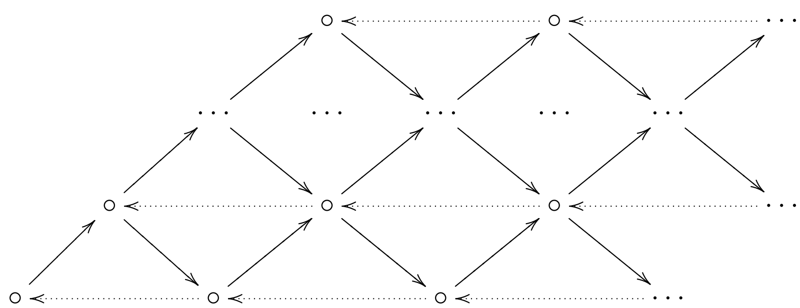
Lemma 4.2.8. *Let* $(\tilde{\Gamma}_{A_{nm-1}}, \tau)$ *be defined as above,* $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ *be one of the connected components of* $(\tilde{\Gamma}_{A_{nm-1}})^m$ *and* $[(i, j), k] \in \mathcal{Q}_0$ *situated in row* $s \in \mathbb{N}$ *of* $\tilde{\Gamma}_{A_{nm-1}}$. *Then:*

1. *If* $s < m$, *then* $\mathcal{Q}$ *is isomorphic to* $\mathbb{Z}A_n$.
2. *If* $s = m$, *then* $\mathcal{Q}$ *is isomorphic to* $\mathbb{Z}A_{n-1}$.

Proof. Consider the quiver of simply-laced Dynkin type $\overrightarrow{A}_n$,

$$\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \,.$$

and its associated translation quiver $(\mathbb{Z}\overrightarrow{A}_n, \tau)$, i.e. $\mathbb{Z}\overrightarrow{A}_n$ has the form:



We number the n rows of $\mathbb{Z}\overrightarrow{A}_n$ from bottom to top. By Lemma 4.2.7 we know that for $s < m$ the vertices of $\mathcal{Q}$ lie in row $s + \alpha m$ where $0 \leq \alpha \leq n-1$, that is that the vertices are on n different rows. Moreover it is straightforward to see that the quiver $\mathcal{Q}$ has the same form as $\mathbb{Z}\overrightarrow{A}_n$, hence both quivers are isomorphic. The same reasoning can be applied for the case where $s = m$: in that case $0 \leq \alpha \leq n-2$, so $\mathcal{Q}$ is isomorphic to the quiver $\mathbb{Z}\overrightarrow{A}_{n-1}$.

Finally, in Chapter 2 we have seen that for Q a quiver of simply-laced Dynkin type, $\mathbb{Z}Q$ doesn't depend on the orientation of Q. Hence in our case we have $\mathcal{Q} \cong \mathbb{Z}A_{n'}$ where $n' = n$ if $s < m$ and $n' = n - 1$ if $s = m$.

□

Proposition 4.2.9. *Let $[(i,j),k]$ and $[(i,j),k']$ in $(\tilde{\Gamma}_{A_{nm-1}})_0$ be two vertices lying in the same row. Then $[(i,j),k]$ and $[(i,j),k']$ are in the same connected component of $(\tilde{\Gamma}_{A_{nm-1}})^m$ if and only if*

$$\tau^{-\gamma m(nm+2)}[(i,j),k] = [(i,j),k']$$

where $\gamma \in \mathbb{Z}$ for m odd and $2\gamma \in \mathbb{Z}$ for m even.

Proof. Notice that the number of different vertices of $\Gamma^1_{A_{nm-1}}$ in each row of $\tilde{\Gamma}_{A_{nm-1}}$ is equal to $nm+2$ and that these vertices appear consecutively in $(\Gamma^1_{A_{nm-1}}, 2k) \cup (\Gamma^1_{A_{nm-1}}, 2k+1)$. This implies that for $\alpha \in \mathbb{Z}$:

$$\tau^{-\alpha(nm+2)}([(i,j),k]) = ([(i,j),k+2\alpha].$$

Let us focus on the row of $[(i,j),k]$ and $[(i,j),k']$:

$$\cdots \xrightarrow{\tau^{-1}} [(i,j),k] \xrightarrow{\tau^{-1}} [(i+1,j+1),k] \xrightarrow{\tau^{-1}} \cdots \xrightarrow{\tau^{-1}} [(i',j'),k'] \ .$$

By Proposition 4.2.6 we have that if

$$\begin{array}{rcl} \tau^{-\alpha(nm+2)}([(i,j),k]) & = & [(i+\alpha(nm+2), j+\alpha(nm+2)), k+2\alpha] \\ & = & [(i,j),k+2\alpha] \end{array}$$

belongs to the same component as $[(i,j),k]$ then

$$\alpha(nm+2) \equiv 0 \mod m$$

i.e $\alpha = \gamma m$ where $\gamma \in \mathbb{Z}$ for m odd and $2\gamma \in \mathbb{Z}$ for m even. On the other side, using the same reasoning as in the proof of Proposition 4.2.6, two vertices $([i,j),k]$, $[(i',j'),k']$ lying in the same row belong to the same connected component if and only if $\tau^{\pm\beta m}([i,j),k] = [(i',j'),k']$ ($\beta \in \mathbb{N}$). This proves the proposition.

□

Corollary 4.2.10. *Two vertices $[(i,j),k]$ and $[(i,j),k']$ of $(\tilde{\Gamma}_{A_{nm-1}})_0$ belong to the same connected component of $(\tilde{\Gamma}_{A_{nm-1}})^m$ if and only if one of the following conditions is met:*

1. *$k' = k \pm 2\delta m$ where $\delta \in \mathbb{N}$ for m odd and $2\delta \in \mathbb{N}$ for m even and $[(i,j),k]$ and $[(i,j),k']$ are in the same row.*
2. *$k' = k + 1 \pm (2\delta \pm 1)m$ where $2\delta \in \mathbb{N}$ and m is even.*

Proof. Let $\delta \in \mathbb{N}$ for m odd and $2\delta \in \mathbb{N}$ for m even. Proposition 4.2.9 implies that

$$\cdots, [(i,j),k], \cdots, [(i,j),k \pm 2\delta m]$$

belong to the same connected component. Moreover if m is odd then by Proposition 4.2.6 the vertices of type

$$[(i,j),k+1 \pm (2\delta \pm 1)m]$$

do NOT belong to the same connected component as the vertices of type

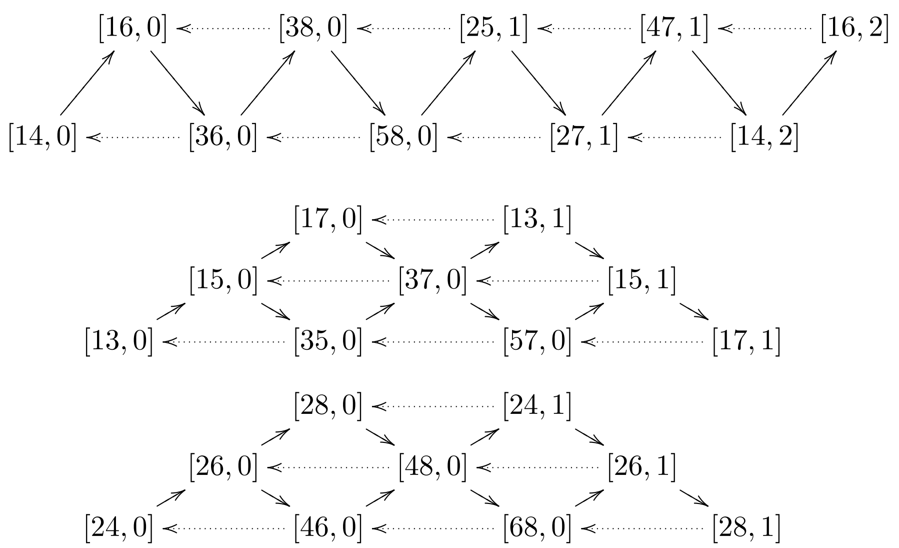
$$[(i,j),k \pm (2\delta)m]$$

as $k \pm 2\delta m \neq k+1 \pm (2\delta' \pm 1)m \mod 2$ whereas if m is even, then it is possible that the vertices of type

$$[(i,j),k+1 \pm (2\delta \pm 1)m] \text{ and } [(i,j),k \pm (2\delta)m]$$

are in the same connected component. □

Example 18. Let $m = 2$ and $n = 3$, then the second power of $\tilde{\Gamma}_{A_{nm-1}}$ is equal to:



It is an example where for $\delta = \frac{m}{2}$ Corollary 4.2.10 part 1 and part 2 both hold.

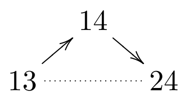
Definition 4.2.11. *The* fundamental region of $(\Gamma^1_{A_{nm-1}}, \tau_1)$, Δ_m, *is defined to be the triangular region formed by:*

1. $(\Delta_m)_0$, *the vertices* $\{(i,j) \mid 1 \leq i \leq m \text{ and } 3 \leq j \leq m+2\}$ *of* $\Gamma^1_{A_{nm-1}}$*;*
2. $(\Delta_m)_1$, *the arrows of* $\Gamma^1_{A_{nm-1}}$ *which have both their endpoints in* $(\Delta_m)_0$.

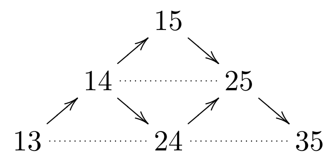
Furthermore the fundamental region of $(\tilde{\Gamma}_{A_{nm-1}}, \tau)$ *is defined to be the subquiver formed by* $\{[\Delta_m, k] \mid k \in \mathbb{Z}\}$ *in* $\tilde{\Gamma}_{A_{nm-1}}$.

Remark. Δ_m contains $\frac{m(m+1)}{2}$ vertices.

Example 19. Consider again the quiver $\Gamma^1_{A_5}$. Then the fundamental region of $\tilde{\Gamma}_{A_5}$ of Example 17 is given by disjoint union of the copies of:



for $m = 2$ and



for $m = 3$.

Lemma 4.2.12. *Let* $(\tilde{\Gamma}_{A_{nm-1}}, \tau)$ *be defined as above and* $\mathcal{Q}$ *be one of the connected components of* $(\tilde{\Gamma}_{A_{nm-1}})^m$*. Then:*

1. *For each* $k \in \mathbb{Z}$*, there is at most one vertex of* $\mathcal{Q}$ *in* $[\Delta_m, k]$*.*

2. *There exists* $[(i, j), k'] \in \mathcal{Q}$ *which belongs to* $[\Delta_m, k']$*.*

Proof.

1. By Proposition 4.2.6 if two vertices $[(i, j), k]$ and $[(i', j'), k]$ of $\Gamma^1_{A_{nm-1}}$ are not congruent modulo m, then they are not in the same connected component. Moreover by construction, $\forall\, k \in \mathbb{Z}$ the vertices in $[\Delta_m, k]$ are not congruent modulo m. Therefore there is at most one vertex of $\mathcal{Q}$ in $[\Delta_m, k]$ for each $k \in \mathbb{Z}$.

2. By construction, $\forall\, k \in \mathbb{Z}$ $[\Delta_m, k]$ contains one representative for each congruence class of pairs (i, j) modulo m. Therefore Proposition 4.2.6 implies that there exists $[(i, j), k'] \in \mathcal{Q}$ which belongs to $[\Delta_m, k']$.

□

Lemma 4.2.13. *Let* $(\tilde{\Gamma}_{A_{nm-1}}, \tau)$ *be defined as above. Then* $((\tilde{\Gamma}_{A_{nm-1}})^m, \tau^m)$ *has* m^2 *connected components.*

Proof. We define the rectangular region of $\Gamma^1_{A_{nm-1}}$, $\square_m$, formed by:

1. $(\square_m)_0$, the vertices $\{(i, j) \mid 1 \le i \le m \text{ and } 3 \le j \le 2m+1\}$ of $\Gamma^1_{A_{nm-1}}$;

2. $(\square_m)_1$, the arrows of $\Gamma^1_{A_{nm-1}}$ which have both endpoints in $(\square_m)_0$.

Furthermore $\forall\, k \in \mathbb{Z}$, we can define the subquiver of $\tilde{\Gamma}_{A_{nm-1}}$, $[\square_m, k]$. Let $\mathcal{Q}$ be one of the connected components of $(\tilde{\Gamma}_{A_{nm-1}})^m$. We claim that there exists exactly one vertex of $\mathcal{Q}$ which belongs to $[\square_m, k]$. By Proposition 4.2.6 there is at most one vertex of $\mathcal{Q}$ in $[\square_m, k]$ as the vertices of $\square_m$ are not congruent modulo m. We show that there exists a vertex of $\mathcal{Q}$ in $[\square_m, k]$. Let $[(i, j), k] \in \mathcal{Q}$. We can distinguish three different cases:

1. $[(i,j),k]$ belongs to $[\square_m, k]$.
2. $[(i,j),k]$ doesn't belong to $[\square_m, k]$ and is situated in row s, $s \leq m$. Then there exists $(i',j') \in \square_m$ such that $i = i' + \gamma m$ and $j = j' + \gamma m$, that is $\tau^\gamma[(i,j),k] = [(i',j'),k] \in [\square_m, k]$.
3. $[(i,j),k]$ doesn't belong to $[\square_m, k]$ and is situated in row s, $s = m + \gamma$ $(1 \geq \gamma \geq nm-1-m)$. Then there exists $[(i, j-\gamma), k]$ and we are back in case 1. or 2.

We can conclude the proof by noticing that there are m^2 vertices in $[\square_m, k]$. □

And thus we get:

Proposition 4.2.14. *There exists an isomorphism of translation quivers between $(\tilde{\Gamma}_{A_{nm-1}})^m$ and the disjoint union*

$$\mathbb{Z}A_{n-1} \coprod \left(\coprod_{i=1}^{r} \mathbb{Z}A_n\right)$$

where $r = m^2 - 1$.

Proof. Clear from Lemma 4.2.13 and Lemma 4.2.8. □

Let us focus again on the quiver $\Gamma^1_{A_{nm-1}}$ and its mth power. By construction of $\tilde{\Gamma}_{A_{nm-1}}$, the quiver $\Gamma^1_{A_{nm-1}}$ is isomorphic to

$$\tilde{\Gamma}_{A_{nm-1}}/\eta_1,$$

where η_s is defined to be the automorphism of translation quivers given by

$$\eta_s[(i,j),k] = [(i,j),k+s]$$

for $s \in \mathbb{Z}$. Consider the functor $\mathcal{P}^m$ introduced in Definition 3.2.2. Then $\mathcal{P}^m(\eta_s) := \eta_s^{(m)}$ is an automorphism of translation quivers for all $s \in \mathbb{Z}$. Moreover, Corollary 4.2.10 implies that $\eta_{2m}|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ is an automorphism of translation quivers for all m and for every connected component $\mathcal{Q}$ of $(\tilde{\Gamma}_{A_{nm-1}})^m$.

Lemma 4.2.15. *Let $(\tilde{\Gamma}_{A_{nm-1}}, \tau)$ be defined as above and $\mathcal{Q}$ be one of the connected components of $(\tilde{\Gamma}_{A_{nm-1}})^m$. Then:*

1. *for m odd, $2m$ is the minimal natural number such that*

$$\eta_{2m}|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$$

is an automorphism of translation quivers.

2. *for m even, $m' = m$ or $\frac{m}{2}$ is the minimal natural number such that*

$$\eta_{m'}|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$$

is an automorphism of translation quivers.

Proof. Clear from Corollary 4.2.10. □

Therefore:

Proposition 4.2.16. *Assume that the vertex $[(i,j),k]$ of $\tilde{\Gamma}_{A_{nm-1}}$ belongs to the connected component $\mathcal{Q}$ of $(\tilde{\Gamma}_{A_{nm-1}})^m$. Then the connected component of $(\Gamma^1_{A_{nm-1}})^m$ containing (i,j) is isomorphic to*

1. *$\mathcal{Q}/\eta_{2m}$ if m is odd.*
2. *$\mathcal{Q}/\eta_m$ or $\mathcal{Q}/\eta_{\frac{m}{2}}$ if m is even.*

Proof. Clear from Lemma 4.2.15.

In the following, we will use the properties of $(\tilde{\Gamma}_{A_{nm-1}})^m$ to find how many connected components there are in $(\Gamma^1_{A_{nm-1}})^m$. In order to clarify the notation, $\mathcal{Q}$ will always designate a connected component of $(\tilde{\Gamma}_{A_{nm-1}})^m$ whereas $\mathcal{X}$ will designate a connected component of $(\Gamma^1_{A_{nm-1}})^m$.

Lemma 4.2.17. *Every connected component of $(\Gamma^1_{A_{nm-1}})^m$ has at least one vertex in Δ_m.*

Proof. Clear from Lemma 4.2.12 part 2. as

$$\Gamma^1_{A_{nm-1}} \cong \tilde{\Gamma}_{A_{nm-1}}/\eta_1.$$

□

Lemma 4.2.18. *Let (i,j) some vertex in Δ_m situated in row s. Then the component $\mathcal{X}$ of $(\Gamma^1_{A_{nm-1}})^m$ to which (i,j) belongs has vertices in all rows*

$$s + \alpha m \text{ and } nm - (s + \alpha m) \text{ for } 0 \le \alpha \le \lfloor \frac{nm-1-s}{m} \rfloor$$

and only in these.

Proof. From Lemma 4.2.7, it is clear that $\mathcal{X}$ has vertices in rows $s + \alpha m$ for $0 \le \alpha \le \lfloor \frac{nm-1-s}{m} \rfloor$. Finally we can complete the proof by noticing that if $[(i,j),k]$ (k even) is in row s in $\tilde{\Gamma}_{A_{nm-1}}$, then $[(i,j),k+1]$ is in row $nm - s$ and vice versa. □

Finally this implies that:

Lemma 4.2.19. *Each connected component of* $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ *has vertices in exactly one row out of the set of rows* $\mathcal{R} = \{1, 2, \cdots, \lceil \frac{m-1}{2} \rceil\} \cup \{m\}$.

Proof. From Lemma 4.2.20, we know that each component has at least one vertex in the fundamental region Δ_m of $\Gamma^1_{A_{nm-1}}$ and Δ_m lies in the rows $\{1, 2, \cdots, m\}$. Furthermore if the vertex (i, j) of Δ_m lies in row $s \notin \mathcal{R}$, then the same reasoning as in the proof of Lemma 4.2.18 implies that there are vertices in the same component of $(\Gamma^1_{A_{nm-1}})^m$ in row

$$(nm - s) - \lfloor \frac{nm - 1 - s}{m} \rfloor m \text{ which is in } \mathcal{R}.$$

Finally Lemma 4.2.18 implies that there are vertices of a connected component of $(\Gamma^1_{A_{nm-1}})^m$ in only one row of $\mathcal{R}$. □

Thus we have limited the number of connected components of $(\Gamma^1_{A_{nm-1}})^m$ to the number of vertices in the fundamental region which are in the set of rows $\mathcal{R} = \{1, 2, \cdots, \lceil \frac{m-1}{2} \rceil\} \cup \{m\}$. The following lemma shows how many different components there are in each row:

Lemma 4.2.20. *The number of connected components of* $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ *per row is*

1. *equal to* 1 *for* m *odd.*
2. *at most equal to* 2 *for* m *even.*

Proof. The proof of Proposition 4.2.6 implies that $\tau^{-\alpha}(i, j) = (i+\alpha, j+\alpha)$ belongs to the same component $\mathcal{X}$ of $(\Gamma^1_{A_{nm-1}})^m$ as (i, j) if and only if either

$$i \equiv i + \alpha \text{ and } j \equiv j + \alpha \mod m$$

or

$$i \equiv j + \alpha \text{ and } j \equiv i + \alpha \mod m,$$

i.e. $\alpha = \beta m$, for $\beta \in \mathbb{Z}$. Moreover

$$\tau^{-\beta m}(i, j) = (i, j) \text{ if and only if } \beta m \equiv 0 \mod nm + 2,$$

i.e:

1. for m odd, the minimal natural number β_{min} such that

$$\beta_{min} m \equiv 0 \mod nm + 2$$

   is equal to $nm + 2$, that is, in each row of $\mathcal{X}$ there are at least consecutively $nm + 2$ different vertices of $\Gamma^1_{A_{nm-1}}$.

2. for m even, the minimal natural number β_{min} such that

$$\beta_{min} m \equiv 0 \mod nm+2$$

is equal to $\frac{nm+2}{2}$, that is, in each row of $\mathcal{X}$ there are at least consecutively $\frac{nm+2}{2}$ different vertices of $\Gamma^1_{A_{nm-1}}$.

And as per row of $\mathcal{X}$ there are at most consecutively $nm+2$ different vertices of $\Gamma^1_{A_{nm-1}}$ we get:

1. for m odd, the maximal number of connected components of $(\Gamma^1_{A_{nm-1}})^m$ per row is equal to 1.
2. for m even, the maximal number of connected components of $(\Gamma^1_{A_{nm-1}})^m$ per row is equal to 2.

□

Furthermore:

Lemma 4.2.21.

1. *For all m, the connected components of $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ having vertices in the same rows are isomorphic.*
2. *For all m, the connected components of $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ having vertices in row s, $s \leq \lfloor \frac{m-1}{2} \rfloor$ are isomorphic.*
3. *For m even, if there are two (not one) connected components in row $\frac{m}{2}$, then the two connected components of $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ having vertices in row $\frac{m}{2}$ are not isomorphic to the connected components having vertices in row s, $s < \frac{m}{2}$.*

Proof.

1. Consider the translation quiver $((\tilde{\Gamma}_{A_{nm-1}})^m, \tau^m)$. Then for 2 connected components $\tilde{X}$, $\tilde{X}'$ having vertices in the same rows, there exists $\alpha \in \mathbb{Z}$ such that $\tau^\alpha(\tilde{X}) = \tilde{X}'$ is an isomorphism of translation quivers. Furtermore $\tau^\alpha(\tilde{X}/\eta_{\gamma m}) = \tilde{X}'/\eta_{\gamma m}$ for all $\gamma \in \mathbb{Z}$ (τ^α and $\eta_{\gamma m}$ commutes), in particular there exists an isomorphism of translation quivers between the components of $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ having vertices in the same row.
2. As every component having vertices in the same row are isomorphic, it suffices to look at the components of $(\tilde{\Gamma}_{A_{nm-1}})^m$, $\tilde{X}_3$, $\tilde{X}_4$, $\cdots$, $\tilde{X}_{\lfloor \frac{m-1}{2} \rfloor}$ where the component $\tilde{X}_t$ contains the vertex $[(1,t),0]$.
For $3 \leq t \leq \lfloor \frac{m-1}{2} \rfloor - 1$, we define the morphism of translation quivers

$$\mu_t : \tilde{X}_t \to \tilde{X}_{t+1}$$

given by

$$\mu_t[(i,j),k] = \begin{cases} [(i,j+1),k] & \text{if } j+1 < nm+2 \\ [(j+1,i),k+1] & \bmod nm+2 \text{ else.} \end{cases}$$

It can be checked that μ_t is well-defined and is an isomorphism of translation quivers. Furthermore

$$\mu_t(\tilde{X}_t/\eta_{\gamma m}) = \tilde{X}_{t+1}/\eta_{\gamma m}, \text{ for all } \gamma m \text{ even,}$$

(as $\eta_{\gamma m}$ and μ_t commutes for γm even) in particular there exists an isomorphism of translation quivers between the components of $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ having vertices in row s, where $s \leq \lfloor \frac{m-1}{2} \rfloor$.

3. For m even, if there are two (not one) connected components in row $\frac{m}{2}$, then we claim that the two components of $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ having vertices in row $\frac{m}{2}$ are isomorphic but they are not isomorphic to the other connected components. Consider the components of $(\Gamma^1_{A_{nm-1}})^m$, X_3 and $X_{\frac{m}{2}}$, where X_3 is the component which contains vertex $(1,3)$ and $X_{\frac{m}{2}}$ is the component which contains vertex $(1, \frac{m}{2})$. We want to show that the number of vertices in X_3 is bigger that the number of vertices in $X_{\frac{m}{2}}$: By Lemma 4.2.8, X_3 has vertices in rows $1+\alpha m$ and $nm-1-\alpha m$ where $0 \leq \alpha \leq n-1$ and $1+\alpha m \neq nm-1-((n-1)\alpha)m$ i.e in $2n$ different rows, whereas $X_{\frac{m}{2}}$ has vertices in rows $\frac{m}{2}+\alpha m$ and $nm - \frac{m}{2} - \alpha m$ where $\frac{m}{2} + \alpha m = nm - \frac{m}{2} - ((n-1)-\alpha)m$ i.e in n different rows.

   By construction there are $nm-s+1$ vertices in row s in $\Gamma^1_{A_{nm-1}}$. This implies that the total number of vertices in X_3 is bigger or equal to

$$\frac{\sum_{\alpha=0}^{n-1} nm+1-(1+\alpha m)+nm+1-(nm-1-\alpha m)}{2} = \frac{(n-1)nm}{2}.$$

   Whereas the total number of vertices in $X_{\frac{m}{2}}$ is equal to:

$$\frac{\sum_{\alpha=0}^{n-1} nm+1-(1+\alpha m)}{2} = \frac{(n-1)nm}{4}.$$

□

Remark.

1. In the proof of Lemma 4.2.21, we need to assume that there are two connected components in row $\frac{m}{2}$, as if this is not the case, the inegality between the number of vertices of a component having vertices in row s, $s \leq \lfloor \frac{m-1}{2} \rfloor$ and of a component having vertices in row $\frac{m}{2}$ is not strict.

2. The reader can check (for example by using the algorithm given in Annexe B) that for $n = 2$, $m = 6$ there are two connected components in row 3, whereas for $n = 2$ or 3 and $m = 4$ there are only one connected component in row 2. Furthermore for the two last cases the connected component having vertices in row 2 is isomorphic to the two connected components having vertices in row 1.

For the rest of this section, we will only consider the case where m is odd. In this case $\lfloor \frac{m-1}{2} \rfloor = \lceil \frac{m-1}{2} \rceil = \frac{m-1}{2}$ and by Lemma 4.2.20 we know that there is only one connected component per row. Furthermore by Lemma 4.2.21, we know that the connected components of $(\Gamma^1_{A_{nm-1}})^m$ with vertices in rows i, $1 \leq i \leq \frac{m-1}{2}$, are isomorphic. Let $\mathcal{X}$ be such a connected component and $\mathcal{X}_m$ be the connected component with vertices in row m. Then by Proposition 4.2.16 we know that:

$$\mathcal{X} \cong \mathcal{Q}/\eta_{2m} \text{ and } \mathcal{X}_m \cong \mathcal{Q}_m/\eta_{2m}$$

where $\mathcal{Q}$ designates a connected component of $(\tilde{\Gamma}_{A_{nm-1}})^m$ isomorphic to $\mathbb{Z}A_n$ and $\mathcal{Q}_m$ designates a connected component isomorphic to $\mathbb{Z}A_{n-1}$.

We would like to find the automorphisms of $\mathbb{Z}A_n$ and $\mathbb{Z}A_{n-1}$ corresponding to η_{2m}, that is an automorphism of $\mathbb{Z}A_n$, φ, such that:

$$\mathcal{Q}/\eta_{2m} \cong \mathbb{Z}A_n/\varphi$$

and an automorphism of $\mathbb{Z}A_{n-1}$, φ_m, such that:

$$\mathcal{Q}_m/\eta_{2m} \cong \mathbb{Z}A_{n-1}/\varphi_m.$$

In Proposition 3.3.1, we have proved that $\mathcal{X}_m$ corresponds to the translation quiver of m-diagonals, $\Gamma^m_{A_{n-1}}$. Therefore in the following we will substitute $\mathcal{X}_m$ by $\Gamma^m_{A_{n-1}}$. Furthermore, by Proposition 4.1.6

$$\Gamma^m_{A_{n-1}} \cong \mathbb{Z}A_{n-1}/\tau^{-1} \circ [m]$$

and we can conclude that $\varphi_m = \tau^{-1} \circ [m]$.

We would also like to express φ in function of the shift $[-]$ and of the translation τ. In order to restrict the possibilities for φ, we look in a first step for $r \in \mathbb{N}$ such that:

1. $\mathbb{Z}A_n/[r]$ is a subquiver of $\mathbb{Z}A_n/\varphi$;

2. $\mathbb{Z}A_n/[r+1]$ is a quiver having $\mathbb{Z}A_n/\varphi$ as subquiver.

First notice that if the number of vertices in $\mathbb{Z}A_n/\varphi$ is equal to $\mathbf{y}$, then the number of vertices of $\mathbb{Z}A_n/[r]$ is less than $\mathbf{y}$ and the number of vertices of $\mathbb{Z}A_n/[r+1]$ is more than $\mathbf{y}$. Hence one method to find r is to calculate the number of vertices of $\mathbb{Z}A_n/\varphi$ (i.e of $\mathcal{X}$), of $\mathbb{Z}A_n/[r]$ and of $\mathbb{Z}A_n/[r+1]$.

Lemma 4.2.22. *Consider* $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$*, m odd and its connected components*

$$\Gamma^m_{A_{n-1}},\, \mathcal{X}_t\, (t \in \{1, \cdots, \frac{m-1}{2}\}).$$

Then:

1. $(\Gamma^1_{A_{nm-1}})^m$ *has* $(nm-1)(\frac{nm}{2}+1)$ *vertices;*
2. $\Gamma^m_{A_{n-1}}$ *has* $(n-1)(\frac{nm}{2}+1)$ *vertices;*
3. $\mathcal{X}_t$ *has* $n(nm+2)$ *vertices* $\forall\, t \in \{1, \cdots, \frac{m-1}{2}\}$.

Proof. We know that the number of vertices in $(\Gamma^1_{A_{nm-1}})^m$ is equal to the number of vertices in $\Gamma^1_{A_{nm-1}}$. Hence it is equal to:

$$\sum_{i=1}^{nm-1} i + (nm-1) = \frac{(nm-1)(nm)}{2} + (nm-1) = (nm-1)\Big(\frac{nm}{2}+1\Big).$$

Whereas the number of vertices in $\Gamma^m_{A_{n-1}}$ is equal to:

$$\Big(\sum_{i=1}^{n-1} i\Big)m + (n-1) = (n-1)\Big(\frac{nm}{2}+1\Big).$$

Hence the total number of vertices in the remaining connected components is equal to

$$(nm-1)\Big(\frac{nm}{2}+1\Big) - (n-1)\Big(\frac{nm}{2}+1\Big) = n(m-1)\Big(\frac{nm}{2}+1\Big).$$

By Lemma 4.2.21 each of these $\frac{m-1}{2}$ component has the same number of vertices. Hence the number of vertices of $\mathcal{X}_t$, $1 \le t \le \frac{m-1}{2}$, is equal to:

$$n(nm+2).$$

□

Lemma 4.2.23. *The number of vertices in* $\mathbb{Z}A_n/[r]$ *for* $r \in \mathbb{N}^*$ *is equal to*

$$\Big(\sum_{i=1}^{n} i\Big)r = \frac{n(n+1)}{2}r.$$

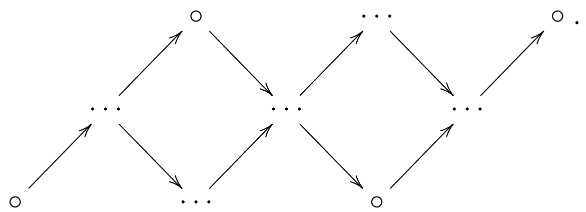
Proof. Clear. □

Theorem 4.2.24. *The maximal natural number r such that for all* $n \in \mathbb{N}$*:*

1. $\mathbb{Z}A_n/[r]$ *is a subquiver of* $\mathbb{Z}A_n/\varphi$
2. $\mathbb{Z}A_n/[r+1]$ *has* $\mathbb{Z}A_n/\varphi$ *as a subquiver,*

is equal $m+1$ *for* m *odd.*
Moreover, for m *odd,* $\varphi = \tau^{-\alpha} \circ [r]$ *where* $\alpha = \frac{m-1}{2}n - \frac{m-3}{2}$.

Proof. First note that as $\mathbb{Z}A_n/\varphi$ has $n(nm+2)$ vertices arranged in n rows, then it has to have the general shape of a parallelogram:



and since, $\mathbb{Z}A_n$ has the shape of r triangulated regions arranged side by side, r is even.

Using Lemma 4.2.22 and 4.2.23 we claim that $r = 4 + (m-3)$ is the maximal even natural number such that for all $n \in \mathbb{N}$

$$n(nm+2) = r\frac{n(n+1)}{2} + s$$

with $0 \leq s < \frac{n(n+1)}{2}$. If we decompose $n(nm+2)$ as follows

$$4\frac{n^2+n}{2} + (m-2)n^2$$

for $m \neq 1$ (case $m = 1$ is trivial), then we get the decomposition

$$(m-2)n^2 = (m-3)\frac{n^2+n}{2} + s', \text{ where } s' = \frac{m-1}{2}n^2 - \frac{m-3}{2}n.$$

Hence $r = 4 + (m-3) = m+1$ satisfies the above conditions.

Finally from the decomposition

$$n(nm+2) = (m+1)\frac{\mathbf{n^2+n}}{\mathbf{2}} + (\frac{m-1}{2}n - \frac{m-3}{2})\mathbf{n},$$

we get that $\varphi = \tau^{-\alpha} \circ [m+1]$ where $\alpha = \frac{m-1}{2}n - \frac{m-3}{2}$. □

Remark. The expression of φ in function of the shift and the translation is not unique. In Theorem 4.2.24, for example, we have looked for the expression of $\varphi = \tau^{-\alpha} \circ [r']$ with r' the maximal even number. But we could also express φ in function of τ^{-1} only, that is

$$\varphi = \tau^{-(nm+2)}.$$

Example 20. 1. Let $m = 3$ then $n(n3+2) = 3n^2 + 2n = \frac{4n^2+4n}{2} + n^2$. Hence $r = 4$ and $\alpha = n$.

2. Let $m = 5$ then $n(n5+2) = 5n^2+2n = \frac{4n^2+4n}{2}+3n^2$ but $3n^2 \geq \frac{n(n+1)}{2}$. Actually, we can always decompose it as follows:

$$3n^2 = 2\frac{n(n+1)}{2} + 2n^2 - n$$

with $2n^2 - n \geq 0$ for all n. Hence $r = 6$ and $\alpha = 2n - 1$.

3. Let $m = 7$ then $n(n7+2) = 7n^2+2n = \frac{4n^2+4n}{2}+5n^2$ but $5n^2 \geq \frac{n(n+1)}{2}$. Actually, we can always decompose it as follows:

$$5n^2 = 4\frac{n(n+1)}{2} + 3n^2 - 2n$$

with $3n^2 - 2n \geq 0$ for all n. Hence $r = 8$ and $\alpha = 3n - 2$.

4.3 Final remarks

As stated in the introduction, the goal of this thesis is to get a better understanding of the connected components of $((\Gamma^1_{A_{nm-1}})^m, \tau^m)$. For the case where m is odd we think that we have achieved this goal. In Proposition 4.2.16, part 1. we were able to express the connected components of $(\Gamma^1_{A_{nm-1}})^m$ in function of the connected components of $(\tilde{\Gamma}_{A_{nm-1}})^m$, that is:

$$\mathcal{X} \cong \mathcal{Q}/\eta_{2m} \text{ and } \mathcal{X}_m \cong \mathcal{Q}_m/\eta_{2m}$$

where $\mathcal{X}$ (respectively $\mathcal{Q}$) is some connected component of $(\Gamma^1_{A_{nm-1}})^m$ (resp. of $(\tilde{\Gamma}_{A_{nm-1}})^m$) having vertices in row s, $s \leq \frac{m-1}{2}$ and $\mathcal{X}_m$ (resp. $\mathcal{Q}_m$) is the connected component of $(\Gamma^1_{A_{nm-1}})^m$ (resp. of $(\tilde{\Gamma}_{A_{nm-1}})^m$) having vertices in row m. Then in Theorem 4.2.24, we have shown that $(\Gamma^1_{A_{nm-1}})^m$ is isomorphic to the disjoint union

$$\mathbb{Z}A_{n-1}/\tau^{-1} \circ [m] \bigcup \Big(\cup_{i=1}^{\frac{m-1}{2}} \mathbb{Z}A_n/\tau^{-\alpha} \circ [m+1] \Big)$$

where $\alpha = \frac{m-1}{2}n - \frac{m-3}{2}$.

For the case m even, due to time constraints, we were not such successful. The main remaining difficulty is to know when there is only one connected component per row and when there are two components. Here is a summary of what we could prove so far:

1. $\mathcal{X} \cong \mathcal{Q}/\eta_m$ or $\mathcal{X} \cong \mathcal{Q}_m/\eta_{\frac{m}{2}}$ where $\mathcal{X}$ (resp. $\mathcal{Q}$) is some connected component of $(\Gamma^1_{A_{nm-1}})^m$ (resp. of $(\tilde{\Gamma}_{A_{nm-1}})^m$) having vertices in row s, $s \leq \frac{m}{2}$ and $\mathcal{X}_m \cong \mathcal{Q}_m/\eta_m$ where $\mathcal{X}_m$ (resp. $\mathcal{Q}_m$) is the connected component of $(\Gamma^1_{A_{nm-1}})^m$ (resp. of $(\tilde{\Gamma}_{A_{nm-1}})^m$) having vertices in row m (see Proposition 4.2.16, part 2.).

2. The number of connected components of $(\Gamma^1_{A_{nm-1}})^m$ per row is at most equal to 2 (see Lemma 4.2.20, part 2).

3. The connected components of $(\Gamma^1_{A_{nm-1}})^m$ having vertices in row s, where $s \leq \frac{m}{2} - 1$, are isomorphic (see Lemma 4.2.21, part 2).

4. If there are two (not one) connected components in row $\frac{m}{2}$, then the two connected components of $((\Gamma^1_{A_{nm-1}})^m, \tau_1^m)$ having vertices in row $\frac{m}{2}$ are isomorphic but they are not isomorphic to the connected components having vertices in row s, $s < \frac{m}{2}$ (see Lemma 4.2.21, part 3).

We have also seen examples where the number of connected components having vertices in row $\frac{m}{2}$ is equal to one or two, thus showing that both cases are possible. Our intuition is that for the connected components having vertices in row s, $s \leq \frac{m}{2} - 1$, the number of components per row is always equal to two. This intuition is confirmed by several examples calculated with the algorithm of Annexe B.

For further analysis, here are two possible extensions of the problems treated here:

1. The understanding of the mth power of $\Gamma^1_{A_{N-3}}$ for a given N-gon.

2. The understanding of the m-cluster category $\mathcal{C}^m_Q$ of type Q, for Q the Dynkin graph of type E_6, E_7 or E_8 (the type D has already been studied by Baur-March [BMD]).

Appendix A

A.1 The Auslander-Reiten translation

In this section we present the construction of the Auslander-Reiten translation on the category $\operatorname{mod} kQ$ first and secondly on the bounded derived category $D^b(kQ)$ (where Q is an acyclic quiver). We then show how these two constructions are related.

Convention A.1.1. *Throughout this section, we will assume that Q is an acyclic quiver. The associated path algebra kQ will be denoted by A.* $\operatorname{mod} A$ *will denote the category of finitely generated left A-modules and* $\operatorname{mod}^r A$ *the category of finitely generated right A-modules. We denote by* $\operatorname{proj} A$ *(resp.* $\operatorname{proj}^r A$*) the full subcategory of* $\operatorname{mod} A$ *whose objects are the projective left (resp. right) A-modules and by* $\operatorname{inj} A$ *(resp.* $\operatorname{inj}^r A$*) the full subcategory of* $\operatorname{mod} A$*, whose objects are the injective left (resp. right) A-modules.*

Lemma A.1.2.

1. *If X is a left or right A-module, then*

$$DX := Hom_k(X, k) \text{ and } X^t := Hom_A(X, A)$$

are A-modules on the other side.

2. *D is a duality functor between* $\operatorname{mod} A$ *and* $\operatorname{mod}^r A$.

3. *D gives a duality between injective left modules and projective right modules.*

4. *The restriction of $(-)^t = Hom_A(-, A) : \operatorname{mod} A \to \operatorname{mod}^r A$ to* $\operatorname{proj} A$ *induces an equivalence between* $\operatorname{proj} A$ *and* $\operatorname{inj}^r A$.

Proof. See [CB] page 21. □

In the following we present the DTr-construction of the Auslander-Reiten translation. But first we need to give the construction of the duality called *transposition* and denoted by Tr:

1. Let $X \in \operatorname{mod} A$. Consider

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X$$

a *minimal projective presentation* of X, that is, an exact sequence such that $p_0 : P_0 \to M$ and $p_1 : P_1 \to Ker\, p_0$ are projective covers.

2. Apply the functor $(-)^t$ (left exact, contravariant) to the sequence such that we get:

$$0 \longrightarrow X^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow Coker\, p_1^t \longrightarrow 0\ .$$

Then $Coker\, p_1^t$ is denoted by $Tr\, X$ and called the *transposition* of X.

The transposition Tr maps modules of $\operatorname{mod} A$ to modules of $\operatorname{mod}^r A$ but does not define a duality between $\operatorname{mod} A$ and $\operatorname{mod}^r A$, because it annihilates the projectives. In order to make this correspondence a duality, we thus need to forget the projectives from $\operatorname{mod} A$ and $\operatorname{mod}^r A$.

Definition A.1.3. *Let $M,\ N \in \operatorname{mod} A$. We define $\mathcal{P}(M,N)$ to be the subset of $Hom_A(M,N)$ consisting of all homomorphisms that factor through a projective A-module. It is easily proved that $\mathcal{P} := \{\mathcal{P}(M,N)\,|\, M,\ N \in \operatorname{mod} A\}$ is an ideal in the category $\operatorname{mod} A$. Thus we can define the quotient category:*

$$\underline{mod}\, A = \operatorname{mod} A/\mathcal{P}$$

called the projectively stable category.

Proposition A.1.4. *The correspondence $M \mapsto Tr\, M$ induces a k-linear duality functor $Tr : \underline{mod}\, A \to \underline{mod}^r\, A$.*

Proof. See [ASS] Proposition 2.2 Chapter 4. □

Definition A.1.5. *The DTr-*Auslander-Reiten translation on $\operatorname{mod} A$ *is defined by:*

$$\tau = DTr \text{ and } \tau^{-1} = TrD.$$

Remark. If we apply the functor D to the above sequence

$$0 \longrightarrow X^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow Coker\, p_1^t \longrightarrow 0\ ,$$

we get:

$$0 \longrightarrow DTr(X) \xrightarrow{p_0^t} I_1 \xrightarrow{D(p_1)^t} I_0 \longrightarrow D(X^t) \longrightarrow 0\ ,$$

where $I_0 = D(P_0)^t$ and $I_1 = D(P_1)^t$ are injective modules.

In the following, we state without proving them some useful results linked to the DTr-Auslander-Reiten translation:

Proposition A.1.6. *Let M and N be indecomposable modules in* $\operatorname{mod} A$.

(a) The module τM is zero if and only if M is projective.

(a') The module $\tau^{-1}N$ is zero if and only if N is injective.

(b) If M is a non-projective module, then τM is an indecomposable non-injective module and $\tau^{-1}\tau M \cong M$.

(b') If N is a non-injective module, then $\tau^{-1}N$ is an indecomposable non-projective module and $\tau\tau^{-1}N \cong N$.

(c) If M and N are non-projective modules, then $M \cong N$ if and only if there is an isomorphism $\tau M \cong \tau N$.

(c') If M and N are non-injective modules, then $M \cong N$ if and only if there is an isomorphism $\tau^{-1}M \cong \tau^{-1}N$.

Proof. See [ASS] Proposition 2.10 Chapter 4. □

We define now the *Auslander-Reiten translation on* $D^b(A)$, τ_D. In order to do that, we need to use the Nakayama automorphism

$$\nu(-) := DHom_A(-, A).$$

Let $Z \in ind\, D^b(A)$. By Happel [HD] page 49, we know that Z is a stalk complex with X an indecomposable module of A in some degree of Z. We can thus identify Z with $\dot{X}$. If X is an indecomposable projective module, i.e $X = Ae_i := P(i)$ for some $i \in Q_0$, then we have that

$$\nu(P(i)) = Hom_k(e_iA, k) := I(i)$$

(see Example 5) and we set $\tau_D(\dot{X})$ to be $I(i)[-1]$.

Assume now that $Z = \dot{X}$ is a stalk complex concentrated in degree zero where X is a non-projective module. To compute $\tau_D(\dot{X})$ in the derived category, we use the following construction:

1. Replace X by its projective resolution

$$\cdots \longrightarrow 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots,$$

regarded as a complex with P_0 in degree 0, P_1 in degree -1 and zeros everywhere else.

2. Apply ν to get

$$0 \longrightarrow I_1 \longrightarrow I_0 \longrightarrow 0$$

with I_0 in degree 0.

3. Apply $[-1]$ to get $0 \longrightarrow I_1 \longrightarrow I_0 \longrightarrow 0$ with I_1 in degree 0. This is then the injective resolution of a module Y

4. Set $\tau_D(\dot{X}) = \dot{Y}$, where $\dot{Y}$ is the stalk complex with the module Y of 3. in degree 0.

Definition A.1.7. *The* Auslander-Reiten translation on $D^b(A)$ *is defined by:*

$$\tau_D = [-1]\nu.$$

Remark.

1. In the construction of τ, $\tau(X) = Y$ is the kernel of $I_1 \xrightarrow{D(p_1)^t} I_0$.

2. In the construction of $\tau_D(\dot{X}) = \dot{Y}$ with $\dot{X}$ a stalk complex concentrated in degree zero and X a non-projective module, Y is the kernel of $I_1 \longrightarrow I_0$.

It can be shown that $I_1 \xrightarrow{D(p_1)^t} I_0$ of 1. and $I_1 \longrightarrow I_0$ of 2. are the same. Hence we have the following lemma:

Lemma A.1.8. *Let X,Y be non-zero indecomposable A-modules. Then*

$$\tau(X) = Y \text{ if and only if } \tau_D(\dot{X}) = \dot{Y}$$

where $\dot{X}$, $\dot{Y}$ are stalk complexes concentrated in degree zero.

A.2 Irreducible morphisms

We recall here the notion of irreducible, minimal and almost split morphisms in the category $\operatorname{mod} A$ of finite-dimensional left A-modules along with some interesting proprieties of left or right almost split sequences. The following results can be found in Chapter 4 of [ASS]. Notice that [ASS] considers the category of right A-modules instead of left A-modules.

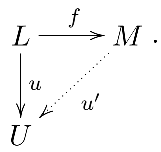
Definition A.2.1. *Let L, M be in* $\operatorname{mod} A$ *and let $f : L \to M$ be a homomorphism of A-modules.*

1. *f is called* left minimal *if every $h \in End\, M$ satisfying $hf = f$ is an automorphism.*

2. *f is called* left almost split *if:*

   (a) *f is not a section and*

(b) *for every A-module homomorphism* $u : L \to U$ *that is not a section there exists* $u' : M \to U$ *such that* $u'f = u$*, that is,* u' *makes the following triangle commutative:*



3. f *is called* left minimal almost split *if it is both left minimal and left almost split.*

The notions dual to left minimal, left almost split and left minimal almost split notions are called right minimal, right almost split and right minimal almost split.

In the following we only state the result for the "left-hand" notions. The dual results are true for the "right-hand" ones.

Proposition A.2.2. *If the homomorphisms* $f : L \to M$ *and* $f' : L \to M'$ *are left minimal almost split, then there exists an isomorphism* $h : M \to M'$ *such that* $f' = hf$.

Proof. As f and f' are almost split, then there exist $h : M \to M'$ and $h' : M' \to M$ such that $f' = hf$ and $f = h'f'$. Hence $f = h'hf$ and $f' = hh'f'$. As f and g' are minimal, hh' and $h'h$ are automorphisms. Hence h is an isomorphism. □

There is a strong relation between almost split morphisms and indecomposable modules:

Lemma A.2.3. *If* $f : L \to M$ *is a left almost split morphism in* $\operatorname{mod} A$*, then the module* L *is indecomposable.*

Proof. Assume that L is decomposable, that is there exist L_1 and L_2 nonzero A-modules such that $L = L_1 \oplus L_2$. Denote by $p_i : L \to L_i$ $(i = 1, 2)$ the corresponding projections. For both i, p_i is not a section. Hence there exists a homomorphism $u_i : M \to L_i$ such that $u_i f = p_i$. But then

$$u = [u_1, u_2]^T : M \to L$$

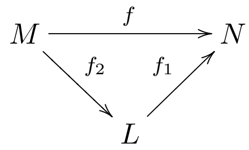
satisfies $uf = 1_L$, which is a contradiction to f not being a section. □

Another useful propriety of an A-module homomorphism is its potential irreducibility. More exactly:

Definition A.2.4. *A homomorphism* $f : M \to N$ *in* $\operatorname{mod} A$ *is said to be* irreducible *if:*

1. *f is neither a section nor a retraction and*

2. *if $f = f_1 f_2$, either f_1 is a retraction or f_2 is a section.*



Remark. An irreducible morphism in $\operatorname{mod} A$ is either a proper monomorphism or a proper epimorphism.

We can reformulate the notion of irreducible morphisms by using the notion of *radical* rad_A of the category $\operatorname{mod} A$ where

$$\operatorname{rad}_A := \{\operatorname{rad}_A(X,Y) \mid X, Y \in \operatorname{mod} A \text{ indecomposable}\};$$

$\operatorname{rad}_A(X,Y)$ is the k-vector space of all non-invertible homomorphisms from X to Y. Similarly we define $\operatorname{rad}^2_A(X,Y)$ to consist of all A-module homomorphisms of the form gf, where $f \in \operatorname{rad}_A(X,Z)$ and $g \in \operatorname{rad}_A(Z,Y)$ for some (not necessarily indecomposable) A-module Z.

Lemma A.2.5. *Let X, Y be indecomposable modules in $\operatorname{mod} A$. A morphism $f : X \to Y$ is irreducible if and only if $f \in (\operatorname{rad}_A(X,Y) \setminus \operatorname{rad}^2_A(X,Y)$.*

Proof. Assume that f is irreducible. Then clearly, $f \in \operatorname{rad}_A(X,Y)$. If $f \in \operatorname{rad}^2_A(X,Y)$, then f can be written as $f = gh$, with $h \in \operatorname{rad}_A(X,Z)$ and $g \in \operatorname{rad}_A(Z,Y)$ for some Z in $\operatorname{mod} A$. We show that h (resp. g) is not a section (resp. not a retraction), thus contradicting f being irreducible.

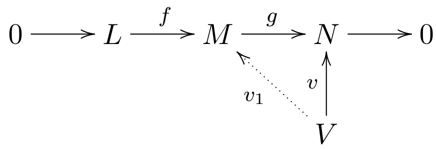
Without loss of generality, assume that Z is indecomposable. Assume h is a section and let $h' : Z \to X$ be such that $1_X = h'h$. Because h is non-invertible, so is hh'. Hence $h'h \in \operatorname{rad}_A(X,X) = \operatorname{rad} End\,X$. Because X is indecomposable, $End\,X$ is local and it follows that $1_X \in \operatorname{rad} End\,X$, a contradiction. It can be shown similarly that g is not a retraction.

Conversely assume $f \in \operatorname{rad}_A(X,Y) \setminus \operatorname{rad}^2_A(X,Y)$. Because X and Y are indecomposable and f is not an isomorphism, it is clearly neither a section nor a retraction. Suppose $f = gh$, where $h : X \to Z$, $g : Z \to Y$. Without loss of generality, we assume that Z is indecomposable. As $f \notin \operatorname{rad}^2_A(X,Y)$, either h or g are invertible. In the first case h is a section; in the second case, g is a retraction. □

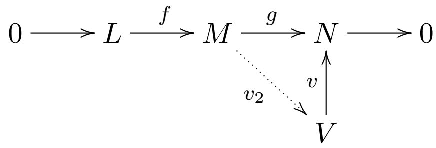
In the following, we characterize the irreducible morphisms by their kernel (or cokernel):

Lemma A.2.6. *Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a non-split short exact sequence in $\operatorname{mod} A$.*

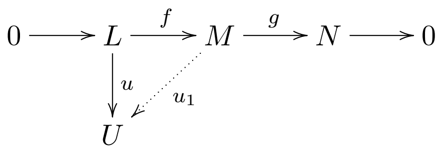
1. *The homomorphism f is irreducible if and only if, for every non-trivial homomorphism $v : V \to N$, there exists either $v_1 : V \to M$ such that $v = gv_1$*



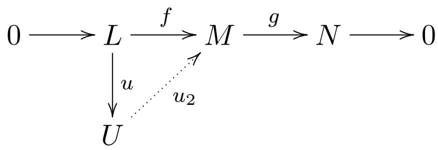
or there exists $v_2 : M \to V$ such that $g = vv_2$.



2. *The homomorphism g is irreducible if and only if, for every non-trivial homomorphism $u : L \to U$, there exists either $u_1 : M \to U$ such that $u = u_1 f$*



or there exists $u_2 : U \to M$ such that $f = u_2 u$.



Proof. See [ASS], Chapter 4, Lemma 1.7. □

We show now the relation between minimal almost split homomorphisms and irreducible one. But first we need an easy lemma which will be useful later on:

Lemma A.2.7. *1. Let $f : L \to M$ be a non-zero A-module homomorphism, with L indecomposable. Then f is not a section if and only if $Hom_A(f, L) \subseteq \operatorname{rad} End\, L$, where $Hom_A(f, L)(h) := hf$ for all $h : Z \to L$.*

2. *Let $g : M \to N$ be a non-zero A-module homomorphism, with N an indecomposable module. Then g is not a retraction if and only if $Im\, Hom_A(N, g) \subseteq \operatorname{rad} End\, N$.*

Proof. We prove (1); the proof of (2) is dual. Because L is indecomposable, $End\,L$ is local. If $Hom_A(f, L) \nsubseteq \operatorname{rad} End\,L$, there exists $h : M \to L$ such that $\ell := Hom_A(f, L)(h) = hf$ is invertible. But then $\ell^{-1}hf = 1_L$ shows that f is a section. Conversely, if there exists h such that $hf = 1_L$, then $Hom_A(f, L)(h) = 1_L$ shows that $Hom_A(f, L)$ is an epimorphism. □

Theorem A.2.8. *Let $f : L \to M$ be left minimal almost split in* $\operatorname{mod} A$. *Then f is irreducible. Further, a homomorphism $f' : L \to M'$ in* $\operatorname{mod} A$ *is irreducible if and only if $M' \neq 0$ and there exists a direct sum decomposition $M \cong M' \oplus M''$ and a homomorphism $f'' : L \to M''$ such that*

$$\begin{bmatrix} f' \\ f'' \end{bmatrix} : L \to M' \oplus M''$$

is left minimal almost split.

Proof. Let $f : L \to M$ be a left minimal almost split homomorphism in $\operatorname{mod} A$. By definition f is not a section. Furthermore f is not a retraction by Lemma A.2.3. Assume now that $f = f_1 f_2$, where $f_2 : L \to X$ and $f_1 : X \to M$. We suppose that f_2 is not a section and prove that f_1 is a retraction. As f is left almost split, there exists $f_2' : M \to X$ such that $f_2 = f_2' f$. Hence $f = f_1 f_2 = f_1 f_2' f$. Because f is left minimal, $f_1 f_2'$ is an automorphism and so f_1 is a retraction.

Let now $f' : L \to M'$ be an irreducible morphism in $\operatorname{mod} A$. Then clearly $M' \neq 0$. Also f' is not a section, hence there exists $h : M \to M'$ such that $f' = hf$. Because f' is irreducible and f is not a section, h is a retraction. We set $M'' = ker\,h$. Then there exists a homomorphism $q : M \to M''$ such that $\begin{bmatrix} h \\ q \end{bmatrix} : M \to M' \oplus M''$ is an isomorphism. It follows that

$$\begin{bmatrix} h \\ q \end{bmatrix} f = \begin{bmatrix} f' \\ qf \end{bmatrix} : L \to M' \oplus M''$$

is left minimal almost split.

Conversely, assume that f' satisfies the stated conditions. We will show that it is irreducible. Because L is indecomposable and f' is not an isomorphism, f' is not a retraction. We show that f' is not a section either. Assume that there exists h such that $hf' = 1_L$, then $\begin{bmatrix} h & 0 \end{bmatrix} \begin{bmatrix} f' \\ f'' \end{bmatrix} = 1_L$ implies that $\begin{bmatrix} f' \\ f'' \end{bmatrix}$ is a section which is a contradiction. Hence f is not a section. Assume that $f' = f_1 f_2$, where $f_2 = L \to X$ and $f_1 : X \to M'$. We suppose that f_2 is not a section and show that f_1 is a retraction. We have $\begin{bmatrix} f' \\ f'' \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_2 \\ f'' \end{bmatrix}$, where $\begin{bmatrix} f_2 \\ f'' \end{bmatrix} : L \to X \oplus M''$ and $\begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix} :$ $X \oplus M'' \to M' \oplus M''$. By Lemma A.2.7, $Hom_A(f_2, L) \subseteq \operatorname{rad} End\,L$. Similarly

$Hom_A(f'', L) \subseteq \operatorname{rad} End\, L$. Consequently, $Hom_A(\begin{bmatrix} f_2 \\ f'' \end{bmatrix}, L) \subseteq \operatorname{rad} End\, L$, hence again by Lemma A.2.7, $\begin{bmatrix} f_2 \\ f'' \end{bmatrix}$ is not a section. Because $\begin{bmatrix} f' \\ f'' \end{bmatrix}$ is left minimal almost split and hence irreducible, $\begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix}$ is a retraction, and this implies that f_1 is a retraction. □

We now define a special type of short exact sequence, which is particularly useful in the representation theory of algebras.

Definition A.2.9. *A short exact sequence in* mod A

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called an almost split sequence if:

1. *f is left minimal almost split and*

2. *g is right minimal almost split.*

Remark. 1. If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an almost split sequence, then, by Lemma A.2.3 L and N are indecomposable.

2. An almost split sequence is never split as f is not a section and g is not a retraction. This implies that L is not injective and that N is not projective.

3. An almost split sequence is uniquely determined (up to isomorphism) by each of its end terms; indeed, if $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ and $0 \longrightarrow L' \longrightarrow M' \longrightarrow N' \longrightarrow 0$ are two almost split sequences in mod A, then Proposition A.2.2 (and its dual definition) implies that the following assertions are equivalent:

   (a) The two sequences are isomorphic.

   (b) There is an isomorphism $L \cong L'$ of A-modules.

   (c) There is an isomorphism $N \cong N'$ of A-modules.

Theorem A.2.10. *Let* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *be a short exact sequence in* mod A*. The following assertions are equivalent:*

1. *The given sequence is almost split.*

2. *L is indecomposable and g is right almost split.*

3. *N is indecomposable and f is left almost split.*

4. *The homomorphism f is left minimal almost split.*

5. *The homomorphism g is right minimal almost split.*

6. *L and N are indecomposable and f and g are irreducible.*

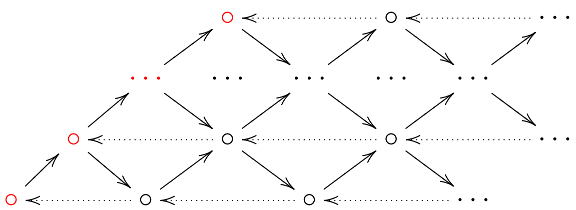
Proof. See [ASS], Chapter 4, Theorem A.2.10. □

Appendix B

B.1 Algorithm

We introduce here an algorithm which calculates the connected components of $(\Gamma^1_{A_{nm-1}})^m$. This algorithm is implemented in the programming language Python. It takes as argument the natural numbers n and m and outputs the list of connected components of $(\Gamma_{A^1_{nm-1}})^m$, each of the components being given as a an ordered list of sectional paths. The construction of the algorithm is based on the following remarks:

1. Each connected component is isomorphic to $\mathbb{Z}A_{n'}/\varphi$ where $n' = n$ or $n-1$ and φ is the automorphism corresponding to the automorphism of $\tilde{\Gamma}_{A_{nm-1}}$, $\eta_{\gamma m}$ for some $\gamma \in \mathbb{Z}$, that is, it has the general form:

   where the longest sectional paths in $\mathbb{Z}A_{n'}/\varphi$ have the same form as the red sectional path in the above figure.

2. Each component has at least one vertex in the fundamental region Δ_m defined in Definition 4.2.11.

3. Starting at a vertex $(i, j) \in \Delta_m$, there exits an arrow in the mth power of $\Gamma^1_{A_{nm-1}}$, $D : (i, j) \to (i', j')$ if and only if $(i', j') = (i, j + m)$ (see proof of Proposition 3.3.1).

4. Starting at a vertex $(i, j) \in \Delta_m$, the ordered list of vertices

   $$(i, j + \alpha(i)m)$$

   (where $0 \leq \alpha(i) \leq \lfloor \frac{nm+1-j}{m} \rfloor$ if $i = 1$ and $0 \leq \alpha(i) \leq \lfloor \frac{nm+2-j}{m} \rfloor$ otherwise) form a longest sectional path in the connected component containing (i, j) (since the component is either isomorphic to $\mathbb{Z}A_n$ or to $\mathbb{Z}A_{n-1}$).

5. In each connected component there are arrows between two sectional paths of 4., X, Y if and only if $\tau^{\pm\alpha m}(Y) = X$ in $\Gamma^1_{A_{nm-1}}$ where $\alpha \in \mathbb{N}$. Moreover given two sectional paths X and Y such that $\tau^{-m}(X) = Y$ and two vertices (i, j), (i', j') such that $(i, j) \in X$ and $(i', j') \in Y$, there is an arrow $D' : (i, j) \to (i', j')$ if and only if $(i', j') = (i + m, j)$.

Hence in order to find the structure of a connected component it suffices to have an ordered list of its longest sectional paths $v_1, v_2, \cdots$ such that

$$\tau^m(v_{i+1}) = v_i.$$

Algorithm 1: Calculation of the connected components of $(\Gamma_{A_{nm-1}})^m$

Input: The natural numbers n and m
Output: The set of the connected components of $(\Gamma_{A^1_{nm-1}})^m$, where each connected component is given as an ordered list of its longest sectional paths $v_1, v_2, \cdots$ such that $\tau^m(v_{i+1}) = v_i$.
Δ_m = set of vertices in the fundamental region of $\Gamma^1_{A_{nm-1}}$;
$\mathcal{C} = \emptyset$, the set which will contain the connected components;
foreach *vertex* (i_0, j_0) *in* Δ_m **do**
  v_{ref} = ordered list of the vertices of the longest sectional path starting at (i_0, j_0);
  $C = v_{ref}$, the ordered list which will contain all the longest sectional paths of the connected component to which (i_0, j_0) belongs;
  $(i, j) = (i_0, j_0)$;
  while *All the vertices of* v_{ref} *have not appeared* **do**
    v = ordered list of the vertices of the longest sectional path starting at $(i + m, j + m)$;
    Add v to C ;
    if *one of the vertices* (i', j') *of* v *belongs to* Δ_m **then**
      $\Delta_m = \Delta_m \backslash (i', j')$;
    end
  end
  $\mathcal{C} = \mathcal{C} \cup C$;
end
return $\mathcal{C}$;

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