ON THE NILRADICAL OF A PARABOLIC SUBGROUP

KARIN BAUR

Abstract. We present various approaches to the understanding of the structure of the nilradical of parabolic subgroups in type A. In particular, we consider the complement of the open dense orbit and describe its irreducible components.

1. Introduction

This report is an extended version of a talk at the conference "Lie Theory and Its Applications" held at UCSD in March 2011.

Nolan Wallach had ignited interest in parabolic subalgebras (cf. Section 1.4). During the time I was a post-doc at UCSD and also during later visits, I have enjoyed numerous lectures by and discussions with Nolan Wallach. I am very grateful for these. This report allows me to display joint work with N. Wallach and to give a view on related recent progress. I will present several approaches towards the understanding of the nilradical of a parabolic subgroup of a reductive algebra group.

1.1. **Classical situation.** Let $\mathfrak{g}$ =End($\mathbb{C}^n$) be the Lie algebra of endomorphisms of $\mathbb{C}^n$. The nilpotent endomorphisms among them are well known. Up to conjugacy by G =GL$_n(\mathbb{C})$, they are given by partitions, i.e. by the Jordan canonical form. In particular, these orbits are well understood. There are finitely many and we have an order on them, namely by inclusion of orbit closures.

If we put this in a more formal language, we are in the following situation: Let G be a classical algebraic group over $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra[1]. G acts on the cone $\mathcal{N} \subset \mathfrak{g}$ of nilpotent elements by conjugation. This action breaks $\mathcal{N}$ up into finitely many orbits. The nilpotent orbits are parametrized by certain partitions. The exact description can be found in [9]. More generally, Jacobson-Morozov theory tells us that every nilpotent element e of $\mathfrak{g}$ can be embedded in an $\mathfrak{sl}_2$-triple (e, f, h). The action of the semi-simple element h of this triple gives rise to a labeled Dynkin diagram associated to the nilpotent orbit, with labels from $\{0, 1, 2\}$. This is the so-called Dynkin-Kostant classification of nilpotent orbits, cf. [9].

Since $\mathcal{N}$ is irreducible, there exists an open dense orbit, called the regular nilpotent orbit. We can give a representative of this orbits in a very nice way: If we take a generator X_α for each simple root space (with respect to a given Cartan subalgebra of $\mathfrak{g}$), we obtain a regular nilpotent element,

$$X = \sum_{\alpha \text{ simple}} X_\alpha$$

The labeled Dynkin diagram of this orbit has a 2 at every node.

[1]Often it would be enough to assume that G is defined over an algebraically closed field.

Example. Let G be $\mathrm{SL}_{n+1}(\mathbb{C})$. We choose the diagonal matrices in its Lie algebra as the Cartan subalgebra. The root spaces are then spanned by the elementary matrices $E_{i,j}$, $i \neq j$, where the only non-zero entry is a 1 at position (i,j). With this choice, the above representative takes the form

$$X = \sum_{i=1}^{n} E_{i,i+1} = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

1.2. **Flags in n-space.** We now consider nilpotent endomorphisms of $\mathbb{C}^n$ which preserve flags of vector spaces. For this and the following section let $G = \mathrm{GL}_n(\mathbb{C})$ and let $\mathcal{F}$ be a partial flag in $\mathbb{C}^n$:

$$\mathcal{F}: \quad 0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{r-1} \subsetneq V_r = \mathbb{C}^n$$

for some $r \geq 1$. Then we define $G \supset P$ to be the parabolic subgroup which is the stabilizer of the flag $\mathcal{F}$,

$$P := \{g \in G \mid gV_i = V_i \ \forall\ i\}$$

We will sometimes write $P = P(\mathcal{F})$ for the parabolic subgroup corresponding to $\mathcal{F}$. We will write $\mathfrak{p}$ to denote the Lie algebra of P.

What can we say about the nilpotent endomorphisms of $\mathbb{C}^n$, which preserve the flag $\mathcal{F}$? In order words, how can wie describe the $X \in \mathcal{N}$ with

$$XV_i \subset V_{i-1} \ \ \forall\ i$$

Example. Consider

$$\mathcal{F}: \ V_0 = 0 \subset \ V_1 \ \subset \ V_2 \ \subset \ V_3 = \mathbb{C}^4.$$

with $V_1 := \langle e_1 \rangle$ and $V_2 := \langle e_1, e_2 \rangle$. Then $\mathfrak{p}$ consists of the 4×4 matrices of the form

$$\begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{pmatrix}$$

(with arbitrary entries at the positions of the $\bullet$'s and $*$'s). If X is a nilpotent element satisfying $XV_i \subset V_{i-1}$ for all i, X has to have the form

$$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1.3. **The Richardson orbit.** The above example is an instance of the following situation. Let $G \supset P$ be a parabolic subgroup, $P = L \cdot U$ with L reductive (called Levi factor) and U the unipotent radical of P. The Lie algebra $\mathfrak{n}$ of U is called the nilradical of P. It is convenient to assume that P contains the Borel subgroup of the upper triangular matrices and that L contains the diagonal matrices. Such P and its Lie algebra $\mathfrak{p}$ are called standard. The direct sum decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ is called Levi decomposition of $\mathfrak{p}$. In the example above, the nilradical $\mathfrak{n}$ consists of the matrices with non-zero entries only at the positions of the $*$'s, and X belongs to $\mathfrak{n}$.

The Levi part $\mathfrak{l}$ of $\mathfrak{p}$ consists of the matrices with non-zero entries only at the $\bullet$'s. It is a general feature that $\mathfrak{l}$ consists of matrices with non-zero entries only in square blocks on the diagonal. The sizes of these square blocks are the differences $\dim V_i - \dim V_{i-1}$. We will later denote them by $d_1, \ldots, d_r$.

The parabolic subgroup P acts on its nilradical $\mathfrak{n}$ by conjugation. It is known that the nilradical can be written as the union of the intersections of $\mathfrak{n}$ with the nilpotent G-orbits in $\mathfrak{g}$, cf. [11] (Satz 4.2.8). Recall that the nilpotent G-orbits are parametrized by partitions of n. If λ is a partition of n, we will write $C(\lambda)$ to denote the corresponding nilpotent G-orbit.

Since there are only finitely many nilpotent G-orbits in $\mathfrak{g}$, one of the intersections of $\mathfrak{n}$ with the nilpotent G-orbits, say $C(\lambda) \cap \mathfrak{n}$, is open and dense in $\mathfrak{n}$. If μ is any partition of n, we get

- $C(\mu) \cap \mathfrak{n} \ \subset \ \mathfrak{n} \setminus (C(\lambda) \cap \mathfrak{n})$ with $C(\mu) \cap \mathfrak{n} \neq 0 \iff \mu \leq \lambda$
- $\dim(C(\mu) \cap \mathfrak{n}) < \dim \mathfrak{n}$ whenever $\mu \neq \lambda$.

Note that we write $\mu \leq \lambda$ if and only if the closure of the orbit $C(\mu)$ is contained in $C(\lambda)$. In particular, we see that $C(\lambda)$ is the unique nilpotent G-orbit of $\mathfrak{g}$ intersecting $\mathfrak{n}$ in an open dense set.

In fact, Richardson shows[2] that $C(\lambda) \cap \mathfrak{n}$ is a single P-orbit, [15]. We call this P-orbit the Richardson orbit of P and denote it by $\mathcal{O}_R$. Its elements are called the Richardson elements (of $\mathfrak{p}$).

Even though $\mathfrak{n}$ contains an open dense P-orbit, we cannot expect that $\mathfrak{n}$ consists of finitely many P-orbits. The Borel subgroup B of GL_6 already has infinitely many B-orbits in its nilradical.

Example. Let $\mathcal{F} : V_i := \mathbb{C}^i$, $0 \leq i \leq n$, be the complete flag in $\mathbb{C}^n$. The corresponding parabolic subgroup is a Borel subgroup $B \subset G$. The nilradical consists of the strictly upper triangular matrices. One can show that if $n = 6$, there is a 1-parameter family of B-orbits in $\mathfrak{n}$.

A consequence of the above example is that whenever a flag $\mathcal{F}$ consists of at least 6 non-zero vector spaces, there are infinitely many $P(\mathcal{F})$-orbits in the corresponding nilradical. For classical G, the parabolic subgroups with finitely many P-orbits are classified, cf. [12]. Roughly speaking, they are the ones with at most 5 blocks in the Levi factor.

1.4. **(Very) nice parabolic subalgebras.** For the moment, let G be a reductive algebraic group over $\mathbb{C}$. Let $B \subset G$ be a Borel subgroup, T a fixed maximal torus in B and $\mathfrak{b}$ the corresponding Borel subalgebra, $\mathfrak{h} =\mathrm{Lie}(T)$ the corresponding Cartan subalgebra. This determines a basis $\{\alpha_1, \ldots, \alpha_n\}$ of simple roots of $\mathfrak{g}$ (with n the rank of $\mathfrak{g}$).

We assume that $\mathfrak{p}$ is a standard parabolic subalgebra, i.e. that it contains $\mathfrak{b}$. It gives rise to a $\mathbb{Z}$-grading of $\mathfrak{g}$ as follows: The parabolic subalgebra $\mathfrak{p}$ is determined by the simple roots α_i such that $\mathfrak{p}$ does not contain the root subspace $\mathfrak{g}_{-\alpha_i}$, equivalently by the simple roots whose root space does not lie in the Levi factor. Hence $\mathfrak{p}$ gives rise to a tuple $(u_1, \ldots, u_n) \in \{0,1\}^n$ with $u_i = 1$ whenever $\mathfrak{g}_{-\alpha_i}$ is not in $\mathfrak{p}$. Then we set $H \in \mathfrak{h}$ to be the element defined by $\alpha_i(H) = u_i$. The adjoint action of H on $\mathfrak{g}$ then defines the $\mathbb{Z}$-grading:

$$\mathfrak{g}_i := \{x \in \mathfrak{g} \mid [H, x] = ix\}$$

[2] His result holds in much greater generality for reductive algebraic groups.

The grading $\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}_i$ is such that the parabolic subalgebra is the sum of the non-negatively graded parts and that the nilradical the sum of the positively graded part. (cf. eg Section 2 of [6]). In case $\mathfrak{g} = \mathrm{End}(\mathbb{C}^n)$ we can read off the graded parts from the block structure of the matrices. In particular, $\mathfrak{g}_0 = \mathfrak{l}$ is the Levi part, $\mathfrak{g}_1$ consists of the sequence of the rectangular regions to the right of the squares on the diagonal. Any element $X \in \mathfrak{g}_1$ gives rise to a character χ_X on $\mathfrak{g}_{-1}$. In case $X \in \mathfrak{g}_1$ is a Richardson element, the character χ_X is admissible in the sense of Lynch, [14]. Hence the existence of a Richardson element in $\mathfrak{g}_1$ ensures the existence of an admissible character. This is exactly Lynch's vanishing condition of certain Lie algebra cohomology spaces for a generalized Whittaker module (associated with the parabolic subalgebra). If $\mathfrak{p}$ has a Richardson element in $\mathfrak{g}_1$ we say that the parabolic subalgebra is nice. In joint work with N. Wallach, [6], we have classified the nice parabolic subalgebras of simple Lie algebras over $\mathbb{C}$.

It is known that for X in $\mathcal{O}_R$, the identity component G_X^0 of the stabilizer subgroup in G is contained in $P_X \subset G_X$, in particular, $|G_X/P_X|$ is finite. The numbers $|G_X/P_X|$ can be found in the article [10] by Hesselink[3]. Assume that $\mathfrak{p}$ is nice. The condition $G_X = P_X$ corresponds to the birationality of the moment map from the dual of the cotangent bundle of G/P onto its image. Nice parabolic subalgebras with $G_X = P_X$ are called very nice. In [7] we continued our joint work with N. Wallach and described the very nice parabolic subalgebras. The main application of this is that under these conditions, one can prove a holomorphic continuation of Jacquet integrals for a real form of $\mathfrak{g}$, cf. [17], [18].

1.5. **P-orbit structure in $\mathfrak{n}$.** What can we say about the P-orbit structure in the nilradical $\mathfrak{n}$? This is a very difficult question. In general, it is a "wild" problem (in the language of representations of algebras).

A first approach towards understanding the nilradical is the description of the elements of the open dense orbit. One can give representatives for Richardson elements explicitely, as has been done for classical and exceptional group, cf. [6], [1], [2] and [4]. We now go back to $G = \mathrm{GL}_n(\mathbb{C})$. Let $\mathcal{F} : 0 = V_0 \subset V_1 \subset \cdots \subset V_r = \mathbb{C}^n$ be a flag and let $P \subset G$ be the corresponding parabolic subgroup. For $i = 1, \dots, r$ we set $d_i := \dim V_i/V_{i-1}$. The d_i are the block lengths of the Levi factor in the nilradical of P. One can show that they determine $\mathfrak{p}$ and $\mathfrak{n}$. The r-tuple $d = (d_1, \dots, d_r)$ forms a composition of n. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 0)$ be the dual of the partition obtained by ordering the d_i by size. Then λ is the partition of the Richardson orbit.

We finish this section by illustrating how one can construct a representative of the Richardson orbit.

Example. If we assume $d_1 \geq d_2 \geq \cdots \geq d_r$ we obtain a representative[4] of the Richardson orbit by choosing small identity blocks of the size $d_{i+1} \times d_{i+1}$ next to

[3]He attributes them to Spaltenstein, cf. [16]

[4]It is enough to assume that the sequence of the d_i is unimodal, i.e. that it is first increasing and then decreasing.

the ith block in the Levi factor: For $d = (3, 2, 2)$ we get

$$\begin{pmatrix} 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & 1 & & \\ 0 & 0 & 0 & & & & \\ & & & 0 & 0 & 1 & \\ & & & 0 & 0 & & 1 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \end{pmatrix}$$

Clearly, if $\mathcal{F}$ is the complete flag, i.e. if $d_i = 1$ for all i, then the resulting element of the nilradical is the regular nilpotent with 1's next to the diagonal.

1.6. **Two approaches to $\mathfrak{n}$.** As $\mathcal{O}_R$ is open dense in $\mathfrak{n}$, the knowledge about the Richardson orbit already gives a lot of information about the nilradical. However, it is very difficult to get a grasp on the remaining P-orbits, in particular if there are infinitely many of them.

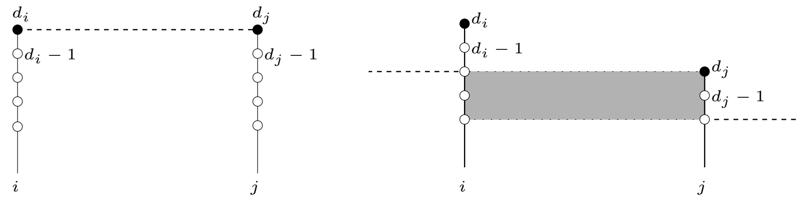
So far, there exist two approaches towards the understanding of the structure of the nilradical. One approach is to study the complement of the open dense orbit. The other approach is an example of the process of categorification: We search for a category of representations for an algebra with the hope of finding a bijection between the P-orbits in $\mathfrak{n}$ and a class of isomorphism classes of modules in this category. Such a correspondence has been established for the general linear groups in [8]. For the orthogonal groups, we have a good candidate for the corresponding algebra, but it is not yet clear what is class of representations corresponding to the P-orbits, [3].

In this article, we explain the first approach: Consider the complement of the open dense orbit in the nilradical, i.e. the variety $Z := \mathfrak{n} \setminus \mathcal{O}_R$. We will describe the irreducible components of Z. In particular, we will see that if the flag $\mathcal{F}$ is composed of r non-zero vector spaces, then Z has at most $r - 1$ components.

2. Complement of the Richardson orbit

2.1. **Notation.** In what follows, we derive the description of the components of Z using rank conditions on matrices. Let $d = (d_1, \dots, d_r)$ be the sizes of the blocks in the Levi factor of the parabolic subgroup. If A is a $n \times n$-matrix, we divide A into rectangular blocks whose sizes are given by the d_i. We let A_{ij} be the $d_i \times d_j$-rectangle formed by the intersection of the d_i rows $(d_1 + \cdots + d_{i-1} + 1), \dots, (d_1 + \cdots + d_i)$ with the d_j columns $(d_1 + \cdots + d_{j-1} + 1), \dots, (d_1 + \cdots + d_j)$: A_{11} is the region formed by the intersection of the first d_1 rows and the first d_1 columns, etc. With this notation, the nilradical $\mathfrak{n}$ consists of the matrices A with $A_{ij} = 0$ whenever $i \geq j$.

Let $X = X(d)$ be a Richardson element and let λ be the partition of X. If A is any element in $\mathfrak{n} \setminus \mathcal{O}_R$, the nilpotency class μ of A is strictly smaller than λ. We can translate this as follows: X is characterized by the fact that the sequence $\operatorname{rk} X$, $\operatorname{rk} X^2$, $\operatorname{rk} X^3$, … of the ranks of its powers decreases as slowly as possible. We will use this observation to characterize the elements of the complement Z.



(a) $d_i = d_j$ (b) $d_i \neq d_j$

FIGURE 1. Dashed lines and shaded areas are not allowed in $\Lambda(d)$

For $A \in \mathfrak{gl}_n$ we write $A[ij]$ for the square formed by the $(j-i+1)^2$ blocks A_{lm}, $i \leq l \leq j$, $i \leq m \leq j$,

$$A[ij] := \begin{matrix} \boxed{A_{ii}} & \dots & \boxed{A_{ij}} \\ \vdots & \ddots & \vdots \\ \boxed{A_{ji}} & \dots & \boxed{A_{jj}} \end{matrix}$$

With this, we are almost ready to state our result. We first need two more definitions:

$$\kappa(i,j) := 1 + |\{l \mid i < l < j,\ d_l \geq \min(d_i, d_j)\}|$$

$$Z_{ij}^k := \{A \in \mathfrak{n} \mid \mathrm{rk}(A[ij]^k) < \mathrm{rk}(X[ij]^k)\}$$

When $k = \kappa(i,j)$, we write Z_{ij} instead of $Z_{ij}^{\kappa(i,j)}$.

2.2. **Decomposition of Z.** In this section, we explain how to get Z as a disjoint union of irreducibe components. We claim that

$$Z = \bigcup_{(i,j) \in \Lambda(d)} Z_{ij}$$

is the decomposition of Z into irreducible components, cf. [5].

It is rather unpleasant to describe the parameter set $\Lambda(d)$. We will first define a larger set $\Gamma(d)$ and then restrict to $\Lambda(d)$. Let $\Gamma(d)$ be

$$\Gamma(d) := \{(i,j) \mid d_l < \min(d_i, d_j) \text{ or } d_l > \max(d_i, d_j)\ \forall\ i < l < j\}$$

Inside $\Gamma(d)$ we define $\Lambda(d)$. In case $d_i \neq d_j$, we put further constraints on the first $i-1$ entries $d_1, \dots, d_{i-1}$ of d and on $d_{j+1}, \dots, d_r$ of d:

$$\begin{aligned} \Lambda(d) \quad := \quad & \{(i,j) \in \Gamma(i,j) \mid d_i = d_j\} \\ & \cup \{(i,j) \in \Gamma(i,j) \mid d_i \neq d_j \text{ and} \\ & \left.\begin{array}{ll} \bullet & \forall\ k \leq r : d_k \leq \min(d_i, d_j) \text{ or } d_k \geq \max(d_i, d_j) \\ \bullet & \text{for } k < i : d_k \neq d_j \\ \bullet & \text{for } k > j : d_k \neq d_i \end{array}\right\} \end{aligned}$$

Figure 1 illustrates this: the vertical lines indicate the entries d_i and d_j of d, the $\bullet$ stand for d_i resp. d_j, the $\circ$ for $d_i - 1$, $d_i - 2$, etc. and $d_j - 1$, $d_j - 2$, etc. Assume that (i,j) belongs to $\Lambda(d)$. Figure (a): if $d_i = d_j$, the conditions on the elements of $\Lambda(d)$ tell us that there is no l between i and j such that d_l equals $d_i = d_j$. That means that the dashed line is ruled out for the d_l (with $i < l < j$). Figure (b): if $d_i \neq d_j$, the shaded area shows that among the $i < l < j$, no d_l is allowed with

$\min(d_i, d_j) < d_l < \max(d_i, d_j)$. The two dashed lines to the left resp. to the right correspond to the two last conditions on elements of $\Lambda(d)$.

Examples. We compare $\Gamma(d)$ and $\Lambda(d)$ for several different choices of d.

a) If d is increasing or decreasing, then $\Gamma(d) = \Lambda(d) = \{(1,2),(2,3),\ldots,(r-1,r)\}$.
b) $d = (1,1,2,1)$: $\Gamma(d) = \{(1,2),(2,3),(2,4),(3,4)\}$ and $\Lambda(d) = \{(1,2),(2,4)\}$.
c) For $d = (7,5,2,3,5,1,2,6,5)$ we have $\Gamma(d) = \{(i,i+1) \mid i = 1,\ldots,8\} \cup \{(1,8),(2,4),(2,5),(3,6),(3,7),(4,6),(4,7),(5,7),(5,8),(5,9),(7,9)\}$ and $\Lambda(d) = \{(1,8),(2,5),(3,7),(5,9)\}$.

A consequence of the result above is that Z has at most $r-1$ irreducible components: when d is increasing or decreasing, it is clear that $\Lambda(d)$ has size $r-1$. If all d_i are different, the same is true. In all other cases, there is at least one pair i,j with $d_i = d_j$, $|i-j| > 1$. Then one can find an index l between i and j such that (i,l) and (l,j) do not belong to $\Lambda(d)$. Example c) above shows that the actual number of irreducible components can be much smaller than $r-1$.

We first illustrate the decomposition of Z on an example before explaining the main ideas behind the proof.

Example. Let $d = (1,2,3,2)$.

$$\mathfrak{n} = \{A \in \mathfrak{gl}_8 \mid A = \begin{pmatrix} 0 & * & * & * & * & * & * & * \\ & 0 & 0 & * & * & * & * & * \\ & 0 & 0 & * & * & * & * & * \\ & & & 0 & 0 & 0 & * & * \\ & & & 0 & 0 & 0 & * & * \\ & & & 0 & 0 & 0 & * & * \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix}\}$$

Then $\Lambda(d) = \{(1,2),(2,4)\}$, with $\kappa(1,2) = 1$ and $\kappa(2,4) = 2$. The matrix $X = E_{1,2} + E_{2,4} + E_{3,5} + E_{4,7} + E_{5,8}$ is a Richardson element for the corresponding parabolic subgroup and $X^2 = E_{1,4} + E_{2,7} + E_{3,8}$. We have to compute the ranks of $X[12]$ (the matrix formed by the first 3 rows and columns) and of the second power of $X[24]$ (the matrix formed by rows and columns 2 to 8): $\operatorname{rk} X[12] = 1$ and $\operatorname{rk} X[24]^2 = 2$.

The component Z_{12} thus consists of all elements of the nilradical whose 1×2-rectangle A_{12} is zero. The component Z_{24} of the matrices $A \in \mathfrak{n}$ with $\operatorname{rk} A[24]^2 \le 1$. Observe that the only non-zero entries of $A[24]^2$ are in the intersection of its first two rows with its last two columns. This square is just $A_{23}A_{34}$.

2.3. **Ideas of the proof.** The main steps in proving that Z is the union of the $Z_{ij} = Z_{ij}^{\kappa(i,j)}$ with $(i,j) \in \Lambda(d)$ are the following:

- Show that $Z_{ij}^k = \emptyset \iff k > j - i$, that $Z_{ij}^l \subsetneq Z_{ij}^{\kappa(i,j)}$ for $1 \le l \le \kappa(i,j)$ and that for l with $\kappa(i,j) < l \le j - i$ there are $i < i_0 \le j_0 < j$ such that $Z_{ij}^l \subset Z_{ij_0}^{\kappa(i,j_0)} \cup Z_{i_0 j}^{\kappa(i_0,j)}$.
- Argue that $Z = \cup_{1 \le i < j \le r} Z_{ij}^{\kappa(i,j)} = \cup_{i<j} \cup_{k \ge 1} Z_{ij}^k$
- Prove that for any two $(i,j) \ne (k,l)$ in $\Lambda(d)$ we have $Z_{ij} \not\subset Z_{kl}$ and that the elements (i,j) in $\Lambda(d)$ are enough to get all components: For $(i,j) \notin \Gamma(d)$, we can find pairs (k_m, l_m) in $\Gamma(d)$ such that Z_{ij} lies in the union of the

corresponding $Z_{(k_m,l_m)}$. If (i,j) is in $\Gamma(d)\backslash\Lambda(d)$, then there exists $k,l\in\Lambda(d)$ such that $Z_{ij}\subset Z_{kl}$.

It then remains to see that the Z_{ij} are irreducible. This can be done using Young tableaux. We first recall a result of Hille, cf. [11]: If $C(\mu)$ is a nilpotent G-orbit, then the irreducible components of $\mathfrak{n}\cap C(\mu)$ are in bijection with a set $\mathcal{T}(\mu,d)$ of Young tableaux of shape μ.

The Young tableaux of $\mathcal{T}(\mu,d)$ are all possible fillings of the Young diagram of shape μ with d_1 ones, with d_2 twos, d_3 threes, etc. If $\mu=\lambda$ is the partition of the Richardson orbit, there is exactly one way to fill the Young diagram of λ with d_1 ones, etc., i.e. $|\mathcal{T}(\lambda,d)|=1$. We write $T(d)$ for this tableau.

Since we want to describe irreducible components of the complement, we aim for degenerations of the Young tableau $T(d)$. These degenerations should be minimal: If not, we might end up taking subsets of irreducible components. The minimal degenerations arise from $T(d)$ by moving a single box down a number of rows as follows.

Let $T(i,j)$ be the tableau obtained from $T(d)$ by removing the box containing the number j from the last row containing i and j and inserting it in the closest row in order to obtain another tableau. We call its partition $\mu(i,j)$. By construction, $\mu(i,j)\le\lambda$. Then we set $\mathfrak{n}(T(i,j))\subset\mathfrak{n}$ to be the irreducible component of $\mathfrak{n}\cap C(\mu(i,j))$ whose tableau is $T(i,j)$ under Hille's bijection.

We proceed by showing that the components Z_{ij} (with $(i,j)\in\Lambda(d)$) are equal to $\mathfrak{n}(T(i,j))$.

We observe that for every row a box from a Young diagram is moved down, the dimension of the GL_n-orbits of the new nilpotent orbit is decreased by two. This can be derived from the formula for the dimension of the stabilizer, [13]. The change in dimension in the nilradical is half of this. This gives us the codimension of Z_{ij} in $\mathfrak{n}$ as the number of rows the box j has been moved down to get $T(i,j)$.

2.4. **Examples and remarks.** (1) If $d=(1,1,1,1,1)$, $P=B$ is a Borel subgroup. Since $\Lambda(d)=\{(1,2),(2,3),(3,4),(4,5)\}$, the complement is the union of four irreducible components. The regular nilpotent elements are the nilpotent 5×5-matrices whose 4th power is non-zero. Thus the Richardson orbit consists of strictly upper triangular matrices $A=(a_{ij})_{ij}$ with

$$A[1,5]^4=\begin{pmatrix}0&0&0&0&x\\ &0&0&0&0\\ &&0&0&0\\ &&&0&0\\ &&&&0\end{pmatrix}$$

where $x:=a_{12}a_{23}a_{34}a_{45}\neq0$. For A to belong to the complement Z of the Richardson orbit, this product has to be zero. In other words, $A[1,5]^4$ is the zero matrix. But this means that A belongs to $Z_{i,i+1}$ for some $i\le4$ as the component $Z_{i,i+1}$ is the set of matrices with $A_{i,i+1}=a_{i,i+1}=0$. So A lies in one of the components $Z_{i,i+1}$. The components $Z_{i,i+1}$ all have codimension one. The Young tableaux in $\mathcal{T}(\mu,d)$ for $\mu=(4,1)$ are in Figure 2.

In the case of a Borel subalgebra, the irreducible components are all orbit closures of B-orbits in $\mathfrak{n}$. So far, it is not known whether this is true in general, though we suspect that it is the case. Another question to which we do not know the answer yet is whether the Z_{ij} are reduced.

$$\begin{ytableau} 1 & 2 & 3 & 4 \\ 5 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 5 \\ 4 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 4 & 5 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 4 & 5 \\ 2 \end{ytableau}$$

FIGURE 2. The four Young tableaux of $\mathcal{T}((5,1),d)$

(2) The smallest interesting case is $d = (1,1,2,1)$. Here, $\Gamma(d) \supsetneq \Lambda(d) = \{(1,2),(2,4)\}$. From the description in Section 2.2 we expect two irreducible components. Z_{12} is the set of the matrices A with $a_{12} = 0$ and Z_{24} the set of matrices A with $A[24]^2 = 0$. We take $A = (a_{ij})_{ij} \in \mathfrak{n}$ and compute A^2, A^3. Then A^3 has $a_{12}(a_{23}a_{35} + a_{24}a_{45})$ as only non-trivial entry, it is in the upper right corner of A^3. A belongs to the Richardson orbit if and only if this product is non-zero. If it is zero, then $a_{12} = 0$ or $a_{23}a_{35} + a_{24}a_{45} = 0$. The case $a_{12} = 0$ clearly corresponds to $A \in Z_{12}$.

By definition, Z_{24} consists of the A with $A[24]^2 = 0$. We have

$$A[24]^2 = \begin{pmatrix} 0 & 0 & 0 & a_{23}a_{35} + a_{24}a_{45} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So $A \in Z_{24}$ if and only if $a_{23}a_{35} + a_{24}a_{45}$. Thus, $A \notin \mathcal{O}_R$ is equivalent to $A \in Z_{12} \cup Z_{24}$.

REFERENCES

[1] K. Baur, *Richardson elements for classical Lie algebras.* J. Algebra 297 (2006), no. 1, 168–185

[2] K. Baur, *A normal form for admissible characters in the sense of Lynch*, Represent. Theory 9 (2005), 30-45.

[3] K. Baur, K. Erdmann, A. Parker, *Δ-filtered modules and nilpotent orbits of a parabolic subgroup in O_N*. J. Pure Appl. Algebra 215 (2011), no. 5, 885–901

[4] K. Baur, S. Goodwin, *Richardson elements for parabolic subgroups of classical groups in positive characteristic.* Algebr. Represent. Theory 11 (2008), no. 3, 275–297.

[5] K. Baur, L. Hille, *On the complement of the Richardson orbit* Math. Z., DOI: 10.1007/s00209-011-0920-9

[6] K. Baur, N. Wallach, *Nice Parabolic Subalgebras of Reductive Lie Algebras*, Represent. Theory 9 (2005), 1-29.

[7] K. Baur, N. Wallach, A class of gradings of simple Lie algebras. In *Lie algebras, vertex operator algebras and their applications*, 3–15, Contemp. Math., 442, AMS, Providence, RI, 2007.

[8] T. Brüstle, L. Hille, C. M. Ringel, G. Röhrle, *The Δ-filtered modules without self-extensions for the Auslander algebra of $k[T]/\langle T^n\rangle$*, Algebr. Represent. Theory 2 (1999), no. 3, 295–312.

[9] D.H. Collingwood, W.M. McGovern, *Nilpotent orbits in semisimple Lie algebras.* Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993. xiv+186 pp

[10] W. Hesselink, *Polarizations in the classical groups*, Math. Zeitschrift, 160 (1978), 217 – 234.

[11] L. Hille, *Aktionen algebraischer Gruppen, geometrische Quotienten und Köcher*, Habilitationsschrift, Hamburg 2003.

[12] L. Hille, G. Röhrle, *A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, Transform. Groups 4 (1) (1999) 35–52

[13] H. Kraft, C. Procesi, *Closures of conjugacy classes of matrices are normal*, Invent. Math. 53 (1979), no. 3, 227–247.

[14] T. E. Lynch, Generalized Whittaker vectors and representation theory, Thesis, M.I.T., 1979.

[15] R.W. Richardson, Conjugacy classes in parabolic subgroups of semisimple algebraic groups, Bulletin London Math. Society 6 (1974), 21–24.

[16] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Mathematics,946. Springer-Verlag, Berlin-New York, 1982.

[17] N. Wallach, *Holomorphic continuation of generalized Jacquet integrals for degenerate principal series.* Represent. Theory 10 (2006), 380–398

[18] H. Yamashita, *Multiplicity one theorems for generalized Gel'fand-Graev representations of semisimple Lie groups and Whittaker models for discrete series*, Adv. Stud. Pure Math. 14, Academic Press Boston, 1988, 31–121.

Karin Baur, Insitute for Mathematics and Scientific Computing, University of Graz, Graz, Austria

E-mail address: karin.baur@uni-graz.at