Geometric construction of $m$-cluster categories

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Joint work with Robert Marsh

Type $A_n$: Transactions (math.RT/0607151)
Type $D_n$: IMRN (math.RT/0610512)
Cluster algebras

Introduced by Fomin/Zelevinsky

Subrings of $\mathbb{Q}(u_1, \ldots, u_m)$.

Defined via generators, the cluster variables (constructed recursively).

Clusters: subsets of fixed cardinality.

Motivation: algebraic framework for

- total positivity
- canonical basis of quantum groups (Lusztig/Kashiwara).
Laurent phenomenon: Cluster variables are in $\mathbb{Z}[u_1^\pm, \ldots, u_m^\pm]$, hence the cluster algebra is in $\mathbb{Z}[u_1^\pm, \ldots, u_m^\pm]$

Connections to

- Poisson geometry, Teichmüller spaces
- Grassmannians
- $Y$-systems
Cluster categories

Introduced by BMRRT, CCS.

$Q$ quiver, underlying graph: ADE.

$D^b(kQ)$: bounded derived category of fin.dim. $kQ$-modules ($k = \overline{k}$).

Cluster category, $\mathcal{C}$: orbit category of $D^b(kQ)$ under canonical automorphism.

Independent of orientation of $Q$.

$\mathcal{C} := D^b(kQ)/\tau^{-1} \circ [1]$
Correspondence:

Gabriel: \( \text{indec. } kQ\text{-mods} \leftrightarrow \text{pos. roots} \)

cluster case: \( \text{indec. obj. of } \mathcal{C} \leftrightarrow \text{almost pos. roots} \)
\[ \leftrightarrow \text{cluster variables} \]

\[ \text{tilting objects of } \mathcal{C} \leftrightarrow \text{clusters} \]

(BMRRT ‘06 types \( A, D, E \); CCS ‘06 type \( A \)).
**m-cluster category, \( \mathcal{C}^m \):** (Keller, ‘05)

\[
\mathcal{C}^m := D^b(kQ)/\tau^{-1} \circ [m].
\]

\( \mathcal{C}^m \) is triangulated (Keller), Krull-Schmidt (BMRRT). Calabi-Yau of dimension \( m + 1 \) (Keller).


**Goal:** Describe \( \mathcal{C}^m \) using diagonals of a polygon (type \( A_n \)) and arcs in a punctured polygon (type \( D_n \)).
Translation Quiver

Definition: A translation quiver is a pair $(\Gamma, \tau)$ where

- $\Gamma = (\Gamma_0, \Gamma_1)$ is a locally finite quiver without loops
- $\tau : \Gamma'_0 \to \Gamma_0$ is an injective map, $\Gamma'_0 \subseteq \Gamma_0$
- $\forall x \in \Gamma_0, y \in \Gamma'_0$:
  $$\# \{\text{arrows } x \to y\} = \# \{\text{arrows } \tau y \to x\}$$

$\tau$ is the translation of $(\Gamma, \tau)$
Example:

\[\tau: \text{indicated by } \longrightarrow, \text{ maps from right to left.}\]

Vertices in \(\Gamma_0 \setminus \Gamma'_0\) are called \textbf{projective}. 
More general example:

A a finite dimensional algebra over a field $k = \overline{k}$.

Quiver: $\Gamma_0$: isom. classes of indecomposable modules in mod $A$

$\Gamma_1$: irreducible maps

**Auslander-Reiten quiver of $A$**

Map $\tau$: “Auslander-Reiten translation”
Example Hexagon:

$\Gamma_0$: diagonals $(ij)$

$\Gamma_1$: arrows $(ij) \rightarrow (i, j + 1), (ij) \rightarrow (i + 1, j)$, provided the image is a diagonal $(i, j \in \mathbb{Z}_6)$.

Translation $\tau$: $(ij) \rightarrow (i - 1, j - 1)$

(anti-clockwise rotation about center, $60^\circ$)
$(\Gamma, \tau)$ is called **stable** if $\Gamma_0' = \Gamma_0$. 
Mesh category

$\Gamma = (\Gamma, \tau)$ a translation quiver, $y \in \Gamma_0$.

Let $\alpha_i : x_i \to y$ be the arrows to $y$ ($i = 1, \ldots, k$) and $\beta_i : \tau y \to x_i$ the corresponding arrows from $\tau y$.

The mesh ending at $y$ is the subquiver
Let $k \langle \Gamma \rangle$ be the free $k$-linear category on $\Gamma$.

Objects $\leftrightarrow \Gamma_0$  
morphisms $\leftrightarrow$ paths, under composition

Extend this $k$-linearly.

**Definition**

The **mesh category** of $\Gamma$ is

$$k \langle \Gamma \rangle / \mathcal{R}$$

where $\mathcal{R}$ is given by the $m_y := \sum_{i=1}^k \beta_i \alpha_i$, over all meshes as above (i.e. for all $y \in \Gamma'_0$)
**Tzanaki complex**

For $n, m \in \mathbb{N}$ let $\Pi$ be an $nm + 2$-gon, label the vertices $1, 2, \ldots, nm + 2$.

An **$m$-diagonal** is a diagonal $(ij)$ dividing $\Pi$ into an $mj + 2$-gon and an $m(n - j) + 2$-gon $(1 \leq j \leq \frac{n-1}{2})$.

Obtain a simplicial complex on the set of $m$-diagonals.

Simplices: collections of non-crossing $m$-diagonals.
Maximal simplex: contains $n - 1$ elements.
Example Octagon: Here $n = 3, m = 2$:

\[ nm + 2 = 8, \quad jm + 2 = 4, 6 \]

A maximal simplex: \{ (16), (36) \}
**Quiver** $\Gamma(n,m)$, type $A_{n-1}$.

We define a quiver $\Gamma(m,n) = (\Gamma, \tau_m)$ as follows:

$\Gamma_0$: $m$-diagonals

$\Gamma_1$: $(ij) \rightarrow (ij')$ if $(ij)$, $B_{jj'}$ and $(ij')$ span an $m + 2$-gon ($B_{jj'}$ is boundary $j$ to $j'$).

$\tau_m$: rotation anti-clockwise (about center), angle $m \frac{2\pi}{nm+2}$.

If $m = 1$: usual diagonals.
Proposition:
\(\Gamma(n, m)\) is a translation quiver.

Let \(\mathcal{C}(n, m)\) be the mesh category of \(\Gamma(m, n)\).

E.g. for \(n = 3, m = 2\):

\[
\begin{align*}
16 & \rightarrow 38 & \rightarrow 25 & \rightarrow 47 & \rightarrow 16 \\
14 & \rightarrow 36 & \rightarrow 58 & \rightarrow 27 & \rightarrow 14
\end{align*}
\]
Equivalence of categories

$Q$ a Dynkin quiver of type $A_{n-1}$ (type $D_n$)
$D^b(kQ)$ bounded derived category of fin. dim. $kQ$-modules
$	au$: Auslander-Reiten translate,
$F_m := \tau^{-1} \circ [m]$.

**Theorem** ($m = 1$: Caldero-Chapoton-Schiffler. $m \geq 1$: B.-Marsh).

$\mathcal{C}(n,m) \cong \text{ind } D^b(kQ)/F_m$

Proof uses Happels description of (AR-quiver of) $D^b(kQ)$ and combinatorial analysis of
$\Gamma(n,m)$ (resp. of $\Gamma_\odot(n,m)$).

In above example: $D^b(A_2)/F_2$. 
Quiver $\Gamma_{\odot}(n,m)$, type $D_n$.

We define a quiver $\Gamma_{\odot}(m,n) = (\Gamma, \tau_m)$:

$\Gamma_0$: tagged $m$-arcs of a punctured $nm - m + 1$-gon

$\Gamma_1$: $m$-moves

E.g $(ij) \rightarrow (ik)$ if $(ij)$, $B_{jk}$ (boundary $j$ to $k$) and $(ik)$ span a (degenerate) $m + 2$-gon.

$\tau_m$: rotation anti-clockwise (about center).
$m$-th power of translation quivers

$(\Gamma, \tau)$ a translation quiver.

A path $x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m$ is **sectional** if $\tau x_{i+1} \neq x_{i-1}$ for $i = 1, \ldots, m - 1$ (for which $\tau x_{i+1}$ is defined).

Define $\Gamma^m$ as the quiver with vertices $\Gamma_0$ and arrows: sectional paths in $\Gamma$ of length $m$.

Let $\tau^m$ be $\tau \circ \tau \circ \cdots \circ \tau$ ($m$ times).

**Theorem:** Let $(\Gamma, \tau)$ be a translation quiver such that if $y$ is projective and $x \to y$ then $x$ is projective.

Then $(\Gamma^m, \tau^m)$ is a translation quiver.

Example: a stable $(\Gamma, \tau)$. 
**Theorem:**  (type $A_{n-1}$)

$\text{ind } D^b(kQ)/F_m$ is a full subcategory of $(\text{ind } D^b(kQ)/F_1)^m$

So: $\Gamma(n, m)$ is a full subquiver of $(\Gamma(nm, 1))^m = (\Gamma(\text{cluster category}))^m$
Example: Type $A_5$, gives $C(3, 2)$ from above.

Obtain two other components: $D^b(A_3)/S$ twice.
Working on: other types, describing other components.

E.g. type $D_4$:
Second power is a torus:

Solution: restrict sectional paths of length $m$. 
**Theorem:** (type $D_{nm-m+1}$)

The restricted $m$-th power

$\mu_m(\Gamma(D_{nm-m+1}, 1), \tau^m)$ is the union of the following connected components:

$$\mu_m(\Gamma(D_{nm-m+1}, 1), \tau^m) = \Gamma_{\odot}(n, m) \cup \bigcup_{k=1}^{m-1} \Gamma(D^b(\mathbb{A}_{n-1})/\tau^{nm-m+1})$$

$(\Gamma(D^b(\mathbb{A}_{n-1})/\tau^{nm-m+1})$ denotes the Auslander-Reiten quiver of $D^b(\mathbb{A}_{n-1})/\tau^{nm-m+1})$. 