Perfect k-colored matchings and $k+2$-gonal tilings

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Abstract

We derive a simple bijection between geometric plane perfect matchings on $2n$ points in convex position and triangulations on $n+2$ points in convex position. We then extend this bijection to monochromatic plane perfect matchings on periodically k-colored vertices and $(k+2)$-gonal tilings of convex point sets. These structures are related to Temperley-Lieb algebras and our bijections provide explicit one-to-one relations between matchings and tilings. Moreover, for a given element of one class, the corresponding element of the other class can be computed in linear time.

1 Introduction

The Fuss-Catalan numbers $f(k,m) = \frac{1}{m}\binom{km+m}{m-1}$ are known to count the number of $k+2$-gonal tilings of a convex polygon of size $km+2$, they go back to Fuss-Euler (cf. [4]). Bisch and Jones introduced k-colored Temperley-Lieb algebras in [1] as a natural generalisation of Temperley-Lieb algebras. These algebras have representations by certain planar k-colored diagrams with $m(k+1)$ vertices on top and bottom. The dimension of such an algebra is $f(k,m)$, with a basis indexed by these diagrams. We call these diagrams plane perfect k-colored matchings or just k-colored matchings, assuming from now on that they are plane and perfect. Since the number of $k+2$-gonal tilings coincides with the number of k-colored matchings, these sets are in bijection. Przytycki and Sikora [4] prove this through an inductive implicit construction but do not give an explicit bijection of the structures.

Furthermore, from work of Marsh and Martin [3], one can derive an implicit correspondence between triangulations and diagrams for $k=1$. However, to our knowledge, no explicit bijection is known.

In this paper, we will give bijections between these two sets of plane graphs on sets of points in convex position. We will first address the case $k=1$ (Section 2) and then treat the general case. Our main theorems are the explicit bijections between the set of k-colored matchings and the $(k+2)$-gonal tilings (Theorems 1 and 8). A key ingredient is the characterization of valid k-colored matchings in Theorem 3.

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Due to lack of space, most proofs are deferred to the appendix.

2 Matchings and triangulations

We will draw the matchings with two parallel rows of n vertices each, labeled v_1 to v_n and v_{n+1} to v_{2n} in clockwise order, and with non-straight edges; see Figure 1(left). We will draw the triangulations (and tilings) on $n+2$ points in convex position, labeled p_1 to p_{n+2} in clockwise order; see Figure 1(right). For the sake of distinguishability, throughout this paper we will refer to $p_1, \ldots, p_{n+2}$ as *points* and to $v_1, \ldots, v_{2n}$ as *vertices*.

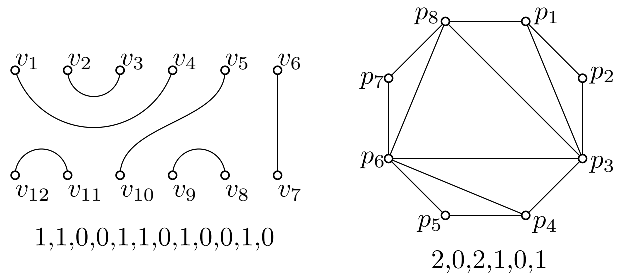

Figure 1: A perfect matching (left) and the corresponding triangulation for $n=6$ (right).

The above defined structures are undirected graphs. We next give an implicit direction to the edges of these graphs: an edge v_iv_j (p_ip_j) is directed from v_i to v_j (p_i to p_j) for $i<j$, that is, each edge is directed from the vertex / point with lower index to the vertex / point with higher index. This also defines the outdegree of every vertex / point, which we denote as b_i for each vertex v_i and as d_i for each point p_i. For technical reasons, we do not count the edges of the convex hull of a triangulation when computing the outdegree of a point p_i, with the exception of the edge p_1p_{n+2}. We call the sequence $(b_1, \ldots, b_{2n})$ of the outdegrees of a matching (or the sequence $(d_1, \ldots, d_n)$ of the first n outdegrees of a triangulation) its *outdegree sequence*; see again Figure 1. We first show that for both structures, this sequence is sufficient to encode the graph.

For matchings, the outdegree sequence is a 0/1-sequence with $2n$ digits, where n digits are 1 and n digits are 0. Moreover, the directions of the edges imply that an incoming edge at a vertex v_j must be outgoing for a vertex v_i with $i<j$. Thus, we have the condition $\sum_{i=1}^{k} b_i \geq k/2$ for any $1 \leq k \leq 2n$, that is, in any subsequence starting at v_1, we have

This is an extended abstract of a presentation given at EuroCG 2017. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear in a conference with formal proceedings and/or in a journal.

at least as many 1s as 0s. Such sequences are called ballot sequences, see [2, p.69]. Obviously, the outdegree sequence of a matching can be computed from a given matching in $O(n)$ time. But also the reverse is true: We consider the outdegrees from b_1 to b_{2n}. We use a stack (with the usual push and pop operations) to store the indices of considered vertices that still need to be processed. Initially, the stack is empty. If $b_i = 1$, we push the index i on the stack. If $b_i = 0$, we pop the topmost index k from the stack and output the edge $v_k v_i$. In this way, always the last vertex with 'open' outgoing edge is connected to the next vertex with incoming edge, implying that the subgraph with vertices v_k to v_i is a valid plane perfect matching. A simple induction argument shows that the whole resulting graph is plane and can be reconstructed from the outdegree sequence in $O(n)$ time.

For triangulations, first note that the outdegrees of p_{n+1} and p_{n+2} are 0. Thus we do not lose information when restricting the outdegree sequence of a triangulation to $(d_1, \ldots, d_n)$. Similar as before, the directions of edges imply that for any valid outdegree sequence, it holds that $\sum_{i=1}^{k} d_{n+1-i} \leq \sum_{i=1}^{k} 1 = k$ for any $1 \leq k \leq n$. This sum is precisely the maximum number of edges which can be outgoing from the 'last' k points p_{n+1-k} to p_n. Recall that we do not consider the edges of the convex hull, except for $p_1 p_{n+2}$, and thus the number of edges which contribute to the outdegree sequence is exactly $n-2$. As before, it is straightforward to compute the outdegree sequence from a given triangulation in $O(n)$ time. For the reverse process, we again use a stack to store the indices of considered points that still need to be processed. We initialize the stack with push$(n+2)$ and push$(n+1)$ and output all the (non-counted) edges $p_i p_{i+1}$ for $1 \leq i \leq n+1$. Then we consider the outdegrees in reversed order, that is, from d_n to d_1. For each degree d_i we perform two steps. (1) d_i times, we pop the topmost index from the stack and then output the edge $p_i p_k$, where k is the (new) topmost index on the stack. (2) We push i on the stack. This process constructs the triangulation from back to front. When processing p_i, all points from p_{i+1} to p_{n+2} that are still 'visible' from p_i are in this order on the stack. Thus, drawing the edges in the described way generates a planar triangulation. At the end of the process, the stack contains exactly the two indices $n+2$ and 1, which can be ignored.

So far we have shown that there exist one-to-one relations between outdegree sequences on the one side and matchings respectively triangulations on the other side. We now present a bijective transform between outdegree sequences of matchings and those of triangulations.

For a given outdegree sequence $B = (b_1, \ldots, b_{2n})$ of a perfect matching, we compute the outdegree d_i for the corresponding point of the triangulation as the number of 1s between the $(i-1)$-st 0 and the i-th 0 in B for $i > 1$, and set d_1 to the number of 1s before the first 0 in B.

For the reverse transformation, we process the outdegree sequence $(d_1, \ldots, d_n)$ of a triangulation from d_1 to d_n and set the entries of B in order from b_1 to b_n in the following way: For each entry d_i we first set the next d_i consecutive elements (possibly none) of B to 1; then we set the next element of B to 0.

It is an easy excercise to see that the two transformations are inverse to each other, and that they form a bijection between valid outdegree sequences of triangulations and outdegree sequences of matchings. Moreover, each transformation can be performed in $O(n)$ time. Figure 2 shows all corresponding perfect matchings, triangulations, and outdegree sequences for $n = 3$.

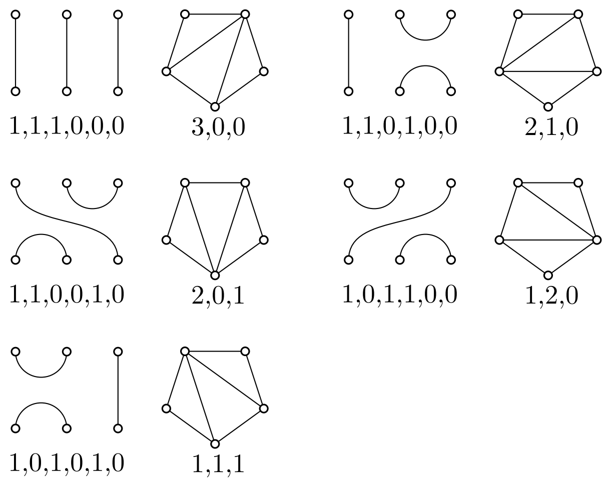

Figure 2: All perfect matchings, triangulations, and outdegree sequences for $n = 3$.

Theorem 1 *There exists a bijection between geometric plane perfect matchings on $2n$ points in convex position and geometric triangulations on $n+2$ points in convex position. Further, for an element of one structure, the corresponding element of the other structure can be computed in linear time.*

3 k-colored matchings

In this section we add colors to the vertices of the perfect matchings and require the matching edges to be monochromatic. For $k \geq 2$, let $c_1, \ldots, c_k$ be the k colors. We color the vertices in a bitonic way, that is, in the order $c_1, c_2, \ldots, c_{k-1}, c_k, c_k, c_{k-1}, \ldots, c_2, c_1, c_1, c_2, \ldots$ and so on. In a *perfect k-colored matching*, all matching edges connect vertices of the same color, and hence n is a multiple of k; see Figure 3 for an example of a k-colored matching with $k = 3$ colors and $n = 9$.

Clearly, any k-colored matching fulfills all conditions of the previous section. But not every match-

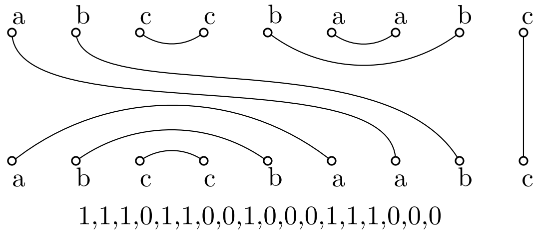

Figure 3: Perfect k-colored matching for $k = 3$ colors and $n = 9$ and its outdegree sequence.

ing obtained in the previous section is a k-colored matching and hence not every outdegree sequence of a matching is an outdegree sequence of a valid k-colored matching. Thus we now derive additional properties to determine which outdegree sequences of matchings correspond to k-colored matchings.

We denote k consecutive vertices $v_i, \dots, v_{i+k-1}$ that are colored with either $c_1, \dots, c_k$ or $c_k, \dots, c_1$ as a *block*. In total we have $2n/k$ such blocks and they form a partition of $2n$ vertices. Observe that within a block, there cannot be a vertex with an incoming edge after a vertex with an outgoing edge, as this would cause a bichromatic edge. Hence, in a k-colored matching, the outdegree sequence of any block has to be of the form $|0, \dots, 0, 1, \dots, 1|$ (where it can consist entirely of 0 or 1 entries). For better readability, we sometimes mark block boundaries in an outdegree sequence with vertical lines. We say that an outdegree sequence (and the matching) fulfilling this property has a *valid block structure*.

Lemma 2 *Let M be a perfect matching with valid block structure that is not a k-colored matching. Then there exists an edge $e = v_s v_e$ in M with the following properties:*

(i) The vertices v_s and v_e lie in different blocks, say $v_s \in S$ and $v_e \in E$.

(ii) The subsequence from v_{s+1} to v_{e-1} contains no bichromatic matching edge.

(iii) The number of blocks between S and E is odd.

(iv) Let v_s be the i-th vertex in S. Then v_e is the $(i+1)$-st vertex in E.

Togther with the previous observations, Lemma 2 (whose proof can be found in the appendix) implies the following theorem.

Theorem 3 *A matching is a k-colored matching if and only if it has a valid block structure and does not contain an edge as described in Lemma 2.*

Remark: For a given outdegree sequence we can check in linear time if it is an outdegree sequence of a k-colored matching by using the reconstruction algorithm described in Section 2.

4 t-gonal tilings

For any $t \geq 3$, a *t-gonal tiling* T on $n + 2$ points in convex position, labeled p_1 to p_{n+2} in clockwise order, is a plane graph where every bounded face is a t-gon and the vertices along the unbounded face are $p_1, p_2, \dots, p_{n+2}$ in this order; see Figure 4 for an example. For the special case of $t = 3$, T is a triangulation. In the next section, we will show that the k-colored matchings on $2n$ vertices of the previous section correspond to $k+2$-gonal tilings of $n + 2$ points in convex position, where $n = km$ for some integer $m > 0$. This is a generalization of the fact that matchings (i.e., $k = 1$) correspond to triangulations. To this end we first derive several properties of t-gonal tilings of convex sets.

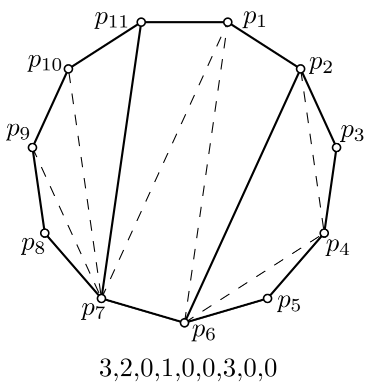

Figure 4: 5-gonal tiling corresponding to the 3-colored matching of Figure 3 and the outdegree sequence of its k-color valid triangulation.

The *dual graph* of a t-gonal tiling T has a vertex for each bounded face T and two vertices are connected by an edge if the according faces share a common edge in T (every pair of bounded faces shares at most one edge). An *ear* of T is a t-gon which shares all but one edge with the unbounded face and can thus be cut off of T (along this edge) so that the remaining part is a valid t-gonal tiling of $n+2-(t-2) = n+4-t$ points.

As the dual graph of any t-gonal tiling T is a tree, as every tree has at least two leaves (where the minimal case is obtained by a path), and as a leaf in the dual graph of T corresponds to an ear in T, we have the following obsevation.

Observation 1 *Every t-gonal tiling with at least $2t-2$ points has at least two ears. At least one of these ears is not incident to the edge $p_1 p_{n+2}$.*

Lemma 4 *Any triangulation $\mathcal{T}$ on $n + 2$ points in convex position contains at most one t-gonal tiling as a subgraph.*

A proof by induction, using Observation 1 can be found in the appendix.

Obviously, if a triangulation $\mathcal{T}$ on $n+2$ points contains a t-gonal tiling T as a subgraph, then n is divisible by $t-2$. Further, as T has at least two ears, $\mathcal{T}$ contains at least two edges that cut off a triangulated t-gon from $\mathcal{T}$. We call such a t-gon that can be split off from a triangulation $\mathcal{T}$ a *t-ear* of $\mathcal{T}$, and the edge along which the t-ear can be split off the *ear-edge* (of the t-ear). Note that for $t > 3$, not every triangulation contains t-ears.

Let $\mathcal{T}$ be a triangulation that contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$. Let B be the outdegree sequence of the corresponding matching. If $s < n+2$, then in B, the t-ear corresponds to a subsequence W of B of length $2t-3$ that starts with a 1 (for $p_r p_s$), ends end with two 0s (as the last point p_{s-1} of the ear cannot have outgoing edges), and has $t-1$ 0s and $t-2$ 1s in total. If $s = n+2$, then in B, the last 0 (the one 'after' p_{n+1}) is not existing. Then the according sequence is $W = (b_{2n-2t+5}, \ldots, b_{2n})$, which must be a ballot sequence.

5 k-colored matchings and $k+2$-gonal tilings

We say that a triangulation on $n+2$ points in convex position is *k-color valid* if it corresponds to a k-colored matching as defined in Section 3. The outdegree sequence of such a triangulation is then also called k-color valid. A $k+2$-gonal tiling of $n+2$ points is called k-color valid if it can be completed to (i.e., is a subgraph of) a k-color valid triangulation. In the following, let $t = k+2$.

Observation 2 *Let $\mathcal{T}$ be a k-color valid triangulation that contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$. Let the first entry of the subsequence W of B that corresponds to this t-ear be the i-th entry within its block, for $1 \leq i \leq k$. If $s = n+2$ then $i = 1$ and $W = (|1, \ldots, 1|0, \ldots, 0|) = (|1^k|0^k|)$. Otherwise, $W = (1, \ldots, 1|0, \ldots, 0, 1, \ldots, 1|0, \ldots, 0) = (1^{k-i+1}|0^{k-i+1}, 1^{i-1}|0^i)$.*

The following three lemmas can be derived using Observation 2. See the appendix for the proofs. The proof of Lemma 5 also shows that the extension is uniquely determined.

Lemma 5 *Any k-color valid t-gonal tiling T on $n+2$ points can be extended by an ear at any edge $e = p_r p_{r+1}$, $1 \leq r \leq n+1$, so that the resulting t-gonal tiling on $n+k$ points is k-color valid.*

Lemma 6 *Let $\mathcal{T}$ be a k-color valid triangulation that contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$. Then the triangulation $\mathcal{T}'$ that results from removing the t-ear from $\mathcal{T}$ is again k-color valid.*

Lemma 7 *Let $\mathcal{T}$ be a k-color valid triangulation. Then $\mathcal{T}$ contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$.*

Combining Lemmas 4 – 7 and Observations 1 – 2, we obtain our main result. See the appendix for a proof.

Theorem 8 *There exists a bijection between geometric plane perfect k-colored matchings on $2n$ points in convex position and t-gonal tilings on $n+2$ points in convex position. Further, for an element of one structure, the corresponding element of the other structure can be computed in linear time.*

6 Future Work

The Temperley-Lieb algebras arising from matchings on $2n$ vertices can be generated by n distinguished elements: An element I (consisting of n propagating lines $v_j v_{2n-j+1}$, $1 \leq j \leq n$, from top to bottom) and $n-1$ elements U_i, $1 \leq i < n$, consisting of a pair of lines $v_i v_{i+1}$ and $v_{2n-i} v_{2n-i+1}$ plus the remaining $n-2$ propagating lines.

It is natural to search for a characterization of these generators in terms of triangulations (and for the generators for the k-colored Temperley-Lieb algebras in terms of k+2-gonal tilings). We plan to use our explicit bijections to study the effect of edge flips in triangulations respectively in tilings on the corresponding matchings and to find out how the actions of generators of the (k-colored) Temperley-Lieb algebra can be interpreted in terms of flips in triangulations respectively in tilings. Preliminary results have already been obtained.

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Appendix: Omitted Proofs

Lemma 2 *Let M be a perfect matching with valid block structure that is not a k-colored matching. Then there exists an edge $e = v_s v_e$ in M with the following properties:*

(i) The vertices v_s and v_e lie in different blocks, say $v_s \in S$ and $v_e \in E$.

(ii) The subsequence from v_{s+1} to v_{e-1} contains no bichromatic matching edge.

(iii) The number of blocks between S and E is odd.

(iv) Let v_s be the i-th vertex in S. Then v_e is the $(i+1)$-st vertex in E.

Proof. To prove the lemma we assume that e is a shortest (w.r.t. the difference of the indices) edge which connects two vertices of different color and show that any such edge has to fulfill the four properties.
(i) As the matching has a valid block structure, no bichromatic edge within a block can exist.
(ii) If the subsequence from v_{s+1} to v_{e-1} contains a bichromatic matching edge, then this edge is shorter, a contradiction.
(iii) Assume there is an even number of blocks between S and E. Then each color shows up in these blocks an even number of times. Hence, by Property (ii), the set of vertices in S after v_s has the same set of colors as the set of vertices in E before v_e. As S and E are colored in reversed order, this implies that v_s and v_e have the same color, a contradiction.
(iv) As there is an odd number of blocks between S and E, by Property (ii), the set of vertices in S after v_s together with the set of vertices in E before v_e have exactly one vertex of each color. As further S and E are colored in the same order, we conclude that the position of v_e in E is 'right after' the position of v_s in S. ◻

Lemma 4 *Any triangulation $\mathcal{T}$ on $n + 2$ points in convex position contains at most one t-gonal tiling as a subgraph.*

Proof. We prove the lemma by induction on n. For $n = t$ the statement is obviously true, so let $n \geq 2t-2$ and let T_1 and T_2 be two t-gonal tilings which are subgraphs of $\mathcal{T}$. By Observation 1 there exists an ear E in T_1. Let $p_a p_b$, $a < b$, be the edge of $\mathcal{T}$ such that E can be separated from the rest of T_1 by this edge. Moreover let e be an edge that is incident to E and to the unbounded face of $\mathcal{T}$. Then the (unique) t-gon in T_2 that is incident to e must be E: Otherwise there is an edge connecting a point p_x between p_a and p_b to a point p_y outside the sequence from p_a to p_b. Then $p_a p_b$, which is part of T_1, crosses $p_x p_y$, which is part of T_2. This is a contradiction to the planarity of $\mathcal{T}$ (recall that T_1 and T_2 are subgraphs of $\mathcal{T}$). Thus we can remove E from both, T_1 and T_2, and obtain two t-gonal tilings of a smaller set of points. By induction, these smaller t-gonal tilings are the same, and hence T_1 and T_2 are the same as well. ◻

Lemma 5 *Any k-color valid t-gonal tiling T on $n+2$ points can be extended by an ear at any edge $e = p_r p_{r+1}$, $1 \leq r \leq n+1$, so that the resulting t-gonal tiling on $n + k$ points is k-color valid.*

Proof. Let $e = p_r p_{r+1}$ be the edge where we add the ear, and let B be the outdegree sequence of the k-colored maching corresponding to T. If $r \leq n$, then in B, e corresponts to the 0, denoted here by $0'$, between the 1s (if any) that correspond to the outdegrees of p_r and p_{r+1}, respectively. Let $0'$ be the i-th entry within its block R, for $1 \leq i \leq k$. Then $R = |0^{i-1}, 0', m|$, where m is an abitrary but valid subsequence. We extend $0'$ to an ear (by inserting k 1s and k 0s before $0'$ according to Observation 2, by this extending R to $|0^{i-1}, 1^{k-i+1}|0^{k-i+1}, 1^{i-1}|0^{i-1}, 0', m|$. If $r = n+1$, then e is not represented in B. In this case, we extend B by adding a block of 1s followed by a block of 0s; see again Observation 2. In both cases, all k new edges in the matching are local within the new blocks and it is straight forward to check by Theorem 3 that the extended outdegree sequence is also color valid. Note that once e is fixed, by Observation 2 the extension is uniquely determined. ◻

Lemma 6 *Let $\mathcal{T}$ be a k-color valid triangulation that contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$. Then the triangulation $\mathcal{T}'$ that results from removing the t-ear from $\mathcal{T}$ is again k-color valid.*

Proof. Let B be the outdegree sequence of the k-colored maching M corresponding to $\mathcal{T}$ and let W be the subsequence of B corresponding to the t-ear. In B, the removal of the ear is equivalent to removing W from B (except for the last 0 for $s < n+2$). Let W' be this sequence to be removed. To show that the resulting triangulation $\mathcal{T}'$ is again k-color valid, we need to prove that the shortened outdegree sequence B' corresponds to a k-colored matching. To this end, first note that in M, removing W' from B is equivalent to removing $2k$ consecutive vertices of the point set. Hence the remaining vertices with the original k-coloring are properly colored. Second, note that the number of 0s and 1s in W' are exactly k each, implying that B' corresponds to some matching M'. It remains to show that M' is k-colored, that is, that there is no bichromatic edge in M'. By Observation 2, we have $W' = (1^{k-i+1}|0^{k-i+1}, 1^{i-1}|0^i)$ for some $1 \leq i \leq k$. In the matching M, this corresponds to k edges that form a matching of the vertices to be removed. Hence all edges in M' also exist in M, implying that none of them is bichromatic. ◻

Lemma 7 *Let $\mathcal{T}$ be a k-color valid triangulation. Then $\mathcal{T}$ contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r + t - 1 \leq n + 2$.*

Proof. Let B be the outdegree sequence of the k-colored matching corresponding to $\mathcal{T}$. Further, let W_i be the subsequence of W that starts at b_i and has length $2k + 1$, for $1 \leq i \leq 2n - 2k$, and let $w_i = \sum_{j=i}^{i+2k} b_j$ be the *weight* of W_i. As $\mathcal{T}$ is k-color valid, we have $w_1 > k$ and $w_{2n-2k} \leq k$. Further, we also have $w_{i+1} - w_i \in \{0, \pm 1\}$. We will show that either at least one of the W_is or the last two blocks of B represents a k-ear of $\mathcal{T}$. To this end, we proceed through the W_is from $i = 1$ to $2n - 2k$ as long as $w_i \geq k$. Whenever $w_i > k$, we continue to the next subsequence (as a necessary condition for W_i to be a k-ear is $w_i = k$). For $w_i = k$ and $w_{i-1} > k$, W_{i-1} starts with $b_{i-1} = 1$ and W_i ends with $b_{i+2k} = 0$. We distinguish the following cases:

Case 1. W_i starts with $b_i = 1$. Let $1 \leq a \leq k$ be such that the block containing b_i ends right before b_{i+a}. Then we have $W_i = 1^a|0^a 1^{k-a}|0^{k-a+1}$, where the 1s in the first block are forced by $b_i = 1$, the 0s in the last block are forced by $b_{i+2k} = 0$, and the form of the middle block stems from $w_i = k$. Hence, W_i is a k-ear by Observation 2.

Case 2. W_i starts with $b_i = 0$. As W_{i-1} starts with $b_{i-1} = 1$, there is a block boundary directly before b_i, and by $w_i = k$ we have $W_i = |0^a 1^{k-a}|0^{k-a} 1^a|0$ for some $1 \leq a \leq k$. Hence, W_j is no ear and $w_j \geq k$ for $i \leq j \leq \min\{i + a, 2n - 2k\}$.

Case 2.1. If $i + a \leq 2n - 2k$ and $w_{i+a} > k$ then $i + a < 2n - 2k$ and we continue to the next subsequence whose weight is equal to k again.

Case 2.2. If $i + a \leq 2n - 2k$ and $w_{i+a} = k$ then all entries in $W_{i+a} \setminus W_i$ are 0s and hence $W_{i+a} = 1^{k-a}|0^{k-a} 1^a|0^{k-a+1}$ is a k-ear.

Case 2.3. If $i + a > 2n - 2k$, then all 1s in W_i must also be in W_{2n-2k}. Due to the k-color validity, this implies that $a = k$, that $W_{2n-2k} = 0|1^k|0^k|$, and that hence the last two blocks of B form a k-ear. ☐

Theorem 8 *There exists a bijection between geometric plane perfect k-colored matchings on $2n$ points in convex position and t-gonal tilings on $n+2$ points in convex position. Further, for an element of one structure, the corresponding element of the other structure can be computed in linear time.*

Proof. We first show (by induction on n) that every t-gonal tiling T can be completed to at least one k-color valid triangulation. For $n = t$ the statement is trivially true, so let $n \geq 2t - 2$. By Observation 1 there exists an ear E of T. If we cut this ear of then by induction there exists an completion to a k-color valid triangulation, which by Lemma 5 can be extended to a k-color valid triangulation $\mathcal{T}$ of T.

Next, assume that there exists t-gonal tilings with at least two differnent triangulations. Let T be a minimal such t-gonal tiling and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two different k-color valid triangulations for T. By Lemma 7, $\mathcal{T}_1$ has a t-ear with ear-edge $e = p_r p_s$ for some $r \geq 1$ and $s = r + t - 1 \leq n + 2$. Thus, e must be an edge of T, implying that $\mathcal{T}_2$ also has a t-ear at e. By Lemma 6, removing the t-ear from $\mathcal{T}_1$ results in a k-color valid triangulation $\mathcal{T}'$. Further, as T is minimal, removing the t-ear from $\mathcal{T}_2$ results in the same triangulation $\mathcal{T}'$. But by the proof of Lemma 5, there is exactly one possibility of extending $\mathcal{T}'$ at e with a t-ear, a contradiction.

So far we have shown that a given t-gonal tiling can be completed to exactly one k-color valid triangulation. For proving that there exists a bijection between k-colored matchings and t-gonal tilings, it remains to show that any k-color valid triangulation contains exactly one t-gonal tiling.

We show (by induction on n) that every k-color valid triangulation $\mathcal{T}$ contains at least one t-gonal tiling. For $n = t$ the statement is trivially true, so let $n \geq 2t - 2$. By Lemma 7, $\mathcal{T}$ has a t-ear with ear-edge $e = p_r p_s$ for some $r \geq 1$ and $s = r + t - 1 \leq n + 2$. Further, by Lemma 6, removing the t-ear from $\mathcal{T}$ results in a triangulation $\mathcal{T}'$, which, by induction, contains at least one t-gonal tiling T'. By Lemma 5, we can extend T' with an ear at e, thus obtaining a t-gonal tiling for $\mathcal{T}$.

As by Lemma 4, every k-color valid triangulation $\mathcal{T}$ contains at most one t-gonal tiling T, this completes the proof of the bijection.

To show that the transformation from a k-colored matching to a t-gonal tiling and vice versa can be done in linear time, it remains to show that the t-gonal tiling of a k-color valid triangulation can be found in linear time and vice versa.

Consider first a k-color valid triangulation $\mathcal{T}$, let B be the outdegree sequence of the k-colored matching corresponding to $\mathcal{T}$, and let B be stored in a linked list. Let T be the t-gonal tiling for $\mathcal{T}$ that we want to construct. By the proof of Lemma 7, we find a t-ear of $\mathcal{T}$ whose subsequence W in B starts at b_j and which is the first t-ear of $\mathcal{T}$ in time $O(j + 2k)$. We can remove the t-ear from $\mathcal{T}$ and W (except possibly its last 0) from B in constant time, by this also obtaining one diagonal of T. Further, the first ear in the shortened sequences B' can start at earliest at b_{j-2k}, which implies that we need not re-start our scan at the beginning. Hence, we can iteratively find all diagonals of T in $O(n)$ time.

For the other direction, consider a t-gonal tiling. We recursively cut of all ears in total linear time. Then, using Lemma 5, we re-add them in reverse order, together with their triangulations that are uniquely defined by Observation 2. ☐