RICHARDSON ELEMENTS FOR CLASSICAL LIE ALGEBRAS

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Abstract. Parabolic subalgebras of semi-simple Lie algebras decompose as $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ where $\mathfrak{m}$ is a Levi factor and $\mathfrak{n}$ the corresponding nilradical. By Richardsons theorem [R], there exists an open orbit under the action of the adjoint group P on the nilradical. The elements of this dense orbits are known as Richardson elements.

In these talks we describe a normal form for Richardson elements in the classical case (cf. [B06]). This generalizes a construction for $\mathfrak{gl}_N$ of Brüstle, Hille, Ringel and Röhrle [BHRR] to the other classical Lie algebra and it extends the authors normal forms of Richardson elements for nice parabolic subalgebras of simple Lie algebras to arbitrary parabolic subalgebras of the classical Lie algebras [B05]. As applications we obtain a description of the support of Richardson elements and we recover the Bala-Carter label of the orbit of Richardson elements.

1. Introduction

These are lecture notes from a short course on Richardson elements for classical Lie algebras given at the university of Leicester in May 2006. Most of the material for these lectures can be found in [B06].

Let $\mathfrak{g}$ be a classical Lie algebra, i.e. of type A_n ($\mathfrak{sl}_{n+1}$), B_n ($\mathfrak{so}_{2n+1}$), C_n ($\mathfrak{sp}_{2n}$) or D_n ($\mathfrak{so}_{2n}$) over the complex numbers (in fact, the construction of Richarsdon elements works over an algebraic closed field of characteristic p, p a good prime, cf. [BG]).

To define the orthogonal Lie algebras $\mathfrak{so}_N$, we use the skew diagonal matrix $\Gamma := J_N$ with ones on the skew diagonal and zeroes else. The symplectic Lie algebras, $\mathfrak{sp}_{2n}$ are defined using $\Gamma := \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$. Then the elements of the corresponding Lie algebras are the $\{A \in M_N(\mathbb{C}) \mid A^t\Gamma + \Gamma A = 0\}$ where $M_N(\mathbb{C})$ is the set of $N \times N$ matrices (with the correct N) and A^t is the transpose of A. So $\mathfrak{so}_N$ consists of the $N \times N$-matrices that are skew-symmetric around the skew-diagonal (in particular, the entries on the skew-diagonal are zero). And $\mathfrak{sp}_{2n}$ is the set of $2n \times 2n$-matrices of the form

$$\begin{bmatrix} A & B \\ C & A^* \end{bmatrix}$$

where B and C are symmetric about the skew-diagonal and where A^* denotes the skew-transpose of A about the skew-diagonal. (For details, we refer to [GW, Chapter 1]).

Inside $\mathfrak{g}$ we fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h}$. We will always work with the Borel subalgebra consisting of the upper triangular matrices in $\mathfrak{g}$ and of the Cartan subalgebra given by the diagonal matrices in the Lie algebra.

We denote by $\triangle = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots of $\mathfrak{g}$ relative to $\mathfrak{b}$, $\mathfrak{h}$.

Example 1. It is best to keep the example $\mathfrak{sl}_{n+1}$ of the traceless $(n+1) \times (n+1)$ matrices in mind. There, $\mathfrak{h}$ is the set of traceless diagonal matrices and $\mathfrak{b}$ the trace zero upper triangular matrices. The simple roots correspond to the entries just above the diagonal, the root space of the simple root α_i is given by the matrices with zeroes except in the place $i, i+1$.

A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a Lie subalgebra that contains a Borel subalgebra. We call $\mathfrak{p}$ *standard* if it contains the fixed Borel subalgebra.

A standard parabolic subalgebra $\mathfrak{p}$ is then just given by a sequence of square matrices on the diagonal and all the (rectangular) blocks above (in the upper right corner). $\mathfrak{p}$ decomposes into a direct sum of a Levi factor and the corresponding nilradical,

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$$

(On the level of the associated adjoint groups, we have $G = M \cdot N$, a semi direct product, where the Levi factor M is a reductive group and N is the corresponding unipotent radical.) We can assume that $\mathfrak{m}$ contains the fixed Cartan subalgebra and call such a Levi factor *standard*. The standard Levi factor consists of the sequence of block matrices on the diagonal and $\mathfrak{n}$ is the union of the rectangular blocks above the diagonal.

From now on we will assume that $\mathfrak{p}$ and $\mathfrak{m}$ are standard. A parabolic subalgebra corresponds bijectively to a subset of the simple roots and to a coloring of the Dynkin diagram:

$$\begin{aligned} \mathfrak{p} \quad &\leftrightarrow \quad S := \{\alpha_i \in \triangle \mid \mathfrak{g}_{\alpha_i} \subset \mathfrak{n}\} \\ &\leftrightarrow \quad \text{coloring of the Dynkin diagram of } \mathfrak{g} \end{aligned}$$

The coloring of the Dynkin diagram is obtained by the rule: the vertex i is colored if and only if $\mathfrak{g}_{\alpha_i}$ is a subspace of $\mathfrak{m}$.

From the block structure of the parabolic subalgebra or of the Levi factor we can read off a $\mathbb{Z}$-grading of the Lie algebra. For $\alpha \in S$ we set $\deg(\mathfrak{g}_\alpha) = 1$ and from there, the grading of the whole Lie algebra is obtained. Formally, let α be a positive root and write $\alpha = \sum_{j=1}^n k_j \alpha_j$ with coefficients $k_j \in \mathbb{N}$. Then $\mathfrak{g}_\alpha$ lies in the ith graded part $\mathfrak{g}_i$ if $\sum_{j:\alpha_j \in S} k_j = i$.

Alternatively, the grading is given by an element of the Cartan subalgebra: let $h_1, \dots, h_n \in \mathfrak{h}$ be "dual to the simple roots", i.e. given by the requirement $\alpha_i(h_j) = \delta_{ij}$. Then set $H = \sum_{\alpha_i \in S} h_i$. The map $ad(H)$ acts on $\mathfrak{g}$ with integer eigenvalues, denote the corresponding eigenspaces by $\mathfrak{g}_i$, $i \in \mathbb{Z}$.

This gives

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \quad \mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i, \quad \mathfrak{m} = \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{n} = \bigoplus_{i > 0} \mathfrak{g}_i$$

Example 2. We consider the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}(\alpha_2, \alpha_6, \alpha_8)$ in $\mathfrak{sl}_{11}$ as pictured in Figure 1. So $S = \{\alpha_2, \alpha_6, \alpha_8\} \subset \triangle$. The colored Dynkin diagram is shown in Figure 2.

FIGURE 1. The parabolic subalgebra $\mathfrak{p}(\alpha_2, \alpha_6, \alpha_8)$ in $\mathfrak{sl}_{11}$.

The corresponding grading is

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3.$$

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ and let P be the associated parabolic subgroup in the adjoint classical group G. We recall a fundamental result of Richardson, [R].

FIGURE 2. The colored Dynkin diagram for $\mathfrak{p}(\alpha_2, \alpha_6, \alpha_8)$

Theorem 1.1 (Richardson)**.** *There is always an open dense P-orbit under the adjoint action in $\mathfrak{n}$.*

In other words, there exists $X \in \mathfrak{n}$ such that $[\mathfrak{p}, X] = \mathfrak{n}$.

We call this P-orbit the *Richardson orbit* and its elements *Richardson elements* (for $\mathfrak{p}$). An alternative definition of Richardson elements is given by Hesselink in [H]. We state it here in full generality.

Definition 1. Let G be a conected reductive group over an algebraically closed field, let P be a parabolic subgroup. Denote their Lie algebras by $\mathfrak{g}$, $\mathfrak{p}$, where $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$. An element $X \in \mathfrak{n}$ is called a *Richardson element of* $\mathfrak{n}$ if $\dim(GX) = 2\dim(G/P)$.

Now the dimension of GX is just the codimension of the stabilizer of X in G (i.e. $\dim G = \dim GX + \dim G_X$) and we have $\dim \mathfrak{g} = \dim M + 2\dim(G/P)$ (where M is a Levi factor of P). So we get the following useful criterion:

$$(1) \qquad X \in \mathfrak{n} \text{ is a Richardson element for } \mathfrak{p} \iff \dim \mathfrak{g}^X = \dim \mathfrak{m}$$

($\mathfrak{g}^X = \{y \in \mathfrak{g} \mid [X, y] = 0\}$ denotes the stabilizer of X in $\mathfrak{g}$).

The purpose of these talks is to construct Richardson elements for the classical Lie algebras and to explain the use of these elements.

2. Applications

The construction of Richardson elements for parabolic subalgebras has various applications. Among them are

- If there is a Richardson element x in $\mathfrak{g}_1$, one has Lynch's vanishing theorem for induced Lie algebra cohomology, $H^i(\bar{\mathfrak{n}}, V \otimes \mathbb{C}_{-\psi_x}) = 0$ for $i > 0$ (ψ_x is the character induced by the chosen element in $\mathfrak{g}_1$) (cf. [L] - generalizing earlier work of Kostant for Borel subalgebras).
- If there is a Richardson element in $\mathfrak{g}_1$ and if furthermore, the stabilizer subgroups of a Richardson element in G and in P are the same, a multiplicity one theorem for generalized Whittaker vector holds (cf. [W1], [W2], [BK]).
- In the case of $\mathfrak{sl}_{n+1}$, there is a $1-1$ correspondence between $\triangle$-filtered modules without self extension for an A_r-type Auslander Reiten quiver and Richardson orbits (cf. [BHRR]).
- The canonical forms of Richardson elements are of interest for understanding the ring of invariants of (bi-)parabolic subalgebras ([J]).

For the first two applications, the construction of Richardson elements is used to show that there exist Richardson elements in the first graded parts. For many applications it is of interest to find a "simple form" of a Richardson element. We can take simple quite literally: the most useful form of a Richarson element is a nilpotent element whose support forms a simple system of roots. We will see that there is not always such an element and that there is not a unique element with simple support.

3. First examples

We include a couple of examples. We'll need the notion of the support of a nilpotent element. Denote the positive roots of $\mathfrak{g}$ (with respect to the chosen $\mathfrak{b}$, $\mathfrak{h}$) by Φ^+.

Definition 2. Let $X \in \mathfrak{g}$ be a nilpotent element, write $X = \sum_{\alpha \in \Phi^+} k_\alpha X_\alpha$ (where X_α spans the root subspace $\mathfrak{g}_\alpha$). Then the *support of* X, $\operatorname{supp}(X)$, is the set of roots $\alpha \in \Phi^+$ with $k_\alpha \neq 0$.

Example 3. If $\mathfrak{p} = \mathfrak{b}$ is the Borel subalgebra, then any element $\sum_{i=1}^n C_{\alpha_i} X_{\alpha_i}$ (with $C_{\alpha_i} \neq 0$) is a Richardson element. Especially the element with $C_{\alpha_i} = 1$ for all i. Since the Richardson orbit is dense, a randomly picked element of $\mathfrak{n}$ is also a Richardson element. We illustrate this for $\mathfrak{sl}_5$:

$$X_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & * & * & * & * \\ & 0 & * & * & * \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix} \text{with } \operatorname{rank}(X_2) = 4.$$

The support of X_1 only involves the simple roots of $\mathfrak{g}$, $\operatorname{supp}(X_1) = \triangle$. However, the support of X_2 may involve all the positive roots.

We have already mentioned that a good candidate of a Richardson element should use as few root spaces as possible. The next example shows that there are different possibilities to do so.

Example 4. If $\mathfrak{p} \subset \mathfrak{sl}_{n+1}$ is parabolic subalgebra with two blocks of the same size d, then the nilradical is the $d \times d$ - block in the upper right corner. Any matrix of rank d in that corner gives a Richardson element. So in particular, any permutation matrix of size d in the upper right corner is a Richardson element with only d roots in its support. We illustrate this with $\mathfrak{sl}_6$, $d = 3$. Let $X_i := \begin{bmatrix} 0 & B_i \\ 0 & 0 \end{bmatrix}$ where B_i $(1 \leq i \leq 6)$ is one of the following square matrices,

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

Each *supp*(X_i) consists of three positive roots which do not add up nor subtract from each other. Therefore, these simple roots all span a factor A_1 and so for each i, $\operatorname{supp}(X_i)$ is a basis of a Lie algebra of type $A_1 \times A_1 \times A_1$. We have $\operatorname{supp}(X_1) = \{\alpha_{123}, \alpha_{234}, \alpha_{345}\}$ and $\operatorname{supp}(X_6) = \{\alpha_3, \alpha_{234}, \alpha_{12345}\}$. Note that we usually abbreviate the sum $\alpha_{i_1} + \cdots + \alpha_{i_l}$ by $\alpha_{i_1,\dots,i_l}$.

In our construction of Richardson elements we will mostly stick to X_1. I.e. we will use some variant of an identity matrix in the nilradical. From a representation theoretic view, the last matrix, is also interesting. If we consider the adjoint action of the Levi factor $\mathfrak{m}$ on the nilradical (more precisely, on the space $\mathfrak{g}_1$), then α_{12345} is the highest root and α_3 is the lowest root appearing in that representation. A way to obtain this element is to start with the highest (or lowest) root in each component of $\mathfrak{g}_1$ and successively take the next lower (or next higher) root. (Such a procedure is called the "Kostant cascade" by A. Joseph).

Remark 3.1. In example 4 we have jumped ahead. We explain a bit more now. The adjoint action of P on $\mathfrak{g}$ induces an action of the Levi factor M on $\mathfrak{g}$. In particular, each $\mathfrak{g}_i$ is a representation of M, which in general is not irreducible. The irreducible components of $\mathfrak{g}_i$ are the blocks in the matrix picture. This is best seen in the case of $\mathfrak{sl}_{n+1}$. In Figure 1, we can see that $\mathfrak{g}_1$ has 3 blocks, $\mathfrak{g}_2$ has 2 blocks and $\mathfrak{g}_3$ has one block (is irreducible). As representations of M, we obtain thus

$$\begin{aligned} \mathfrak{g}_1 &= \mathfrak{g}_{1,2} \oplus \mathfrak{g}_{2,3} \oplus \mathfrak{g}_{3,4} \\ \mathfrak{g}_2 &= \mathfrak{g}_{1,3} \oplus \mathfrak{g}_{2,4} \\ \mathfrak{g}_3 &= \mathfrak{g}_{1,4} \end{aligned}$$

where we label the block in position i, j by $\mathfrak{g}_{i,j}$.

4. Special linear Lie algebra

We recall the beautiful construction of [BHRR, §8].

We have already mentioned that the parabolic subalgebras (or the Levi factors) consists of a sequence of square matrices on the diagonal and all the blocks above the diagonal (resp. the square matrices on the diagonal). That means that we can give $\mathfrak{p} \subset \mathfrak{sl}_{n+1}$ by specifying the sizes of the square matrices. So $\mathfrak{p} = \mathfrak{p}(d)$, where $d = (d_1, \dots, d_r)$ with $\sum d_i = n+1$ is the vector of the block lengths in the Levi factor.

We construct a Richardson element for $\mathfrak{p}(d)$ as follows. We draw a (horizontal) line diagram $L(d)$: Write the numbers $1, \dots, n, n+1$ in r columns of sizes $d_1, d_2, \dots, d_r$, top adjusted, from left to right. Then connect the entries with horizontal lines wherever possible.

Define $X = X(d) := \sum_{i-j} E_{ij}$ where E_{ij} is the elementary matrix in $\mathfrak{gl}_{n+1}$ whose entries are zero except for a one in position i, j.

Example 5. We look at Example 2. The block lenghts are $d = (2,4,2,3)$ and we draw the line diagram

$$
\begin{array}{rccccccc}
L(d) = & 1 & - & 3 & - & 7 & - & 9 \\
 & 2 & - & 4 & - & 8 & - & 10 \\
 & & & 5 & - & - & - & 11 \\
 & & & 6 & & & &
\end{array}
$$

This defines the element $X = E_{13} + E_{24} + E_{37} + E_{48} + E_{5,11} + E_{79} + E_{8,10}$. The partition of X is $(4,4,2,1)$ and the dual of the partition is $(4,3,2,2)$.

Remark 4.1. The following observations are easy to see:

1) By construction, $X(d)$ is an element of the nilradical $\mathfrak{n}(d)$ of $\mathfrak{p}(d)$.

2) The partition of X is given by the length of the chains in $L(d)$, say $\lambda = \lambda_1 \geq \dots \geq \lambda_s$. We set the *length* of the chain $i_1 - i_2 - \dots - i_l$ in $L(d)$ to be l, i.e. to be equal to the number of vertices that are connected.

3) The dual of the partition is just the dimension vector d, with ordered entries, say $\mu = \mu_1 \geq \dots \geq \mu_r$.

We use criterion (1) to check that $X = X(d)$ is really a Richardson element. To do that we need formulae to calculate the dimension of the centralizer of X. They are given by Kraft and Procesi in [KP] and in Jantzens book [Ja] for positive characteristic:

Theorem 4.2. *Let $(n_1, \dots, n_r)$ be the partition of the Jordan canonical form of a nilpotent matrix x in the Lie algebra $\mathfrak{g}$, let $(m_1, \dots, m_s)$ be the dual partition. Then the dimension of the centralizer of x in $\mathfrak{g}$ is*

$$
\begin{array}{ll}
\sum_i m_i^2 & \text{if } \mathfrak{g} = \mathfrak{gl}_n \\
\sum_i \frac{m_i^2}{2} + \frac{1}{2}|\{i \mid n_i \text{ odd}\}| & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n} \\
\sum_i \frac{m_i^2}{2} - \frac{1}{2}|\{i \mid n_i \text{ odd}\}| & \text{if } \mathfrak{g} = \mathfrak{so}_N
\end{array}
$$

Corollary 4.3. *The nilpotent element $X = X(d)$ defined above is a Richardson element for $\mathfrak{p}(d)$.*

Proof. From the observations above we obtain that the dimension of the centralizer of X is $\sum_{i=1}^r \mu_i^2 - 1$. On the other hand, it is clear that the Levi factor has dimension $\sum_{i=1}^r d_i^2 - 1$. Hence X is a Richardson element. □

The line diagram also displays the support of the Richardson element $X(d)$:

In the notation of Definition 2, we write $X = X(d) = \sum_{i-j} E_{ij} = \sum_{i-j} X_{\varepsilon_i - \varepsilon_j}$. In other words, whenever there is an edge $i - j$ in $L(d)$, the support $\mathrm{supp}(X)$ of X contains $\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$.

Remark 4.4. Let α, β be positive roots of $\mathfrak{sl}_{n+1}$, $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_k - \varepsilon_l$ (with $i < j, k < l$). Then $\alpha + \beta$ is a (positive) root of $\mathfrak{sl}_{n+1}$ if and only if $j = k$ or $i = l$. Similarly, $\pm(\alpha - \beta)$ are roots of $\mathfrak{sl}_{n+1}$ if and only if $i = k$ or $j = l$.

Corollary 4.5. *Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ be the partition of $X(d)$. Then the support of $X(d)$ is a simple system of type $A_{\lambda_1 - 1} \times \cdots \times A_{\lambda_s - 1}$.*

Proof. Let α, β be in the support of X. So in the diagram $L(d)$, they correspond to two edges, say $i - j$ and $k - l$. By the definition of the line diagram, $\pm(\alpha - \beta)$ are not roots of $\mathfrak{sl}_{n+1}$ (for any $1 \leq p \leq n+1$ there is only one edge $p - q$ with $p < q$). Furthermore, $\alpha + \beta$ is a root of $\mathfrak{sl}_{n+1}$ if and only if their edges are joint, i.e. if and only if $j = k$ or $i = l$. So if there is a chain of length m in the diagram $L(d)$, the support of X has $m - 1$ roots $\beta_1, \ldots, \beta_{m-1}$ that form a factor of type A_{m-1}. Since the parts of the partition λ of X correspond to the chains in $L(d)$, the statement follows. □

5. Orthogonal and symplectic Lie algebras

For the other classical types, the parabolic subalgebras and the Levi factors are symmetric around the skew diagonal. So if we describe them by using the block lengths of the square matrices on the diagonal, we have dimension vectors of the form

$$d = (d_t, d_{t-1}, \ldots, d_1, d_0, d_1, \ldots, d_t)$$

where we allow $d_0 = 0$ for an even number of blocks in the Levi factor $\mathfrak{m}$.

For simplicity, we assume the first and last t entries of d to be ordered: $d_t \leq d_{t-1} \leq \cdots \leq d_1$. (In case the entries $d_s, \ldots, d_1$ are not ordered, we can permute them using the symmetric group Σ_r). This will ensure that the line diagrams have a minimal number of crossing lines.

The approach for the Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_{2n}$ is similar to the approach for $\mathfrak{sl}_{n+1}$. The main differences are

(i) the symmetry (around skew diagonal)

(ii) the slopes (lines are not only horizontal)

(iii) "branching" appears (i.e. multiple edges ending at a single vertex).

(i) and (ii) are due to the (skew) symmetry around the skew diagonal of the matrices: for $\mathfrak{sp}_N$, $\mathfrak{so}_N$ there is a edge $(N-j+1) - (N-i+1)$ whenever there is an edge $i - j$. We sometimes call the edge $(N-j+1) - (N-i+1)$ the *counterpart* of $i - j$.

For the moment, we will avoid (iii) by extra assumptions on the dimension vectors (d).

We now describe the diagrams. In both cases we arrange the numbers $1, \ldots, N$ ($N = 2n$ in the symplectic case, $N = 2n$ or $N = 2n+1$ for the orthogonal Lie algebras) in columns of length $d_t, \ldots, d_1, d_0, d_1, \ldots, d_t$ ($2t$ columns if $d_0 = 0$, $2t+1$ columns if $d_0 > 0$), from left two right, ordered from top to bottom.

5.1. Symplectic Lie algebras.

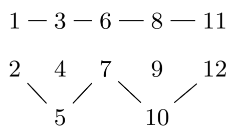
We assume in addition, that there are no repeated odd entries $d_i < d_0$. Otherwise we will have to introduce branched diagrams. We will cover this later.

The central block is of even length, so if its not zero, we have $d_0 \geq 2$. We describe an algorithmic approach to obtain a Richardson element for $\mathfrak{p}(d)$.

Algorithm 1. There are four different cases to consider. In (I), we describe what to do with all these cases and illustrate each of them.

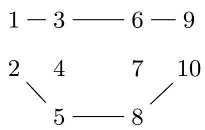
(I) **(a)** If $d_t \geq 2$ and $d_0 \geq 2$: Connect all the vertices on the top and their counterparts on the bottom of each row. The edges form two chains of length $2t+1$.

Example with $t = 2$



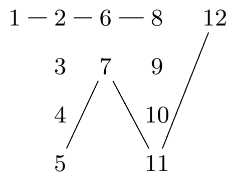
(b) If $d_t \geq 2$ and $d_0 = 0$: Connect the $2t$ vertices on the top and their counterparts, the $2t$ vertices on the bottom. The edges form two chains of length $2t$.

Example with $t = 2$



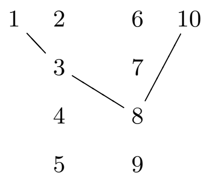
(c) If $d_t = 1$ and $d_0 \geq 2$: Connect the first $2t$ top vertices (from the left) and their counterpart, the last $2t$ bottom vertices (all except the right most bottom vertex).

Example with $t = 2$



(d) If $d_t = 1$ and $d_0 = 0$: Connect the central vertices and next-to-central vertices of each column: If d_k $(k \leq t)$ is even, take the $\frac{d_k}{2}$th vertex of column k and the $\frac{d_k}{2}+1$st vertex of the k to the last column. This gives one chain of length $2t$.

Example with $t = 2$



(II) Rearrange all the remaining vertices to have top adjusted (possibly fewer) columns. We call the new dimension vector also (d), (possibly with a smaller t). Then go back to (I).

The algorithm eventually stops when all the vertices are connected. Denote the resulting diagram by $L(d)$, the line diagram for $L(d)$. Define the corresponding nilpotent element by

$$X(d) := \sum_{i-j,\ i\leq n} E_{ij} - \sum_{i-j,\ i>n} E_{ij}$$

The extra conditions we imposed on the shape (d) of the Levi factor ensured that there is no branching in the diagram. Therefore (as for $\mathfrak{sl}_{n+1}$) the line diagram $L(d)$ gives we can read off the partition of $X(d)$ and the support of the resulting $X(d)$ The partition is given by the lengths of the chains. And each pair of chains obtained as in (a), (b) or (c) corresponds to a factor A_{2t}, A_{2t-1} resp. A_{2t-1} whereas the chain obtained in step (d) corresponds to a factor C_t.

Example 6. Now you can work out a Richardson element for $\mathfrak{sp}_{22}$ with dimension vector $(d) = (2, 3, 5, 2, 5, 3, 2)$.

1	3	6	11	13	18	21
2	4	7	12	14	19	22
	5	8		15	20	
		9		16		
		10		17		

Figure 3 displays the Richardson element for $(d) = (2, 3, 5, 2, 5, 3, 2)$ from example 6. For convenience, the diagonal entries are numbering the rows/columns. The Levi factor has dimension $4 + 9 + 25 + 3 = 41$. You can find the corresponding line diagram in subsection 5.3 below. The partition of $X(d)$ is $\lambda = (7, 7, 4, 2, 2)$ and the dual of λ is $\mu = (5, 5, 3, 3, 2, 2, 2)$. So using the formulae from Theorem 4.2, one computes that the dimension of the centralizer of $X(d)$ is 41 as needed. The support of the corresponding Richardson element is of type $A_6 \times A_1 \times C_2$.

FIGURE 3. Richardson element for $\mathfrak{p}(2, 3, 5, 2, 5, 3, 2)$ in $\mathfrak{sp}_{22}$

5.2. **Orthogonal Lie algebras.** Let $\mathfrak{g} = \mathfrak{so}_N$ with $N = 2n+1$ or $N = 2n$. Again, we will first treat the cases, where there exists a line diagram without branching. The conditions on the shape of the Levi factor for such a "simple" diagram are more complicated for the orthogonal Lie algebras. They are listed here.

We assume furthermore, that there are $0 \le i \le j \le t$, such that
(i) $d_{t-j+1} \le d_0 < d_{t-j}$
(ii) the first i entries $d_t, \dots, d_{t-i+1}$ are even,
(iii) the next entries $d_{t-i}, \dots, d_{t-j+1}$ are odd,
(iv) entries of parity different from the parity of d_0 appear only once among $d_{t-j}, \dots, d_1$,
(v) if $i < j$ then d_0 is odd.

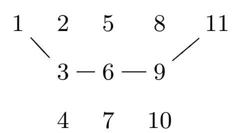
Remark 5.1. In (ii), $i = 0$ means that there is no even entry $\le d_0$. And in (iii), $i = j$ means that there are no odd entries $\le d_0$.

The first condition imples that among the $\{d_t, \dots, d_{t-j+1}\} \cup \{d_0\}$ any even entry is smaller than all the odd entries. In particular, if d_0 is even, we must have $i = j$, i.e. no odd entries smaller than d_0.

Algorithm 2. In (I), we explain, what to do in the different cases and illustrate them.

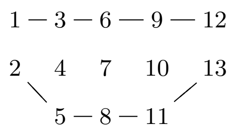
(I) **(a)** If d_0 is odd and $d_1 = 1$: Connect the central vertices and next-to-central vertices of each column, i.e. if d_k is even $1 \le k \le t$, take the $\frac{d_k}{2}$th entry of the kth column and the $\frac{d_k}{2} + 1$st entry of the k to the last column. This gives a $2t+1$-chain.

Example with $t = 2$



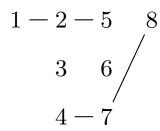
(b) If d_0 is odd and $d_1 > 1$: Connect $2t+1$ vertices at the top and correspondingly the $2t+1$ entries at the bottom. This gives two $2t+1$-chains.

Example with $t = 2$



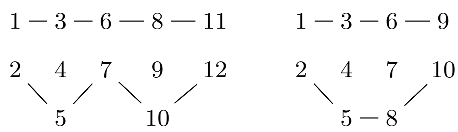
(c) If d_0 is even and $d_1 = 1$ (then $d_0 = 0$ by (v)): Connect the first $2t-1$ vertices of the top column and all their counterparts at the bottom. This gives two $2t-1$-chains.

Example with $t = 2$



(d) If d_0 is even and $d_1 > 1$. Connect all the vertices of the top column and all their counterparts at the bottom. This gives two $2t+1$-chains if $d_0 \geq 2$, otherwise its a pair of $2t$-chains.

Examples with $t = 2$



(II) Rearrange all the remaining vertices to have top adjusted (possibly fewer) columns. We call the new dimension vector also (d), (possibly with a smaller t). Then go back to (I).

The algorithm will eventually stop and we denote the diagram obtained by $L(d)$. From that we can define an element of the nilradical $\mathfrak{n}(d)$ by setting

$$X(d) := \sum_{i-j:\ i+j \leq N} E_{ij} - \sum_{i_j:\ i+j \geq N+2} E_{ij}$$

where $\mathfrak{g} = \mathfrak{so}_N$.

Using the dimension criterion and the formulae for the dimension of the centralizer, one checks that the constructed element $X(d)$ is Richardson.

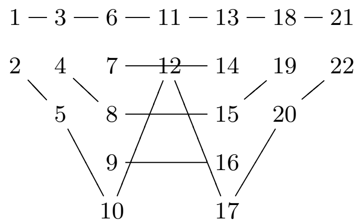
Example 7. You can now work out the line diagram $L(d)$ for the dimension vector $(d) = (2, 3, 4, 3, 4, 3, 2)$

1	3	6	10	13	17	20
2	4	7	11	14	18	21
	5	8	12	15	19	
		9		16		

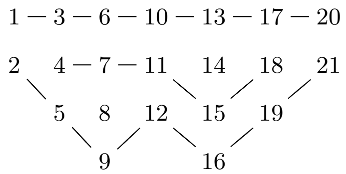
The Richardson element obtained from the line diagram $L(d)$ for Example 7 is displayed in Figure 4. Its Levi factor has dimension 32. And $X(d)$ has partition $\lambda = (7, 7, 5, 1, 1)$ and its dual is $\mu = (5, 3^4, 2^2)$. Using Theorem 4.2 one obtains $\dim \mathfrak{g}^{X(d)} = 32$

FIGURE 4. Richardson element for $\mathfrak{p}(2,3,4,3,4,3,2)$ in $\mathfrak{so}_{21}$

5.3. **Line diagrams for examples.** Here is the line diagram for Example 6



And here the line diagram for Example 7



5.4. **Support of Richardson elements.**

Remark 5.2. If the parabolic subalgebra admits a line diagram without branching, we can read off the partition immediately and the support of the Richardson element $X(d)$. Let $\lambda = \lambda_1 \geq \cdots \geq \lambda_s$ be the partition of $X(d)$.

The roots of the support span a system of type $A_{\lambda_1-1} \times \cdots \times A_{\lambda_s-1}$ if $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

For the symplectic and orthogonal Lie algebras we observe that for each a pair of chains of length t in $L(d)$, the support of $X(d)$ contains a factor A_{l-1}. And for a single chain of length $2t$ in the symplectic case, the support has a factor C_t whereas for a single chain of length $2t+1$ in the odd orthogonal case, the support contains a factor B_t.

If $\mathfrak{g} = \mathfrak{sp}_{2n}$, the support is of the form $A_{i_1} \times \cdots \times A_{i_k} \times C_{j_1} \times \cdots \times C_{j_l}$ (with $l \leq d_0$). From the observations above, it follows that l is equal to the number of single even entries in λ.

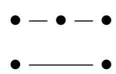
And if $\mathfrak{g} = \mathfrak{sp}_{2n}$, the support is of the form $A_{i_1} \times \cdots \times A_{i_k} \times B_{j_1} \times \cdots \times B_{j_l}$

If there is branching in the diagram, we need a bit more work to obtain the partition of the orbit. This can be done, it only needs the understanding of the three types of branching (resp. the knowledge of the partition that appear for the orthogonal and symplectic cases). (For more details of the branched cases: [BG]).

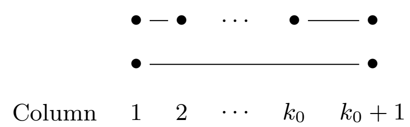
It turns out that in all cases, the support one obtains is the Bala Carter label of the Richardson orbit (for the theory of Bala Carter labels refer to [C] and [P] - the latter gives the labels in terms of the partition of the nilpotent orbits).

6. REMARKS AND OUTLOOK

Remark 6.1. By definition, a Richardson element X lies in the nilradical $\mathfrak{n}$. More precisely, it lies in the subspace $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ for some $k \geq 1$. For example, $X(d) \in \mathfrak{g}_1$ if and only if in the line diagram $L(d)$ there is no "jumped" columns, i.e. no pattern

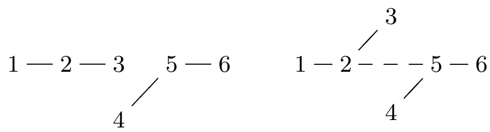


appears. More general, the minimal k_0 such that $X(d) \in \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{k_0}$ is the the size of the largest jump in the diagram, i.e. it is k_0 if the maximal number of columns an edge avoids is $k_0 - 1$:



6.1. **Branching.** The diagrams we have introduced had at most one line to the left and at most one line to the right of a vertex. We call such a diagram a *simple line diagram.* If there is no simple line diagram, there appear three types of branching. We illustrate them here by way of examples. The correct statements are still work in progress (cf. Section 5 in [B06] and [BG]). The extra edges needed in those diagrams are indicated by dashed lines.

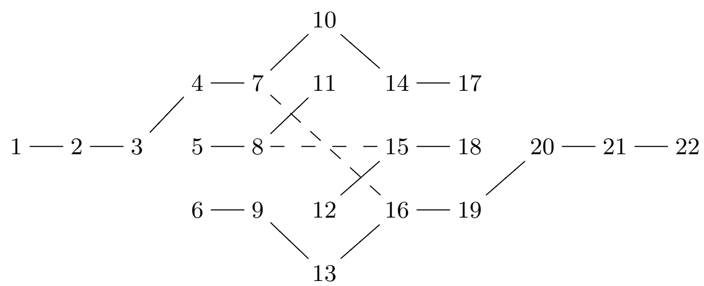
Example 8. Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{sp}_6$ with dimension vector $(1, 1, 2, 1, 1)$. Consider the diagrams



The line diagram to the left is obtained using the rules of Subsection 5.1. The corresponding nilpotent element has a centralizer of dimension 7. However, the Levi factor is five dimensional. In the second diagram, there is one extra line, connecting the vertices 2 and 5. The defined matrix $X = E_{12} + E_{23} + E_{25} - E_{45} - E_{56}$ has a five dimensional centralizer as needed.

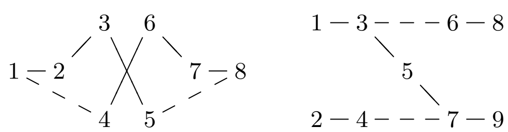
Observe: to make the diagrams look more symmetric, we have arranged the central entries of each column to be on the same height.

Example 9. The following branched line diagram for the parabolic subalgebra of $\mathfrak{sp}_{22}$ with dimension vector $d = (1, 1, 1, 3, 3, 4, 3, 3, 1, 1, 1)$ gives a Richardson element for $\mathfrak{p}(d)$



The Levi factor and the centralizer of the constructed X have dimension 31.

Example 10. For the orthogonal Lie algebras, the smallest example are given by $d = (1, 1, 2, 2, 1, 1)$, i.e. (a)-type of $\mathfrak{g} = \mathfrak{so}_8$ and by $d = (2, 2, 1, 2, 2)$ for an odd number of blocks in $\mathfrak{so}_9$. The following branched diagrams give Richardson elements for the corresponding parabolic subalgebras.



6.2. **Generalizing** [BHRR]**.** The work of Brüstle, Hille, Ringel and Röhrle is for the type A_n Lie algebras. In [BHRR], the authors describe a correspondence between the Richardson orbits and Δ-good modules for the Auslander Reiten quiver of type A_r without self-extension (here, r is the number of blocks in the Levi factor of the standard parabolic subalgebra and not the rank of the Lie algebra).

The goal is to generalize these results to the other classical Lie algebras. The AR-quiver in [BHRR] is a quiver of Dynkin type A_r with arrows $\alpha_i : i \to i+1$ ($i = 1, \dots, n-1$), $\beta_i : i \to i-1$ ($i = 2, \dots, n$) and relations $\alpha_{i-1}\beta_i = \beta_{i+1}\alpha_i$ for $i = 2, \dots, n-1$ together with $\beta_2\alpha_1 = 0$.

Instead of using the category of Δ-good representations for this quiver, we will have to use another category or representations.

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