CLUSTER CATEGORIES FROM GRASSMANNIANS AND ROOT COMBINATORICS

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Abstract. The category of Cohen-Macaulay modules of an algebra $\mathrm{B}_{k,n}$ is used in [JKS16] to give an additive categorification of the cluster algebra structure on the homogeneous coordinate ring of the Grassmannian of k-planes in n-space. We study the Auslander-Reiten translation periodicity for this category, extensions, and we find canonical Auslander-Reiten sequences. Then, we focus on the tame cases and establish a correspondence between certain rigid indecomposable modules of rank 2 and real roots of degree 2 for the associated Kac-Moody algebra. We also an give explicit construction of indecomposable rank 2 modules.

1. Introduction

In this paper we investigate the category $\mathrm{CM}(\mathrm{B}_{k,n})$ of Cohen-Macaulay modules of an algebra $\mathrm{B}_{k,n}$, defined in [JKS16] to categorify the cluster structure of the Grassmannian coordinate rings of k-planes in n-space. We study parts of the Auslander-Reiten quiver of this category, mostly those Auslander-Reiten sequences and components containing rigid modules. Of particular interest are the indecomposable Cohen-Macaulay rigid modules with the same class in the Grothendieck group, i.e. the modules with the same rank 1 modules appearing as composition factors in their filtrations. Since the category in question is, in most cases, of wild type, it is impossible to determine the full Auslander-Reiten quiver. We will focus our attention on the infinite-tame types, in particular, the construction of tubular components of the Auslander-Reiten quiver containing rigid indecomposable modules of rank 1 and rank 2. We show that for each such rigid indecomposable module of rank 2 we can cycle the filtration layers to obtain a new rigid indecomposable module. Also, we give a conjecture about the number or rigid indecomposable rank 2 modules corresponding to real roots in the general case.

In the following we recall the definition of $\mathrm{B}_{k,n}$, the category of Cohen-Macaualy modules $\mathrm{CM}(\mathrm{B}_{k,n})$, and the relation between this category and root systems. We also establish the main results of this paper.

1.1. **The category** $\mathrm{CM}(\mathrm{B}_{k,n})$**.** We follow the exposition from [BKM16] in order to introduce notation and background results. Let n and k be integers such that $1 < k \leq \frac{n}{2}$. Let C be a circular graph with vertices $C_0 = \mathbb{Z}_n$ set clockwise around a circle, and with the set of edges, C_1, also labelled by $\mathbb{Z}_n$, with edge i joining vertices $i-1$ and i. For integers $a, b \in \{1, 2, \dots, n\}$, we denote by $[a, b]$ the closed cyclic interval consisting of the elements of the set $\{a, a+1, \dots, b\}$ reduced modulo n. Consider the quiver with vertices C_0 and, for each edge $i \in C_1$, a pair of arrows $x_i\colon i-1 \to i$ and $y_i\colon i \to i-1$. Then we consider the quotient of the path algebra over $\mathbb{C}$ of this quiver by the ideal generated by the $2n$ relations $xy = yx$ and $x^k = y^{n-k}$. Here, we interpret x and y as arrows of the form x_i, y_i appropriately, and starting at any vertex. For example, when $n = 5$ we have the quiver

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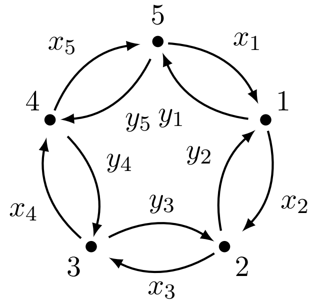


The completion $\mathrm{B}_{k,n}$ of this algebra coincides with the quotient of the completed path algebra of the graph C, i.e. the doubled quiver as above, by the closure of the ideal generated by the relations above (we view the completed path algebra of the graph C as a topological algebra via the m-adic topology, where m is the two-sided ideal generated by the arrows of the quiver, see [DWZ08, Section 1]). The algebra $\mathrm{B}_{k,n}$, that we will often denote by B when there is no ambiguity, was introduced in [JKS16], Section 3. Observe that $\mathrm{B}_{k,n}$ is isomorphic to $\mathrm{B}_{n-k,n}$, so we will always assume that $k \leq \frac{n}{2}$.

The center Z of B is the ring of formal power series $\mathbb{C}[[t]]$, where $t = \sum_{i=1}^{n} x_i y_i$. The (maximal) Cohen-Macaulay B-modules are precisely those which are free as Z-modules. Indeed, such a module M is given by a representation $\{M_i : i \in C_0\}$ of the quiver with each M_i a free Z-module of the same rank (which is the rank of M, cf. [JKS16], Section 3).

Definition 1.1 ([JKS16], Definition 3.5)**.** For any B-module M, if K is the field of fractions of Z, we define its *rank*

$$\mathrm{rank}(M) = \mathrm{rk}(M) = \mathrm{len}(M \otimes_Z K).$$

Note that $B \otimes_Z K \cong M_n(K)$, which is a simple algebra. It is easy to check that the rank is additive on short exact sequences, that $\mathrm{rk}(M) = 0$ for any finite-dimensional B-module (because these are torsion over Z) and that, for any Cohen-Macaulay B-module M and every idempotent e_j, $1 \leq j \leq n$,

$$\mathrm{rk}_Z(e_j M) = \mathrm{rk}(M),$$

so that, in particular, $\mathrm{rk}_Z(M) = n\,\mathrm{rk}(M)$.

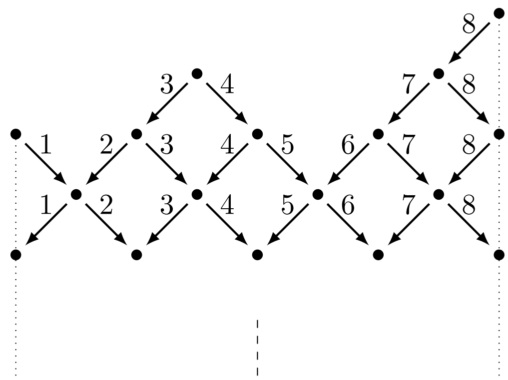
Definition 1.2 ([JKS16], Definition 5.1)**.** For any k-subset I of C_1, we define a rank 1 B-module

$$L_I = (U_i,\ i \in C_0\,;\, x_i, y_i,\ i \in C_1)$$

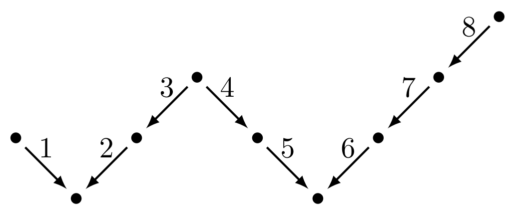
as follows. For each vertex $i \in C_0$, set $U_i = \mathbb{C}[[t]]$ and, for each edge $i \in C_1$, set

$x_i \colon U_{i-1} \to U_i$ to be multiplication by 1 if $i \in I$, and by t if $i \notin I$,
$y_i \colon U_i \to U_{i-1}$ to be multiplication by t if $i \in I$, and by 1 if $i \notin I$.

The module L_I can be represented by a lattice diagram $\mathcal{L}_I$ in which $U_0, U_1, U_2, \ldots, U_n$ are represented by columns from left to right (with U_0 and U_n to be identified). The vertices in each column correspond to the natural monomial $\mathbb{C}$-basis of $\mathbb{C}[t]$. The column corresponding to U_{i+1} is displaced half a step vertically downwards (respectively, upwards) in relation to U_i if $i+1 \in I$ (respectively, $i+1 \notin I$), and the actions of x_i and y_i are shown as diagonal arrows. Note that the k-subset I can then be read off as the set of labels on the arrows pointing down to the right which are exposed to the top of the diagram. For example, the lattice picture $\mathcal{L}_{\{1,4,5\}}$ in the case $k = 3$, $n = 8$, is shown in the following picture



We see from the above picture that the module L_I is determined by its upper boundary, that is, by its rim (this is why we refer to the k-subset I as the rim of L_I), which is the following directed graph with the leftmost and rightmost vertices identified:



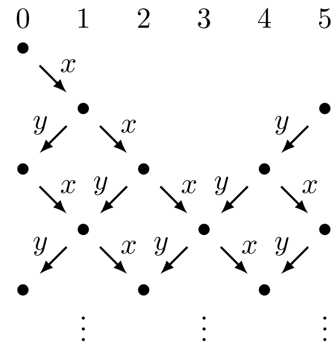
Throughout this paper we will identify a rank 1 module L_I with its rim from the above picture. Moreover, most of the time we will omit the arrows in the rim of L_I and represent it as an undirected graph. We say that i is a peak of the rim I if $i \notin I$ and $i+1 \in I$. In the above example, the peaks of $I = \{1, 4, 5\}$ are 3 and 8.

Remark. We represent a rank 1 module L_I by drawing its rim in the plane and identifying the end points of the rim. Unless specified otherwise, we will assume that the leftmost vertex is labeled by n, and in this case, most of the time we will omit labels on the edges of the rim. Looking at the rim from left to right, the number of downward edges in the rim is k (these are the edges labeled by the elements of I), and the number of upward edges is $n-k$ (these are the edges labeled by the elements of $[1, n] \setminus I$).

Proposition 1.3 ([JKS16], Proposition 5.2)**.** *Every rank* 1 *Cohen-Macaulay* B*-module is isomorphic to* L_I *for some unique* k*-subset* I *of* C_1*.*

Every B-module has a canonical endomorphism given by multiplication by $t \in Z$. For L_I this corresponds to shifting $\mathcal{L}_I$ one step downwards. Since Z is central, $\mathrm{Hom}_B(M, N)$ is a Z-module for arbitrary B-modules M and N. If M, N are free Z-modules, then so is $\mathrm{Hom}_B(M, N)$. In particular, for rank 1 Cohen-Macaulay B-modules L_I and L_J, $\mathrm{Hom}_B(L_I, L_J)$ is a free module of rank 1 over $Z = \mathbb{C}[[t]]$, generated by the canonical map given by placing the lattice of L_I inside the lattice of L_J as far up as possible so that no part of the rim of L_I is strictly above the rim of L_J (see [BKM16, Section 6]).

One sees explicitly that the algebra B has n indecomposable projective left modules $P_j = Be_j$, corresponding to the vertex idempotents $e_j \in B$, for $j \in C_0$. Our convention is that representations of the quiver correspond to left B-modules. The projective indecomposable B-module P_j is the rank 1 module L_I, where $I = \{j+1, j+2, \ldots, j+k\}$, so we represent projective indecomposable modules as in the following picture, where P_5 is pictured ($n = 5$, $k = 3$):



Definition 1.4. A pair I, J of k-subsets of C_1 is said to be non-crossing (or weakly separated) if there are no elements a, b, c, d, cyclically ordered around C_1, such that $a, c \in I \setminus J$ and $b, d \in J \setminus I$.

Definition 1.5. A B-module is *rigid* if $\mathrm{Ext}^1_B(M, M) = 0$.

If I and J are non-crossing k-subsets, then $\mathrm{Ext}^1_B(L_I, L_J) = 0$, in particular, rank 1 modules are rigid (see [JKS16, Proposition 5.6]).

Notation 1.6. Every rigid indecomposable M of rank n in $\mathrm{CM}(B)$ has a filtration having factors $L_{I_1}, L_{I_2}, \dots, L_{I_n}$ of rank 1. This filtration is noted in its *profile*, $\mathrm{prf}(M) = I_1|I_2|\dots|I_n$, [JKS16, Corollary 6.7].

The category $\mathrm{CM}(B)$ provides a categorification for the cluster structure of Grassmannian coordinate rings. As we will discuss later, the stable category $\underline{\mathrm{CM}}(B)$ is 2-Calabi-Yau. Maximal non-crossing collections of k-subsets give rise to cluster-tilting objects T as the corresponding rank 1 modules are pairwise ext-orthogonal. Given a maximal collection of non-crossing k-sets $\mathcal{I}$ (including the projectives, i.e. the successive k-sets), the direct sum $T = \oplus_{I \in \mathcal{I}} L_I$ corresponds to an alternating strand diagram ([Pos06]) whose associated quiver is an example of a *dimer model with boundary* (see [BKM16, Section 3]). If we forget its frozen vertices, we obtain a quiver with potential (Q, P) encoding the endomorphism algebra $\mathrm{End}_{\underline{\mathrm{CM}}}(T)$ as a finite-dimensional Jacobian algebra $J(Q, P)$ in the sense of [DWZ08].

Remark 1.7. Any given k-subset I can be completed to a maximal non-crossing collection $\mathcal{I}$. The arrows in the quiver Q of $\mathrm{End}_{\underline{\mathrm{CM}}}(T)$ represent morphims in $\mathrm{Hom}_{\underline{\mathrm{CM}}}(L_I, L_J)$ that do not factor through L_U with $U \in \mathcal{I}$. There is a strand diagram associated and dimer with boundary, and the quiver Q will not have loops.

Rigid rank 1 modules and the Hom-spaces between them were studied in [BKM16]. In this work we study rank 1 and rigid rank 2 modules from the Auslander-Reiten quiver point of view. We obtain the following result regarding the Auslander-Reiten translation τ.

Theorem 1 (Proposition 2.7, Proposition 2.9). *Set $v = \mathrm{lcm}(k, n)/k$.*
Any rigid indecomposable module of rank ≤ 2 in $\underline{\mathrm{CM}}(\mathrm{B}_{k,n})$ is τ-periodic with period a factor of $2v$.

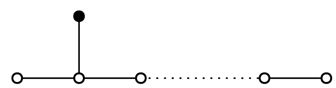
In Section 3.1, we follow closely Keller's argument on quivers with potential to show that every module in $\underline{\mathrm{CM}}(\mathrm{B}_{k,n})$ is τ-periodic, with period (a factor of) $2n$. We also determine certain canonical Auslander-Reiten sequences containing rank 1 modules in Theorem 2.17.

This theorem describes all the Auslander-Reiten sequences containing rank 1 modules in the case of the algebra $B_{3,n}$.

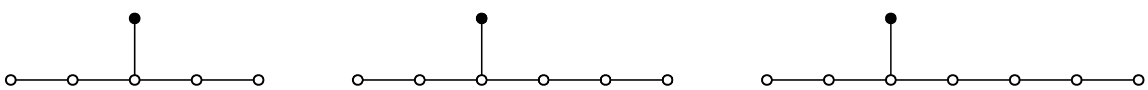
In Section 4 we study the connection between rank 2 modules and positive roots. Among others, we give an explicit construction of indecomposable rank 2 modules, yielding many examples of rigid modules. For this, we include a brief reminder of the link between roots and indecomposable modules.

1.2. **Root combinatorics.** Here we recall the connection between indecomposable modules of $\mathrm{CM}(\mathrm{B}_{k,n})$ for k and n as above and roots for an associated Kac-Moody algebra, as explained in [JKS16].

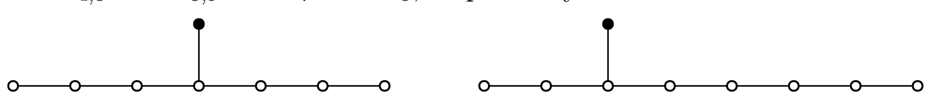
For (k,n) let $J_{k,n}$ be the tree obtained by drawing a Dynkin diagram of type A_{n-1}, labeling the nodes $1,2,\dots,n-1$ and adding a node n with an edge to node k. We consider positive roots for the associated Kac-Moody algebra, denoting the simple root associated with node i by α_i for $i=1,\dots,n-1$ and the simple root associated with n by β. For $k=2$, the resulting diagram $J_{k,n}$ is a Dynkin diagram of type D_n.



For $n=6,7,8$ and $k=3$, we obtain E_6, E_7 and E_8 respectively.



The diagrams $J_{4,8}$ and $J_{3,9}$ are $\widetilde{E}_7$ and $\widetilde{E}_8$, respectively:



There is a grading on the roots of the corresponding Kac-Moody algebra, where the *degree* is given by the coefficient of the root at β, i.e. at the n-th node, the black node in the figures.

Zelevinsky conjectured ([Zel12]) that the number of degree d cluster variables is equal to d times the number of real roots for $J_{k,n}$ of degree d. In the finite types, this is known to hold, [Sco06, Theorems 6,7,8], whereas in the infinite cases it does not hold in this generality. It is expected that one needs to restrict to cluster variables which are associated to real roots. In this spirit, one can ask whether the number of rank d rigid indecomposable modules of $\mathrm{CM}(\mathrm{B}_{k,n})$ is d times the number of real roots for $J_{k,n}$ of degree d. Jensen et al. confirmed that this holds in the finite type cases, cf. [JKS16, Observation 2.3], see also Example 1.8 below. In this article, we provide further evidence for this categorical version of this conjecture by proving that when we restrict to indecomposables corresponding to real roots it holds for $d=2$ in the tame cases, see Theorem 2 below.

We recall a map from indecomposable modules in $\mathrm{CM}(B)$ to roots for $J_{k,n}$ via a map from $\operatorname{ind}\mathrm{CM}(B)$ to $\mathbb{Z}^n$ from [JKS16, Section 8]: If $M=L_1|L_2|\dots|L_d$ is indecomposable, let $\underline{a}=\underline{a}(M)=(a_1,\dots,a_n)$ be the vector where a_i is the multiplicity of i in $L_1\cup\dots\cup L_m$, for $i=1,\dots,n$. Let $e_1,\dots,e_n$ be the standard basis vectors for $\mathbb{Z}^n$. Then we can associate with M a root $\varphi(M)$ for $J_{k,n}$ via the correspondence $\alpha_i \longleftrightarrow -e_i+e_{i+1}$, $i=1,\dots,n-1$, and $\beta\longleftrightarrow e_1+e_2+\dots+e_k$.

Note that the image of M under $\underline{a}$ is in the sublattice $\mathbb{Z}^n(k)=\{a\in\mathbb{Z}^n \mid k \text{ divides } \sum a_i\}$ and that $\varphi(M)$ is a root of degree d. Via these correspondences, we can identify the lattice $\mathbb{Z}^n(k)$ with the root lattice of the Kac-Moody algebra of $J_{k,n}$ and we have the quadratic form $q(\underline{a})=\sum_i a_i^2+\frac{2-k}{k^2}(\sum_i a_i)^2$ on $\mathbb{Z}^n(k)$ which characterizes roots for $J_{k,n}$ as the vectors with $q(\underline{a})\le 2$. Among them, the vectors with $q(\underline{a})=2$ correspond to real roots.

Conjecturally, rigid indecomposable modules correspond to roots and if a module belongs to a homogenous tube, the associated root is imaginary. In the finite types, the correspondence between rigid indecomposable modules and (real) roots is confirmed. Here, we initiate the study of infinite representation types, and in particular, we study rank 2 indecomposable CM-modules. We recall that for $J_{3,9}$ there exist 8 rigid indecomposable modules whose associated root is imaginary, see [JKS16, Figure 13].

Theorem 2 (Section 5.3, Section 6.3)**.** *In the tame cases, when $(k,n) = (3,9)$ or $(k,n) = (4,8)$, for every real root of degree 2 there are two rigid indecomposable modules. Moreover, if M is such a rigid indecomposable rank 2 module and if its filtration by rank 1 modules is $L_I|L_J$, then the rank 2 module with filtration $L_J|L_I$ is also rigid indecomposable.*

Example 1.8. For $k = 2$, there are no indecomposable modules of rank 2. The diagram $J_{2,n}$ is a Dynkin diagram of type D_n for which there are no roots of degree 2.

Let $k = 3$.

(i) The diagram $J_{3,6}$ is a Dynkin diagram of type E_6. The only root where node 6 has degree 2 is the root $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\beta$. It is known that there are exactly two degree 2 cluster variables, cf. [Sco06, Theorem 6]. On the other hand, the only rank 2 (rigid) indecomposable modules in this case are $L_{135}|L_{246}$ and $L_{246}|L_{135}$, cf. [JKS16, Figure 10].

(ii) The diagram $J_{3,7}$ is a Dynkin diagram of type E_7 . The Lie algebra of type E_7 has 7 roots of degree 2. There are 14 cluster variables of degree 2 ([Sco06, Theorem 7]) and, correspondingly, 14 rank 2 (rigid) indecomposable modules for $(3,7)$. These can be found in [JKS16, Figure 11].

(iii) The diagram $J_{3,8}$ is of type E_8, there are 28 roots of degree 2 in the corresponding root system. The number of cluster variables of degree 2, and the number of rank 2 (rigid) indecomposable modules is 56, cf. [JKS16, Figure 12].

Remark 1.9. [Degree two roots in the tame types $(3,9)$ and $(4,8)$]

For $J_{3,9}$ there are 84 real roots of degree 2. For $J_{4,8}$ there are 56 real roots of degree 2. One can find all these roots considering the classical result [Kac90, Theorem 5.6] and playing the so-called *find the highest root game* which is attributed to B. Kostant by A. Knutson [Knu].

2. Homological properties

The algebra $B = \mathrm{B}_{k,n}$ is Gorenstein, i.e. it is left and right noetherian and of finite (left and right) injective dimension. Hence, the category $\mathrm{CM}(B)$ is Frobenius and the projective-injective objects are the projective B-modules. The stable category $\underline{\mathrm{CM}}(B)$ has a triangulated structure in which the suspension [1] coincides with the formal inverse of Ω (see [Buc86, Hap88]).

Let $\Pi_{k,n}$ be the quotient of the preprojective algebra of type $\mathbb{A}_{n-1}$ over the ideal $\langle x^k, y^{n-k}\rangle$. This finite dimensional $\mathbb{C}$-algebra is Gorenstein of dimension 1. The category $\mathrm{CM}(\Pi_{k,n})$ is equivalent to the exact subcategory $\mathrm{Sub}Q_k$ defined in [GLS08]. Analogously to $\mathrm{CM}(B)$, the category $\mathrm{CM}(\Pi_{k,n})$ is Frobenius and the stable category $\underline{\mathrm{CM}}(\Pi_{k,n})$ has a triangulated structure in which [1] coincides with the formal inverse of Ω. We denote by $(\Omega)^{-1}$ the formal inverse of Ω, the usual syzygy. This formal inverse is not the co-syzygy Ω^{-1} since the algebras $\mathrm{B}_{k,n}$ and $\Pi_{k,n}$ are not self-injective, hence the slightly different notation.

By [JKS16, Section 4], there is a (quotient) exact functor $\pi\colon \mathrm{CM}(B) \to \mathrm{CM}(\Pi_{k,n})$ setting a 1-1 correspondence between the indecomposable modules in $\mathrm{CM}(\mathrm{B}_{k,n})$ other than P_n and the indecomposable modules in $\mathrm{CM}(\Pi_{k,n})$. This functor restricts to a triangle equivalence $\underline{\pi}\colon \underline{\mathrm{CM}}(B) \to \underline{\mathrm{CM}}(\Pi_{k,n})$. By construction the standard triangles of $\underline{\mathrm{CM}}(\mathrm{B}_{k,n})$, obtained via push-outs, are of the form

$$A \to B \to C \to A[1],$$

where $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathrm{CM}(B)$. The functor π takes exact sequences to exact sequences. This implies that we can use the additivity of the dimension vector $\underline{\dim}$ on $\mathrm{CM}(\Pi_{k,n})$ to reconstruct triangles, in particular, Auslander-Reiten triangles. We may refer to an Auslander-Reiten triangle

$$A \to B \to \tau^{-1}A \to A[1]$$

also as the associated short exact sequence $A \hookrightarrow B \twoheadrightarrow \tau^{-1}A$.

The category $\underline{\mathrm{Sub}}Q_k$ is triangulated and 2-Calabi-Yau, see [GLS08, Proposition 3.4]. Denote by $[1]_{\mathrm{Sub}}$ the shift in this category. Notice that $[1]_{\mathrm{Sub}}$ can be interpreted as the formal inverse of the syzygy when we are in $\underline{\mathrm{CM}}(\Pi_{k,n})$, see [JKS16, Remark 4.2]. By [RVdB02], $\tau[1]_{\mathrm{Sub}} \simeq S$, where S is the Serre functor. It also holds that $S = [2]_{\mathrm{Sub}}$ from the 2-Calabi Yau condition. Therefore, $\tau = [1]_{\mathrm{Sub}}$ over $\underline{\mathrm{Sub}}Q_k$, and this implies that $\Omega = \tau^{-1}$ in the category $\underline{\mathrm{CM}}(\Pi_{k,n})$. Hence, by the equivalence $\underline{\pi}$, we have $\Omega = \tau^{-1}$ in $\underline{\mathrm{CM}}(B)$.

Example 2.1. Let $(k,n) = (3,8)$. Consider the rank 2 module $M = L_I \mid L_J$ with $I = \{2,5,7\}$ and $J = \{1,3,6\}$. As for rank 1 modules, it is convenient to view higher rank modules as lattice diagrams. The lattice diagram of M is drawn on the left hand side in Figure 1. The image $\pi(M)$ is obtained by taking the quotient of M by the projective P_8. In particular, we can obtain the dimension vector of the $\Pi_{k,n}$-module $\pi(M)$ by cutting out the rim of $L_I \mid L_J$ as in Figure 1 (center) and considering the multiplicities of the vertices 1 to $n-1$.

There is a covering functor from the category of finitely generated complete poset representations for a poset Γ ([Sim93, Chapter 13]), where Γ is a cylindrical covering of the circular quiver of B, to the category $\mathrm{CM}(\mathrm{B}_{k,n})$. The complete poset representations have a vector space at each vertex of Γ and all arrows are subspace inclusions. If $\widetilde{M}$ is a complete poset representation for Γ, it can be identified with a finite subspace configuration of a vector space, say M_*, with $\dim M_* = \operatorname{rank} \widetilde{M}$.

It is important that $\widetilde{M}$, and therefore $M \in \mathrm{CM}(B)$, can be identified with a pull-back from a finite quotient poset of Γ and the indecomposability of M can be deduced from the indecomposability of $\widetilde{M}$. Moreover when M is rigid indecomposable, $\widetilde{M}$ is unique up to a grade shift, [JKS16, Lemma 6.2, Remark 6.3].

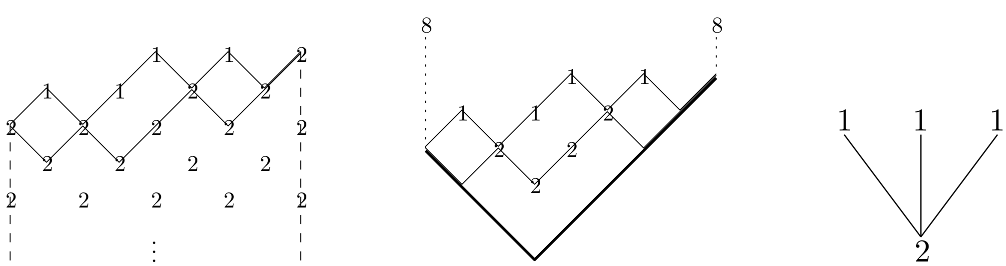


FIGURE 1. Lattice diagram of a module in $\mathrm{CM}(B_{3,8})$ and its image in $\mathrm{CM}(\Pi_{3,8})$ under π, and the corresponding quotient poset $(1^3, 2)$

Remark 2.2. Let M be an indecomposable of rank 2 in $\mathrm{CM}(B)$. Then the corresponding finite quotient poset has to be of the form $(1^r, 2)$ for some r. Since $(1,2)$ and $(1^2,2)$ are dimension vectors for the quivers $1 \to 2$ and $1 \to 2 \leftarrow 3$ respectively, of Dynkin type A_2 and

A_3 respectively, the corresponding representations cannot be indecomposable. Hence $r \geq 3$. These posets are precisely the ones corresponding to indecomposable subspace configurations of rank 2.

Example 2.3. Let $M = L_I \mid L_J$ be a rank 2 module in $\mathrm{CM}(\mathrm{B}_{k,n})$ with $I \neq J$. Then its poset is of the form $(1^r, 2)$ for $0 < r \leq k$. The modules $M = L_I \mid L_J$ with $I = \{2,5,7\}$ and $J = \{1,3,6\}$ from Example 2.1 and the first module in Figure 3 have poset $(1^3, 2)$. Examples for the poset $(1,2)$ are the modules in Figure 2 and the last module in Figure 3. An example for $(1^2, 2)$ is the second module in Figure 3.

For the rest of this paper, we call an indecomposable non-projective M *τ-periodic* if $\tau^n M = M$ for some positive integer n.

Proposition 2.4. *Every module in $\underline{\mathrm{CM}}(\mathrm{B}_{k,n})$ is τ-periodic, with period (a factor of) $2n$.*

This follows from Keller's work [Kel13] (see Section 3.1). We note that the period is expected to be a factor of $2n$, depending on k as well as on n, as we will see below (Proposition 2.7).

Notation 2.5. Let $I_1|\dots|I_n$ be a profile of a Cohen-Macaulay module. Then $(I_1 \mid \dots|I_n) + m$ is the profile $(I_1 + m)|(I_2 + m)|\dots|(I_n + m)$ obtained by adding m to each number in every k-subset appearing in the profile.

Lemma 2.6. [BB16, Proposition 2.7] *Let L_I be a rank 1 module, then $\Omega^2(L_I) = L_{I+k}$.*

Set $v = \mathrm{lcm}(n,k)/k$.

Proposition 2.7. *Let L_I be a non-projective rank 1 module, and let M be a module in the τ-orbit of L_I. Then, M is τ-periodic of period d, for some factor d of $2v$.*

Proof. By Lemma 2.6, we know that $\Omega^{2v}(L_I) = L_{I+vk} = L_{I+[k,n]}$. But, $L_{I+[k,n]} = L_I$, and applying Ω^{2v} is the same as applying τ^{-2v}, so $L_I = \tau^{-2v} L_I$. The statement follows since $M = \tau^s(L_I)$ for some s. □

Lemma 2.8. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathrm{CM}(B)$. Then there exist short exact sequences*

$$0 \to \Omega^2(L) \to \Omega^2(M) \oplus P \to \Omega^2(N) \to 0,$$
$$0 \to (\Omega)^{-2}(L) \to (\Omega)^{-2}(M) \oplus Q \to (\Omega)^{-2}(N) \to 0,$$

where P and Q are projective B-modules.

Proof. From the short exact sequence

$$0 \to L \to M \to N \to 0 \tag{1}$$

we can construct a commutative diagram using the snake lemma and the universal property of projective modules.

$$\begin{tikzcd}
0 \arrow[r] & \Omega(L) \arrow[r] \arrow[d] & \Omega(M) \oplus Q \arrow[r] \arrow[d] & \Omega(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(L) \arrow[r] \arrow[d] & P(L) \oplus P(N) \arrow[r] \arrow[d] & P(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

The middle term in the upper exact sequence $\Omega(M) \oplus Q$ can have a projective summand Q since the epimorphism $P(L) \oplus P(N) \to M$ might not be minimal. We repeat the process but starting with the upper exact sequence to obtain

$$0 \to \Omega^2(L) \to \Omega^2(M) \oplus P \to \Omega^2(N) \to 0$$

The process can be done in direction $(\Omega)^{-1}$ because the projective modules are projective-injective and each module in $\mathrm{CM}(B)$ has a projective-injective envelope. The projective has the injective-lifting property since the initial exact sequence is in $\mathrm{CM}(B)$. This dual construction permits to obtain the second short exact sequence in the statement. □

Proposition 2.9. *Let $M \in \mathrm{CM}(\mathrm{B}_{k,n})$ be a module with no projective summands and having a filtration $L_{I_1}|L_{I_2}$. Then $\tau^{-2}M$ has a filtration by rank 1 modules such that $\mathrm{prf}(\tau^{-2}M) = \mathrm{prf}(M) + k$ and $\tau^2 M$ has a filtration by rank 1 modules such that $\mathrm{prf}(\tau^2 M) = \mathrm{prf}(M) - k$.*

Proof. For a module L_I of rank 1, by Lemma 2.6 we have $\tau^{-2}L_I = \Omega^2(L_I) = L_{I+k}$. Let M be a module with a filtration $L_{I_1}|L_{I_2}$. Then, there is a short exact sequence in $\mathrm{CM}(B)$

$$0 \to L_{I_2} \to M \to L_{I_1} \to 0.$$

Observe that since M has no projective direct summands, the modules L_{I_2} and L_{I_1} are not projective (if either L_{I_1} or L_{I_2} is projective, the sequence splits). By Lemma 2.8, there is a short exact sequence

$$0 \to \Omega^2(L_{I_2}) \to \Omega^2(M) \oplus P \to \Omega^2(L_{I_1}) \to 0$$

where, over the stable category, Ω^2 is τ^{-2}. We have $\tau^{-2}L_{I_j} = L_{I_j+k}$, for $j = 1, 2$, by Lemma 2.6. Thus, we have $\mathrm{rank}(\tau^{-2}M \oplus P) = 2$. Suppose that $P \neq 0$, then $\mathrm{rank}\, \tau^{-2}M = 1$. So we must have $\tau^{-2}M = L_J$ for a k-set J. This could only happen if $M = L_{J-k}$, which is impossible since $\mathrm{rank}\, M = 2$. Then $\mathrm{rank}\, \tau^{-2}M = 2$ and its profile is given by $I_1 + k|I_2 + k$. □

By the uniqueness of $\widetilde{M}$, if $M = L_H|L_J$ is rigid indecomposable it is the only such module with profile $H|J$. So a module M as in the above proposition is τ-periodic of period d, for some factor d of $2v$.

Notice that when all the modules in the category $\mathrm{CM}(B)$ are rigid, as is the case of the finite representation types, Proposition 2.9 can be thought of as the first step of an induction. In this situation, for every indecomposable we have $\mathrm{prf}(\tau^{-2}M) = \mathrm{prf}(M) + k$.

2.1. **Extension spaces between rank 1 modules.** Let I be a rim, and let d_i and l_i respectively be the lengths of disjoint intervals of I and the lengths of the corresponding intervals of the complement of I in $\{1, 2, \dots, n\}$. In other words, let d_i and l_i denote the lengths of downward and upward slopes, respectively, of I. Let m denote the minimum of the numbers d_i and l_i.

Proposition 2.10. *Let I be a rim with two peaks and let J be any rim. If I and J are crossing, then*

$$\mathrm{Ext}^1(L_I, L_J) \cong \mathbb{C}[[t]]/(t^a),$$

where a is less or equal to the minimum m of the lengths of the slopes (both downward and upward) of I.

Proof. Since there are only four slopes on the rim I, when J is placed underneath I in the computation of $\mathrm{Ext}^1(L_I, L_J)$, as in [BB16, Theorem 3.1], there are at most four trapezia appearing. Since we assumed that the rims are crossing, then there are exactly four trapezia with nontrivial lateral sides, and hence, there are exactly two boxes (each box consisting of two trapezia). Each of the lateral sides of the trapezia involved is of length at most equal to the length of the corresponding slope of the rim I. It follows that the matrix D^* is of the form

$$\begin{bmatrix} -t^{m_1} & t^{m_2} \\ t^{m_3} & -t^{m_4} \end{bmatrix},$$

where the numbers m_i denote the lengths of the lateral sides of the trapezia used to compute the extension space $\mathrm{Ext}^1(L_I, L_J)$ (cf. [BB16]). If we choose a to be the minimal m_i, then the proposition follows. □

Corollary 2.11. *If I and J are crossing rims with two peaks each, then*

$$\mathrm{Ext}^1(L_I, L_J) \cong \mathbb{C}[[t]]/(t^a),$$

where a is less or equal to the minimum of the lengths of all slopes of I and J.

Proof. It follows from $\mathrm{Ext}^1(L_I, L_J) \cong \mathrm{Ext}^1(L_J, L_I)$ (see [BB16, Theorem 3.7]). □

Corollary 2.12. *If I and J are crossing rims with I having two peaks and one of the slopes of length 1, then*

$$\mathrm{Ext}^1(L_I, L_J) \cong \mathrm{Ext}^1(L_J, L_I) \cong \mathbb{C}.$$

Proof. If a is as in the previous proposition, then in this case $a \leq 1$, and since I and J are crossing, we must have $1 \leq a$. □

We can use the previous corollary to construct part of the Auslander-Reiten quiver containing rank 1 modules whose rims have two peaks and a slope of length 1. For such a rim I, and J such that $\Omega(L_I) = L_J$, if we can find a non-trivial short exact sequence of the form

$$0 \to L_I \to M \to L_J \to 0,$$

then this sequence must be an Auslander-Reiten sequence.

Example 2.13. If $I = \{1, 2, \ldots, k-1\} \cup \{m\}$, where $n > m > k$, then for any rim J that is crossing with I we have $\mathrm{Ext}^1(L_I, L_J) \cong \mathbb{C}$.

In the following proposition, we deal with a case where the upper bound from the previous proposition is achieved.

Proposition 2.14. *Let I be a rim with two peaks, and let m be as above, i.e. the minimum of the lengths of the slopes of I. Then*

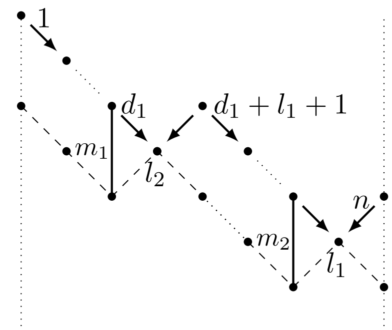
$$\mathrm{Ext}^1(L_I, \Omega(L_I)) \cong \mathbb{C}[[t]]/(t^m).$$

Proof. Since the rim I has two peaks, its first syzygy is also a rank 1 module. As in the proof of the previous proposition, from the proof of [BB16, Theorem 3.1] we know that the matrix of the map D^* from that proof is a 2×2 matrix of the form

$$\begin{bmatrix} -t^{m_1} & t^{m_1} \\ t^{m_2} & -t^{m_2} \end{bmatrix},$$

where the numbers m_1 and m_2 denote the lengths of the lateral sides of the trapezia used to compute the extension space (cf. [BB16]), as in the following picture, and that $m =$

$\min\{m_1, m_2\}$.



We see from the picture that $m_1 = \min\{d_1, l_2\}$ and $m_2 = \min\{d_2, l_1\}$. The proposition now follows.

□

Corollary 2.15. *Let I be a rim with two peaks and one of the slopes (either downward or upward) of length 1. Then*

$$\operatorname{Ext}^1(L_I, \Omega(L_I)) \cong \operatorname{Ext}^1(\Omega(L_I), L_I) \cong \mathbb{C}.$$

2.2. **Auslander-Reiten sequences.** The purpose of this section is to understand Auslander-Reiten sequences of the form $L_I \to M \to L_J$, where L_I and L_J are rank 1 $\mathrm{B}_{k,n}$-modules. To do this we move back and forth from $\mathrm{CM}(\mathrm{B}_{k,n})$ to $\mathrm{CM}(\Pi_{k,n})$ using the quotient functor π.

Remark 2.16. Let L_I and L_J be two rank 1 modules such that $\dim \operatorname{Ext}^1(L_J, L_I) = 1$. Using the quotient functor, the modules $\pi(L_I)$ and $\pi(L_J)$ are rigid modules over $\mathrm{CM}(\Pi_{k,n})$ (or one may consider them as modules in the subcategory $\mathrm{Sub}Q_k$ of the preprojective algebra). Then, by [GLS06, Proposition 5.7] the middle term $\pi(M)$ of the sequence is rigid. So, the middle term M is rigid.

In particular, let L_I be a module whose rim I has two peaks and one of the downward slopes of length 1. From [BB16, Section 2], we know that if $I = \{i, j, \ldots, j+k-2\}$ for some $j \in [i+2, \ldots, n-k+i+1]$, then $\Omega(L_I) = L_J$, where $J = \{i+1, \ldots, i+k-1, j+k-1\}$.

Theorem 2.17. *Let $I = \{i, j, \ldots, j+k-2\}$ for some $j \in [i+2, \ldots, n-k+i+1]$, and $L_J = \Omega(L_I)$, where $J = \{i+1, \ldots, i+k-1, j+k-1\}$. Then the corresponding Auslander-Reiten sequences are as follows:*
(1) *if $j \neq i+2$, then*

$$L_I \hookrightarrow \frac{L_X}{L_Y} \twoheadrightarrow L_J,$$

with $X = \{i+1, j, j+1, \ldots, j+k-3, j+k-1\}$ and $Y = \{i, i+2, i+3, \ldots, i+k-1, j+k-2\}$,
(2) *if $j = i+2$, then*

$$L_I \hookrightarrow P_i \oplus L_U \twoheadrightarrow L_J,$$

with $U = \{i, i+2, i+3, \ldots, k+i-1, k+i+1\}$.

Proof. We will prove the claims for $i = 1$, the statement then follows from the symmetry of B. By Corollary 2.15, we know that $\dim \operatorname{Ext}^1(L_J, L_I) = 1$, so the middle term M is rigid by Remark 2.16.

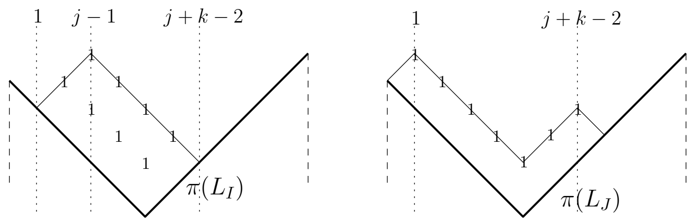
Note that M is a rank 2 module, so it is either a direct sum of two rank 1 modules or indecomposable module of rank 2. The projective cover of M is a direct summand of the direct sum of the projective covers of L_I and L_J, i.e. a summand of $P_0 \oplus P_{j-1} \oplus P_1 \oplus P_{j+k-2}$.

Suppose that $M = L_U \oplus L_V$. By the above, the peaks of M belong to $\{0, 1, j-1, j+k-2\}$.

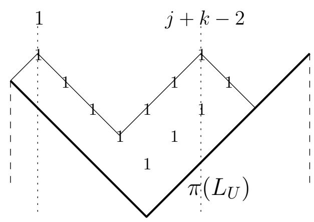
(i) If M has a projective summand, say $L_V = P_a$ for some a, we have an irreducible monomorphism $L_I \hookrightarrow P_a$, hence $L_I = \mathrm{rad}(P_a)$. In that case, $I = \{a, a+2, \dots, a+k\}$, i.e. $a = 1$, $j = 3$ and L_U is as claimed in part (2) of the theorem.

(ii) If none of the summands of M are projective, they have two peaks each, so $U = A \cup B$ and $V = C \cup D$ are two-interval subsets of C_1.

We first claim that if the vertices 1 and $j+k-2$ are the two peaks of L_U, then $L_U \cong L_J$. To see this, let $U = A \cup B = \{2, \dots\} \cup \{j+k-1, \dots\}$. One checks that the dimension of $\pi(L_I \oplus L_J)$ is zero at vertices $j+k-1$ and $j+k, \dots, n$ (see the following two figures).



If $|B| > 1$, then $\pi(L_U)$ has positive dimension at vertex $j+k-1$ (see the following figure).



Hence $|B| = 1$, i.e. $U = J$.

Now we claim that for $U = J$, the exact sequence $L_I \to L_V \oplus L_U \to L_J$ is not an Auslander-Reiten sequence. Assume that L_I is a summand of a middle term of an Auslander-Reiten sequence starting at L_I. Complete L_I to a cluster tilting object T as in Remark 1.7. Then the quiver of the endomorphism algebra $\mathrm{End}_{\underline{\mathrm{CM}}(B)}(T)$ has a loop at the vertex L_I corresponding to the irreducible map $L_I \to L_I$, which is a contradiction.

So U and V both contain one peak from I and one from J. Say U contains the peak at 1 and V the peak at $j+k-2$. Since 0 cannot be a peak of U, we have $U = \{2, \dots\} \cup \{j, \dots\}$ and $V = C \cup D = \{1, \dots\} \cup \{j+k-1, \dots\}$. By the same argument as above, $|D| = 1$. Applying π to our short exact sequence yields that the dimension at vertex 2 of $\pi(L_U)$ is 1, whereas at vertex 2 of $\pi(L_V)$ it is 0. However, $\pi(L_I \oplus L_J)$ has dimension 2 at vertex 2, which is a contradiction.

We assume now that M is indecomposable. By the discussion above (Remark 2.16) M is rigid so it is determined by its profile, and the profile provides a filtration by rank 1 modules, $M = L_X | L_Y$. There are infinitely many injective maps of L_I into M. They differ by the relative positions of L_I and the quotient L_J on the lattice diagram of M covering the dimension vector of M. There are five distinct cases: the rim of L_I can be strictly lower than the rim of L_J, they can touch, intersect properly, or the rim of L_J can be below L_I, touching or being strictly lower, as in Figures 2 and 3.

Note that all these representations have up to three disjoint regions with 1-dimensional vector spaces at finitely many vertices mapping to the infinite region with 2-dimensional vector spaces at every vertex. The only case providing a profile of an indecomposable module is the left one in Figure 3, yielding $X = \{2, j, \dots, j+k-3, j+k-1\}$ and $Y = \{2, 4, \dots, k, j+k-2\}$ $\square$

Remark 2.18. In case $k = 3$, Theorem 2.17 covers all possible Auslander-Reiten sequences where the start and end terms are of rank 1. By [BB16, Section 2], the rank of $\tau^{-1}(L_J)$ is

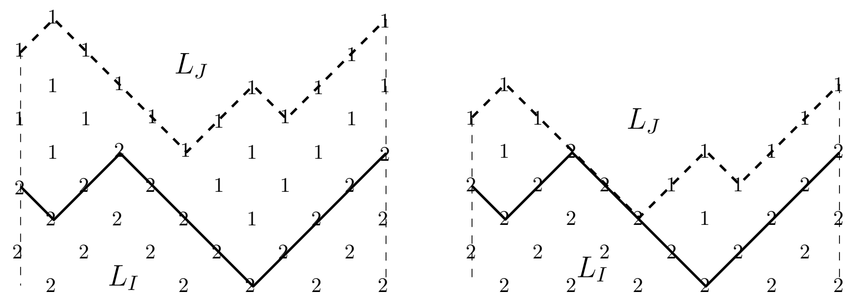


Figure 2. Rank 2 modules with submodule L_I and quotient L_J

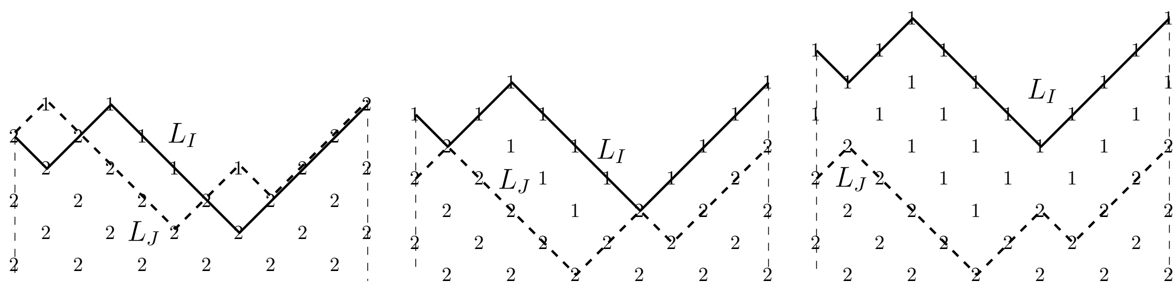


Figure 3. Rank 2 modules with submodule L_I and quotient L_J, continued

one less than the number of peaks of J, so rank $\tau^{-1}(L_J) = 1$ if and only if J is a 2-interval subset. Since $k = 3$, the 2-interval subsets are of the form $\{i, i+1, i+m\}$ with $2 < m < n-1$.

Remark. Assume $k = 3$. If we are given a rank 2 module M with profile $X|Y$, then how do we know if it is the middle term of an Auslander-Reiten sequence with rank 1 modules L_I and $\Omega(L_I)$? From the previous theorem, and the diagram on the left hand side in the previous picture, in order for M to be such a module, X and Y must be 3-interlacing (see the definition in Section 4), and when drawn one above the other (as in the above picture), the diagram we obtain has to contain three consecutive diamonds, with no gaps between them and with the end two diamonds of lateral size 1. In order to recognize the rims of L_I and $\Omega(L_I)$ from the profile $X|Y$, the easiest thing to do is to identify the middle diamond if it happens to be of size greater than 1 (as in the above picture). If the middle diamond is of size 1, then we can identify the right hand side diamond, since it is followed by an upward "tail like" portion of the rim covered by both rim X and rim Y in the above picture. If there is no tail, then we only have three diamonds of size 1, and then we deal with the module $\{1,3,5\}|\{2,4,6\}$.

Other rigid indecomposable rank 2 modules whose profile $X|Y$ does not satisfy these conditions do not appear as the middle term of an Auslander-Reiten sequence with rank 1 modules. In the tame cases, they either appear at the mouth of a tube, or they are meshes of the modules with at least one of them of rank greater than 1 as we will see later.

3. Auslander-Reiten periodicity and tame types

The category $\underline{\mathrm{CM}}(\mathrm{B}_{k,n})$ has cluster-tilting objects whose endomorphism algebras have rectangular quivers built by $(k-1) \times (n-k-1)$ lines of arrows, forming alternatingly oriented squares. These are exactly the quivers of the rectangular arrangements from [Sco06, Section 4]. We will denote them by $Q\Box Q'$, where Q is a Dynkin quiver of type A_{k-1} and Q' of type

A_{n-k-1}, as in [Kel13]. These are quivers with a natural potential P, in the sense of [DWZ08], given by the sum of all clockwise cycles minus the sum of all anti-clockwise cycles. We will write QP to abbreviate 'quiver with potential'.

Example 3.1. For $(3,9)$ and for $(4,8)$ the rectangles $Q\square Q'$ are as follows:

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet & \bullet \arrow[l] \\
\bullet \arrow[u] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[u] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[d] & \bullet \arrow[l] \\
\bullet \arrow[u] \arrow[d] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[u] \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[u] & \bullet \arrow[l]
\end{tikzcd}$$

3.1. **General case.**

Let k be a field. Let Q and Q' be two orientations of Dynkin quivers of type A with Coxeter numbers h and h'. Assume that Q and Q' are linearly oriented. Consider the algebra $A = kQ \otimes kQ'$. This is an algebra of global dimension 2 and it is possible to construct the corresponding (Hom-finite) generalized cluster category $\mathcal{C}_A$ in the sense of [Ami09].

By [KR07, Section 2.1], there exists an equivalence of categories

$$\mathcal{C}_A/(\Sigma T) \leftrightarrow \operatorname{mod} Jac(Q,P) \leftrightarrow \underline{\mathrm{CM}}(\mathrm{B}_{k,n})/((\Omega)^{-1}T'),$$

where (ΣT) is the ideal of morphisms that factor through $\operatorname{add}\Sigma T$ for a cluster tilting object T (respect. T' and $(\Omega)^{-1}T'$) over the 2-CY category. These categories are Krull-Schmidt, so by [KR07, Section 3.5], the Auslander-Reiten quiver of $\operatorname{mod} Jac(Q,P)$ is obtained from the Auslander-Reiten quiver of the 2-CY categories removing the vertices corresponding to ΣT (respect. $(\Omega)^{-1}T'$) and this establishes a connection between the Auslander-Reiten quiver of the 2-CY categories.

There is a cluster-tilting object T in $\mathcal{C}_A$ such that the endomorphism algebra $\operatorname{End}_{\mathcal{C}_A}(T)$ is given by a QP $(Q\boxtimes Q', \widetilde{P})$. The quiver $Q\boxtimes Q'$ has rectangular shape, with Q and Q' as rows and columns, and it has diagonal arrows which are all parallel, forming oriented triangles throughout. In the cases $(3,9)$ and $(3,8)$, these QP are

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] \arrow[dl] & \bullet \arrow[r] \arrow[dl] & \bullet \arrow[r] \arrow[dl] & \bullet \arrow[dl] \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] \arrow[dl] & \bullet \arrow[dl] \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] \arrow[dl] & \bullet \arrow[u] \arrow[dl] \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u]
\end{tikzcd}$$

with potential the sum of all positive 3-cycles minus the sum of all negative 3-cycles. (Note that this agrees with the natural potential of the dimer model from [BKM16, Section 3].) Observe that these are mutation-equivalent to the rectangular QP $(Q\square Q', P)$ from Example 3.1.

In the following, we denote Σ by the shift functor, S the Serre functor, and $\tau_{\mathcal{C}}$ the Auslander-Reiten translation over $\mathcal{C}_A$. The property of being 2-CY means that there is an isomorphism of functors $\Sigma^2 = S$. On the other hand S is $\Sigma\tau$ over $\mathcal{C}_A$, so $\tau_{\mathcal{C}} = \Sigma$. Keller's proof of the periodicity of the Zamolodchikov transformation $(\tau\otimes 1)$ [Kel13, Theorem 8.3] indicates a way to show that $\tau_{\mathcal{C}}$ is $2n$-periodic.

First it is shown that $(\tau\otimes 1)^h = \Sigma^{-2}$ is an isomorphism of functors over $\mathcal{C}_A$. Later, the author shows that $(\tau\otimes 1)^{h+h'} = \mathbb{1}$ is isomorphism of functors of $\mathcal{C}_A$. We can do the following:

$$\mathbb{1} = \mathbb{1}^h = (\tau\otimes 1)^{(h+h')h} = (\tau\otimes 1)^{h^2}(\tau\otimes 1)^{hh'} = (\Sigma^{-2})^h(\Sigma^{-2})^{h'} = \Sigma^{-2h-2h'} = \tau_{\mathcal{C}}^{-2(h+h')}$$

In our situation: $h = k$ and $h' = n-k$, so $2(h+h') = 2n$. Hence, we have that $\tau_{\mathcal{C}}^{2n} = \mathbb{1}$. This implies that the objects of $\underline{\mathrm{CM}}(\mathrm{B}_{k,n})$ are periodic under τ.

3.2. **Tame cases.** For $k = 2$, such $Q\Box Q'$ is of type A_{n-3}, for $(3,6)$ of (mutation) type D_4, for $(3,7)$ of type E_6 and for $(3,8)$ of type E_8. The Dynkin types above are the only cases of finite representation type, whereas the two cases $(3,9)$ and $(4,8)$ in Example 3.1 are the first cases of infinite representation type. The quivers with potential give rise to path algebras with relations whose mutation class representation theory is studied in [GGS15]. They correspond to *elliptic types* $E_{7,8}^{(1,1)}$, known to be tame [GLFS16, Theorem 9.1]. Besides the cases $(3,9),(4,8)$ and the finite types, all other QP algebras obtained from rectangular arrangements are of *wild* type.

In the cases $(3,9)$ and $(4,8)$ the QP's also arise from two cases of 2-Calabi Yau categories $\mathcal{C}(A)$, called *tubular cluster categories.* The case $(3,9)$ corresponds to the type $(6,3,2)$, and the case $(4,8)$ corresponds to $(4,4,2)$. It is known that $\mathcal{C}(A)$ is formed by a coproduct of tubular components $\coprod_{x\in\mathbb{X}}\mathcal{T}_x$ where almost all tubes $\mathcal{T}_x$ are of rank one, and there are finitely many tubes of ranks $6,3,2$ (resp. $4,4,2$) [BKL10]. Therefore, the Auslander-Reiten quiver for the cases $(k,n) = (3,9)$ and $(4,8)$ are formed by finitely many tubes of ranks $6,3,2$ (resp. $4,4,2$), and infinitely many homogeneous tubes.

4. Rank two modules and root combinatorics

In this section, we deal with rigid indecomposable rank 2 modules. Recall that the rank 1 modules in $\mathrm{CM}(\mathrm{B}_{k,n})$ are in bijection with k-subsets of n. The rigid indecomposable rank 2 modules can also be described combinatorially, as we will see here. We will give an explicit construction of rank 2 modules and study their properties.

Definition 4.1 (r-interlacing)**.** Let I and J be two k-subsets of $[1,n]$. I and J are said to be *r-interlacing*, if there exist subsets $\{i_1,i_3,\ldots,i_{2r-1}\}\subset I\setminus J$ and $\{i_2,i_4,\ldots,i_{2r}\}\subset J\setminus I$ such that $i_1<i_2<i_3<\cdots<i_{2r}<i_1$ (cyclically) and if there exist no larger subsets of I and of J with this property.

If I and J are r-interlacing, then the poset of $I \mid J$ is $(1^r,2)$, see Figure 2.1 for $r=3$. The module in question is indecomposable for $r\geq 3$ (see Remark 2.2).

Proposition 4.2. *Let I and J be r-interlacing. Then there exist $0\leq a_1\leq a_2\leq\ldots a_{r-1}$ such that, as Z-modules,*

$$\mathrm{Ext}^1(L_I,L_J)\cong\mathbb{C}[[t]]/(t^{a_1})\times\mathbb{C}[[t]]/(t^{a_2})\times\cdots\times\mathbb{C}[[t]]/(t^{a_{r-1}}).$$

Proof. We assume that we have drawn the rims I and J one above the other, say I above J, as in the proof of Theorem 3.1 in [BB16, Section 3]. For every $i_{2s}\in J\setminus I$ we have that rims I and J are not parallel between points i_{2s-1} and i_{2s}, yielding a left trapezium. Similarly, for every $i_{2s+1}\in I\setminus J$ we have that rims I and J are not parallel between points i_{2s} and i_{2s+1}, yielding a right trapezium. Since I and J are r-interlacing, we have, in alternating order r-left and r-right trapezia, giving us in total r boxes. The statement now follows from the proof of Theorem 3.1 in [BB16] which says that $\mathrm{Ext}^1(L_I,L_J)$ is a product of $r-1$ cyclic Z-modules. □

Corollary 4.3. *Let $k=3$ and I and J be rims. Then $\mathrm{Ext}^1(L_I,L_J)\cong\mathbb{C}\times\mathbb{C}$ if and only if I and J are 3-interlacing.*

Proof. If I and J are 3-interlacing, then they are both unions of three one-element sets. Hence, all the lateral sides in the boxes from the above proof are of length 1 and the statement follows since a_i from the previous proposition are strictly positive, but at most equal to the lengths of the boxes involved. □

Corollary 4.4. *Let $k = 3$. If I and J are crossing but not 3-interlacing, then* $\mathrm{Ext}^1(L_I, L_J) \cong \mathbb{C}$.

Note that if in Corollary 4.4 we have $L_J = \tau(L_I)$, then we are in the situation of Theorem 2.17.

Proposition 4.5. *Assume that $(k, n) = (3, 9)$ or $(k, n) = (4, 8)$. Let $M \in \mathrm{CM}(\mathrm{B}_{k,n})$ be a rigid indecomposable rank 2 module. Then $M \cong L_I \mid L_J$ where I and J are 3-interlacing.*

Proof. Let $M = L_I \mid L_J$ be rigid indecomposable, with I and J r-interlacing. Since M is indecomposable, we get $r \in \{3, 4\}$. If $r = 4$, then we must have $k = 4$, and the only 4-interlacing 4-subsets are $I = \{1, 3, 5, 7\}$ and $J = \{2, 4, 6, 8\}$. Assume that $M = L_I|L_J$ is rigid. Then it has a filtration given by its profile. Moreover, if any other module is rigid with the same profile it is isomorphic to M. On the other hand, if M is rigid, then $\tau(M)$ is also rigid. If we compute $\tau^{-1}M = \Omega(M)$ in $\mathrm{CM}(\Pi_{4,8})$, we obtain that they have the same filtration so $\tau^{-1}M = M$. So M and $\tau^{-1}(M)$ are the end-terms of an Auslander-Reiten-sequence and M is not rigid, a contradiction. Then I and J must be 3-interlacing. □

Corollary 4.6. *Let $M \in \mathrm{CM}(B_{3,9})$ be a rigid indecomposable rank 2 module. Then $\varphi(M)$ is a real root for $J_{3,9}$ of degree 2.*

Proof. By Proposition 4.5, $M \cong L_I \mid L_J$ with I and J 3-interlacing, so $I \cup J$ consists of six distinct elements of $\{1, 2, \ldots, n\}$. But then in $\underline{a}(M) = (a_1, \ldots, a_9)$ there are six entries equal to 1 and three entries equal to 0. So $q(\underline{a}) = \sum a_i^2 - \frac{1}{9}(\sum a_i)^2 = 6 - 4 = 2$. □

We observe that there exist rigid rank 2 modules corresponding to imaginary roots, an example is $L_{2568} \mid L_{1347}$ ([JKS16, Figure 13]). We expect that if we impose that the modules correspond to real roots, we get a counterpart to Proposition 4.5.

Proposition 4.7. *Let M be a rank 2 indecomposable module with profile $I|J$. If M corresponds to a real root of $J_{k,n}$, then $|I \cap J| = k - 3$.*

Proof. Assume that $|I \cap J| = k - 3 - m$ for some $m \geq 0$. Note that $|I \cap J|$ cannot be greater than $k-3$, because in this case, the corresponding poset would be either $(1^2, 2)$ or $(1, 2)$, which do not correspond to an indecomposable module. If there are $k - 3 - m$ common elements in I and J, then the corresponding vector, say $\underline{a}$, has $k - 3 - m$ coordinates equal to 2, $2(m+3)$ coordinates equal to 1, and the rest are equal to 0. If we apply our quadratic form q to this vector $\underline{a}$, then we get that $q(\underline{a}) = 2-2m$. In order for $\underline{a}$ to be real, it has to be that $m = 0$. □

Example 4.8. If $(k, n) = (3, 9)$, and M is a rank 2 indecomposable module with profile $I|J$. The conditions on I and J imply that the form q of such a module evaluates to 2, as in that case, $\underline{a} = (1, 1, 1, 1, 1, 1, 0, 0, 0)$, up to permuting the entries.

Corollary 4.9. *Let M be a rank 2 indecomposable module corresponding to a real root. Then the poset of M is $(1^3, 2)$.*

Proof. If the poset is of the form $(1^r, 2)$, where $r \geq 4$, then $|I \cap J| < k - 3$. If $r \leq 2$, then the module in question is not indecomposable. □

It follows that necessary conditions for a rank 2 indecomposable module M with profile $I|J$ to be rigid are $|I \cap J| = k - 3$ and that the poset of M is $(1^3, 2)$. Unfortunately, we do not know if these conditions are sufficient in the general case. We will show that these conditions are sufficient in the tame cases, and we conjecture that it holds in general.

Notation 4.10. Let I and J be 3-interlacing k-subsets. We say that I and J are *tightly 3-interlacing* if $|I \cap J| = k - 3$.

Lemma 4.11. *Let $M \in \mathrm{CM}(\mathrm{B}_{k,n})$ be a module with $M \cong L_I \mid L_J$ where I and J are tightly 3-interlacing. Then M is indecomposable and $q(M)$ is a real root for $J_{k,n}$.*

Proof. The poset of M is $(1^3, 2)$, thus M is indecomposable. Since $|I \cap J| = k-3$, up to permuting the entries, $\underline{a}(M) = (\underbrace{2,\dots,2}_{k-3}, \underbrace{1,\dots,1}_{6}, 0,\dots,0)$, yielding $q(M) = q(a) = 2$. □

We now give a construction for rank 2 modules in $\mathrm{CM}(\mathrm{B}_{k,n})$ in the spirit of the definition of rank 1 modules (Definition 1.2).

Definition 4.12. Let I and J be two k-subsets which are r-interlacing for $r \geq 3$. Then we define a rank two module $L(I,J)$ in $\mathrm{CM}(\mathrm{B}_{k,n})$. For $i = 1,\dots,n$ let $V_i := e_i L(I,J) = \mathbb{C}[[t]] \oplus \mathbb{C}[[t]]$. We define maps $x_i, y_i : Z^2 \to Z^2$, $x_i : V_{i-1} \to V_i$, $y_i : V_i \to V_{i-1}$.

$$\begin{array}{rcll} & x_i & \text{multiplication by } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \\ & y_i & \text{multiplication by } \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} & \text{if } i \in I \cap J \\ & & & \\ x_i & = y_i & \text{multiplication by } \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} & \text{if } i \in (I \cap J^c) \cup (I^c \cap J) \\ & & & \\ & x_i & \text{multiplication by } \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} & \\ & y_i & \text{multiplication by } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \in I^c \cap J^c \end{array}$$

The module $L(I,J)$ can be represented by a lattice diagram $\mathcal{L}_{I,J}$ obtained by overlaying the lattice diagrams $\mathcal{L}_I$ and $\mathcal{L}_J$ from Section 1 such that $\mathcal{L}_I$ is above $\mathcal{L}_J$ where the two rims are meeting at at least one vertex and possibly share arrows but have no two-dimensional intersections. The vector spaces $V_0, V_1, V_2, \dots, V_n$ are represented by columns from left to right (with V_0 and V_n to be identified). The vertices in each column correspond to some monomial $\mathbb{C}$-basis of $\mathbb{C}[t] \oplus \mathbb{C}[t]$, depending on the two sets I and J. Note that the k-subset I can then be read off as the set of labels on the arrows pointing down to the right which are exposed to the top of the diagram and the k-subset J can be read off as the set of labels on the arrows pointing down to the right between two-dimensional vertices. To illustrate this, the lattice picture $\mathcal{L}_{I,J}$ for $I = \{2,5,8,9\}$ and $J = \{1,3,7,8\}$, $(k,n) = (4,9)$ is shown in Example 4.13.

Example 4.13. Let $(k,n) = (4,9)$. Consider the tightly 3-interlacing 4-subsets $I = \{2,5,8,9\}$ and $J = \{1,3,7,8\}$. The module $L(I,J)$ is illustrated in Figure 4.

Lemma 4.14. *(1) Let I and J be r-interlacing, for $r \geq 3$. Then $L(I,J)$ is a rank 2 module in $\mathrm{CM}(\mathrm{B}_{k,n})$ with filtration $L_I \mid L_J$.*
(2) If I and J are tightly 3-interlacing, then $L(I,J)$ and $L(J,I)$ are indecomposable.

Note that Lemma 4.14 (2) provides modules with cyclically reordered filtration, as discussed in [JKS16, Observation 8.2].

Proof. (1) For abbreviation, let $B = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$. We first note $x_i y_i = t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all i and that $B^2 = t \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we will use this frequently below.

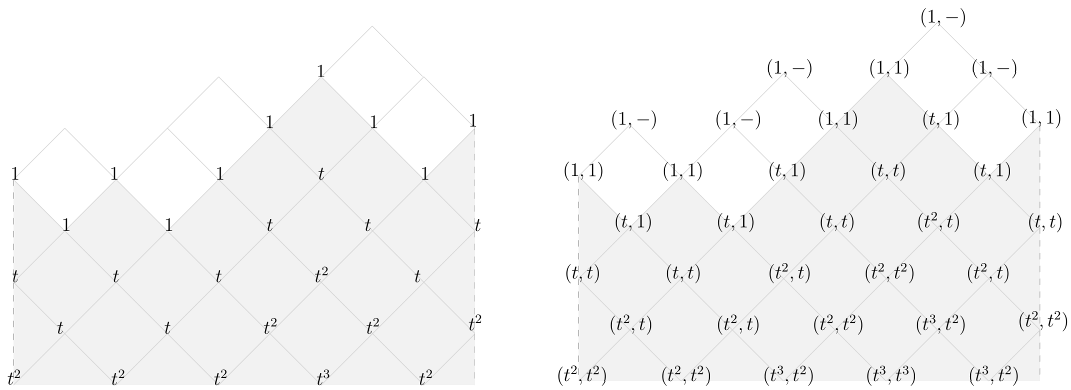


FIGURE 4. Lattice diagrams for $L(\{1,3,7,8\})$ and $L(\{2,5,7,8\},\{1,3,7,8\})$, with $n=9$

Set $u = |I \cap J|$, $v = |I^c \cap J^c|$, so $|(I \cup J) \setminus (I \cap J)| = 2(k-u)$. We check that $x^k = y^{n-k}$ holds everywhere. For this, observe that all the x_i commute among themselves and that all the y_i commute. Consider $x^k = x_{i+k-1}x_{i+k-2}\cdots x_i$ for some i (reducing modulo n) and let a_0 be the number of indices from $I^c \cap J^c$ appearing in x^k, let a_1 be the number of indices from $(I \cap J^c) \cup (I^c \cap J)$ and a_2 the number of indices in $I \cap J$. In particular, $a_0 + a_1 + a_2 = k$. So

$$x^k \quad = \quad t^{a_0} B^{a_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{a_2} = t^{a_0} B^{a_1} = B^{a_1 + 2a_0}$$

On the other hand,

$$\begin{aligned} y^{n-k} \quad &= \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{v-a_0} B^{2(k-n)-a_1} t^{u-a-2} = B^{2(k-n)-a_1} \cdot t^{u-a_2} \\ &= \quad B^{-a_1} t^{k-u} t^{u-a_2} = B^{-a_1} t^{k-a_2} = B^{2k-2a_2-a_1} = B^{a_1+2a_0} \end{aligned}$$

So $L(I,J)$ is a module for $\mathrm{B}_{k,n}$. The module L_J embeds in $L(I,J)$ diagonally via the map $a \mapsto (a,a)$ lattice point wise, as in Figure 4 of Example 4.13, sending column U_i to column V_i in a way to get an image of L_J as high up as possible. This yields an exact sequence $0 \to L_J \to L(I,J) \to L_I \to 0$. So $L(I,J) \in \mathrm{CM}(\mathrm{B}_{k,n})$, with filtration as claimed.

That it is a rank 2 module follows from the construction.

(2) The indecomposability follows from the fact that for tightly 3-interlacing k-subsets both modules have poset $(1^3, 2)$. □

We expect that tightly 3-interlacing subsets always yield rigid modules. Combined with the preceding statements, this would give us:

Conjecture 4.15. *Fix (k,n) with $k \geq 3$. Let I and J be tightly 3-interlacing. Then $L(I,J)$ is a rigid indecomposable rank 2 module.*

Theorem 2.17 provides further evidence for Conjecture 4.15 for arbitrary (k,n): We can use part 1 to find many examples of rigid indecomposable rank 2 modules $L(X,Y) = L_X \mid L_Y$ where X and Y are tightly 3-interlacing k-subsets satisfying $|X \cap Y| = k-3$ by choosing $j = i+3$ in Theorem 2.17.

Conjecture 4.16. *Let M be a rank 2 module with poset $(1^r, 2)$, for $r \geq 4$. Then M is not rigid.*

We recall that in the finite cases with $k=3$, the numbers of (rigid) indecomposable rank 2 modules are 2 (for $n=6$), 14 (for $n=7$) and 56 (for $n=8$). All these correspond to real roots for the associated Kac-Moody algebra $J_{k,n}$.

In the tame cases $(3,9)$ and $(4,8)$, we will show that there are, respectively, 168 and 120 rigid indecomposable modules of rank 2. This follows from Proposition 4.5 and the fact that in these cases, the real root in question has to correspond to a 9-tuple $(a_1,\dots,a_9)$ with six entries equal to 1 and zeros elsewhere, or to an 8-tuple $(a_1,\dots,a_8)$ with one entry equal to 2, six entries equal to 1 and one 0 respectively. We will confirm the above numbers explicitly in the sections 5 and 6 by computing all tubes that contain rank 2 modules in the Auslander-Reiten quiver.

This leads us to a conjectured formula for the number of rigid indecomposable rank 2 modules which correspond to real roots. The 3-interlacing property (Proposition 4.5) yields the factor $\binom{n}{6}$, a choice of 6 elements from $[1,n]$, say $\{1\le i_1<j_1<i_2<j_2<i_3<j_3\le n\}$, with $\{i_1,i_2,i_3\}\subset I$ and $\{j_1,j_2,j_3\}\subset J$. If Conjecture 4.15 is true, each pair I and J of 3-interlacing subsets where the remaining $k-3$ labels are common to I and J yields two rigid indecomposable rank 2 modules. Using the map from indecomposable modules to roots for $J_{k,n}$ (see Section 1.2) we see that these give rise to real roots. So there is a choice of $k-3$ elements from the remaining $n-6$ elements of $[1,n]$, yielding a factor $\binom{n-6}{k-3}$. Finally, there is a factor 2 which arises from the choice of which of these subsets is I and which is J. The above arguments give an upper bound for the number of rigid indecomposable rank 2 modules corresponding to real roots.

Conjecture 4.17. *Let $3\le k\le n/2$. For every real root α of degree 2 there are exactly two non-isomorphic rigid indecomposable rank 2 modules M_1 and M_2 such that $q(M_1)=q(M_2)=\alpha$. There are thus $2\binom{n}{6}\binom{n-6}{k-3}$ rigid indecomposable rank 2 modules corresponding to real roots is.*

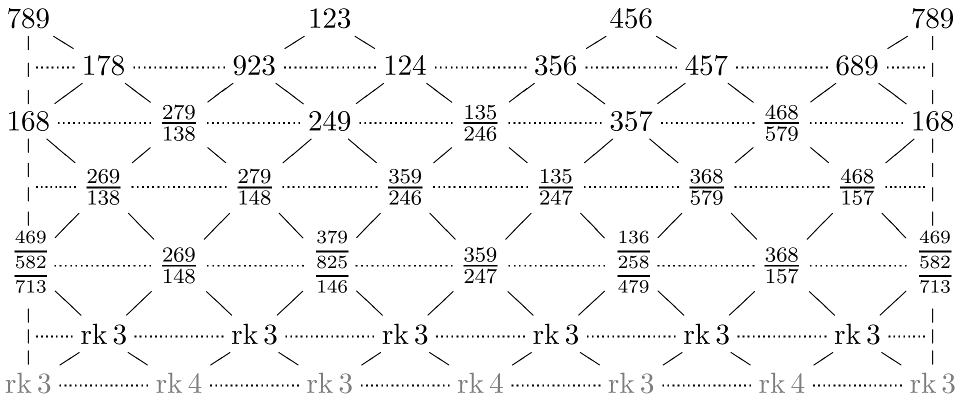
Remark. Note that we have proved (Proposition 4.7, Corollary 4.9) that the number $2\binom{n}{6}\binom{n-6}{k-3}$ is the number of indecomposable rank 2 modules corresponding to real roots. Combinatorially, indecomposable rank 2 modules corresponding to real roots are determined by the conditions $|I\cap J|=k-3$ (which follows from the quadratic form) and the poset of the module is $(1^3,2)$ (which follows from the indecomposability requirement).

5. The case $(k,n)=(3,9)$

Here, the tubes containing rigid indecomposable modules are of rank 2, 3 and 6.

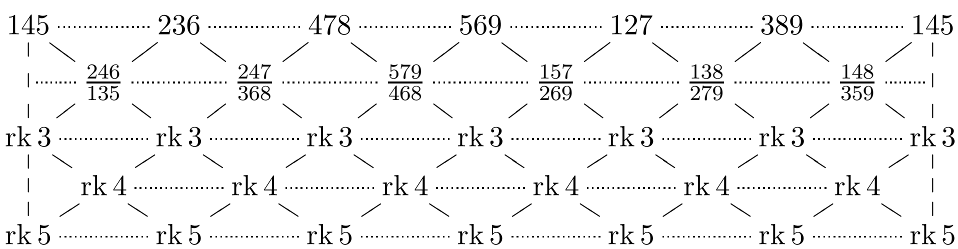
5.1. **Tubes containing rank 1 modules.** In the following, we describe all tubes containing rank 1 modules.

5.1.1. *Tubes containing projective-injectives.*

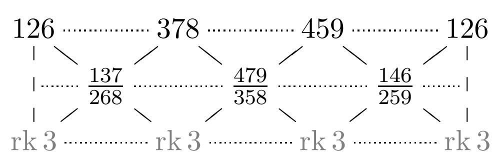


There are three such tubes. The labels of the rank 1 modules appearing are $\{i, i+1, i+2\}$, $\{i, i+1, i+3\}$, $\{i, i+2, i+3\}$ and $\{i, i+2, i+4\}$. The lowest row (with modules written in grey) is the first non-rigid level in this tube.

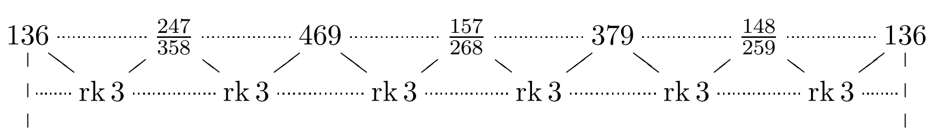
5.1.2. *Tubes with modules L_I for $I = \{i, i+1, i+4\}$ and $I = \{i, i+3, i+4\}$.*
These modules are contained in three rank 6 tubes of the form:



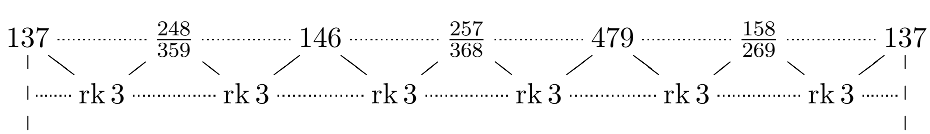
5.1.3. *Tubes with modules L_I for $I = \{i, i+1, i+5\}$.*
These modules are contained in three rank 3 tubes of the form:



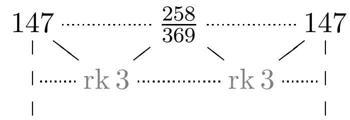
5.1.4. *Tubes with modules L_I for $I = \{i, i+2, i+5\}$ and $I = \{i, i+3, i+5\}$.*
There are three rank 6 tubes containing these modules, as follows.



5.1.5. *Tubes with modules L_I for $I = \{i, i+2, i+6\}$ and $I = \{i, i+6, i+6\}$.*
These modules are contained in three rank 6 tubes.



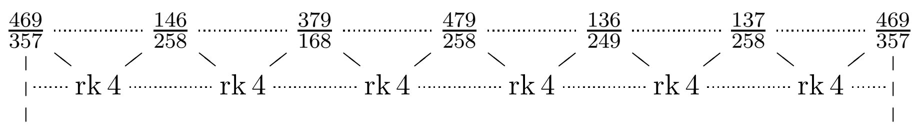
5.1.6. *Tubes with modules L_I for $I = \{i, i+3, i+6\}$.*
These modules are contained in three rank 2 tubes.



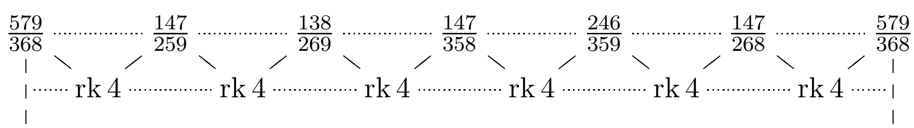
5.2. **Tubes without rank 1 modules, but with rank 2 modules.**

By Proposition 4.5 there are at most 168 rigid indecomposable rank 2 modules in $\mathrm{CM}(B_{3,9})$, because there are 84 3-interlacing pairs (I, J) in that case. The tubes we described above so far cover 84 of them. The remaining 84 rank 2 modules are at the mouths of further tubes of rank 2,3 and 6. There are six tubes of rank 6 where every module at the mouth is a rank 2 module and three tubes of rank 3 where every module at the mouth is a rank 2 module. Then there are twelve tubes of rank 6, where the modules at the mouth alternate between rank 2 and rank 3, and three tubes of rank 2 where the modules at the mouth alternate between rank 2 and rank 4.

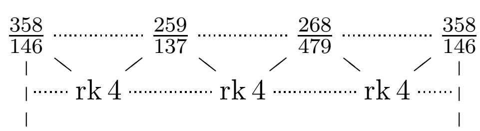
5.2.1. *Tubes with modules with filtration $\{i, i+2, i+5\}|\{i-1, i+1, i+3\}$ and $\{i-3, i, i+2\}|\{i-2, i+1, i+4\}$.* There are three tubes of this form.



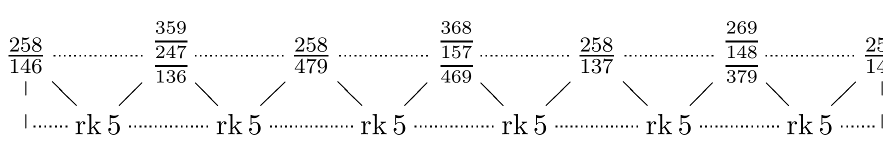
5.2.2. *Tubes with modules with filtration $\{i, i+2, i+4\}|\{i-2, i+1, i+3\}$ and $\{i-4, i-1, i+2\}|\{i-3, i, i+4\}$.* There are three tubes of this form.



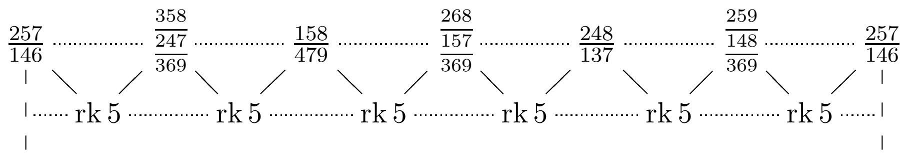
5.2.3. *Tubes with modules with filtration $\{i, i+2, i+5\}|\{i-2, i+1, i+3\}$ and $\{i-1, i+2, i+6\}|\{i-2, i, i+4\}$.* There are three tubes of this form.



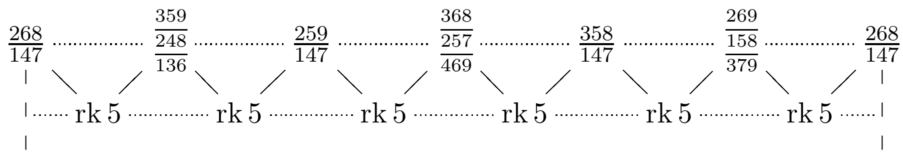
5.2.4. *Tubes with modules with filtration $\{i, i+3, i+6\}|\{i-1, i+2, i+4\}$.* There are three tubes of this form.



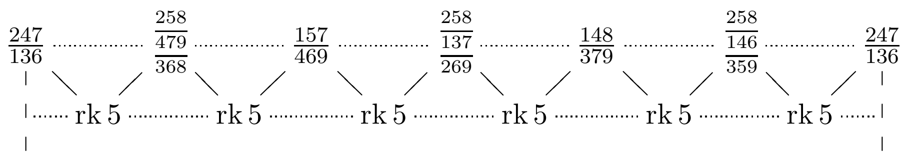
5.2.5. *Tubes with modules with filtration* $\{i, i+3, i+5\}|\{i-1, i+2, i+4\}$. There are three tubes of this form.



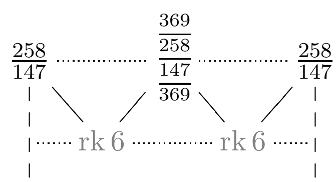
5.2.6. *Tubes with modules with filtration* $\{i, i+4, i+6\}|\{i-1, i+2, i+5\}$. There are three tubes of this form.



5.2.7. *Tubes with modules with filtration* $\{i, i+2, i+5\}|\{i-1, i+1, i+4\}$. There are three tubes of this form.



5.2.8. *Tubes with modules with filtration* $\{i, i+3, i+6\}|\{i-1, i+2, i+5\}$. There are three tubes of this form.



5.3. **Counting rigid indecomposables for** $Gr(3,9)$**.**

To sum up, the number of rigid indecomposable modules of rank 1, 2 and 3 in the above tubes, listed by rank is:

rank	1	2	3
#	84	168	117

We have collected all rank 1 and rank 2 rigid indecomposable modules. Since there are 168 rigid indecomposable modules whose roots are real, and 84 real roots of $J_{3,9}$ of degree 2, we have proved that the number of rigid indecomposable modules whose corresponding roots are real is twice the number of degree 2 cluster variables. Also, for every rank 2 rigid indecomposable module whose root is real, and whose filtration is $L_I|L_J$, there exist a rigid indecomposable module with filtration $L_J|L_I$. Therefore, both Theorem 2 and Conjecture 4.17 hold in this case.

6. The case $(k, n) = (4, 8)$

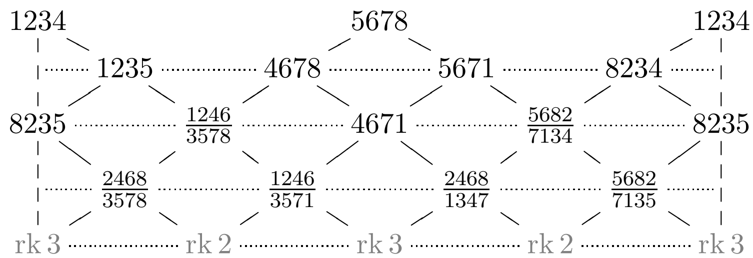
Here, the tubes containing rigid indecomposable modules are of rank 2 and 4.

6.1. **Tubes containing rank 1 modules.**

In the following, we describe all tubes containing rank 1 modules.

6.1.1. *Tubes with projective-injectives.*

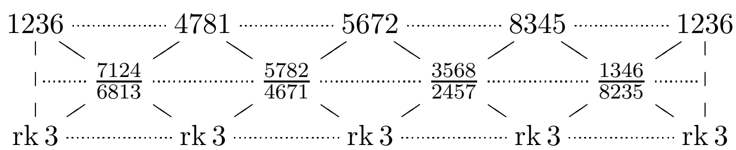
These tubes have rank 4, there are four of them, each contains two projective-injectives, four modules with rims with two peaks and two modules with rims with three peaks.



The labels of the rank 1 modules appearing in these tubes are $\{i, i+1, i+2, i+4\}$, $\{i, i+1, i+2, i+6\}$, $\{i, i+1, i+3, i+6\}$. The lowest row shows the first non-rigid level in this tube. Note that the rank 2 modules appearing in the third row of this tube do not correspond to the real roots, because the rims involved are disjoint.

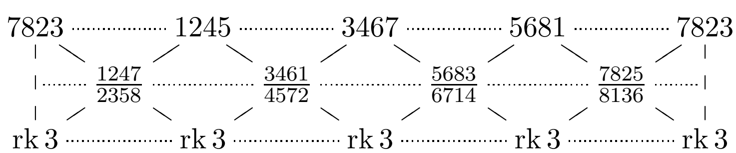
6.1.2. *Tubes with modules L_I for $I = \{i, i+1, i+2, i+5\}$ and $I = \{i, i+3, i+4, i+5\}$.*

These are two rank 4 tubes of the form



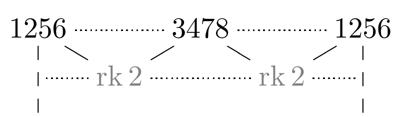
6.1.3. *Tubes with modules L_I for $I = \{i, i+1, i+3, i+4\}$.*

These are two rank 4 tubes of the form



6.1.4. *Tubes with modules L_I for $I = \{i, i+1, i+4, i+5\}$.*

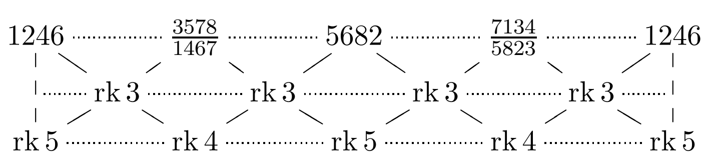
These are two rank 2 tubes of the form



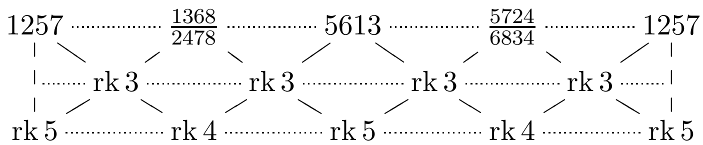
Note that the rank 2 modules appearing here are not rigid since the tube is of rank 2.

6.1.5. *Tubes with modules L_I for $I = \{i, i+1, i+3, i+5\}$.*

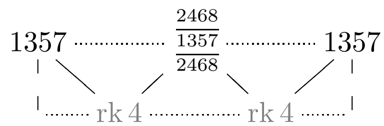
These are four rank 4 tubes of the form



6.1.6. *Tubes with modules L_I for $I = \{i, i+1, i+4, i+6\}$.*
These are four rank 4 tubes of the form



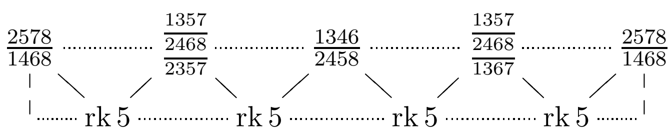
6.1.7. *Tubes with modules L_I for $I = \{i, i+2, i+4, i+6\}$.*
These are two rank 2 tubes of the form



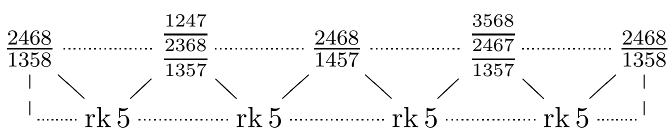
6.2. **Tubes without rank 1 modules containing rigid rank 2 modules.**

The remaining rigid indecomposable rank 2 modules are at the mouths of further tubes of rank 4. There are eight tubes of rank 4 where every module at the mouth is a rank 2 module. Then there are 16 tubes of rank 4, where the modules at the mouth alternate between rank 2 and rank 3.

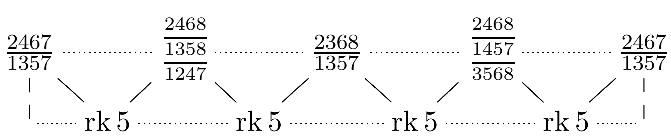
6.2.1. *Tubes with modules with filtration $\{i, i+3, i+5, i+6\}|\{i-1, i+2, i+4, i+6\}$.* There are four rank 4 tubes of the form



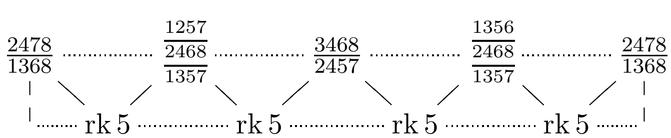
6.2.2. *Tubes with modules with filtration $\{i, i+2, i+4, i+6\}|\{i-1, i+1, i+3, i+6\}$.* There are four rank 4 tubes of the form



6.2.3. *Tubes with modules with filtration $\{i, i+2, i+4, i+5\}|\{i-1, i+1, i+3, i+5\}$.* There are four rank 4 tubes of the form



6.2.4. *Tubes with modules with filtration $\{i, i+2, i+5, i+6\}|\{i-1, i+1, i+4, i+6\}$.* There are four rank 4 tubes of the form



6.2.5. *Tubes with modules with filtration* $\{i, i+1, i+3, i+5\}|\{i, i+2, i+4, i+7\}$ *and* $\{i, i+2, i+5, i+6\}|\{i-1, i+1, i+4, i+5\}$. There are two rank 4 tubes of the form

$$\begin{tikzcd}[column sep=tiny]
\frac{1246}{1358} \arrow[rr, dotted, dash] \arrow[dr, dash] & & \frac{2478}{1367} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{2568}{1457} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{3468}{2357} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{1246}{1358} \arrow[dl, dash] \\
& \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 &
\end{tikzcd}$$

6.2.6. *Tubes with modules with filtration* $\{i, i+1, i+3, i+5\}|\{i+1, i+2, i+4, i+6\}$ *and* $\{i, i+1, i+3, i+5\}|\{i-2, i+1, i+2, i+4\}$. There are two rank 4 tubes of the form

$$\begin{tikzcd}[column sep=tiny]
\frac{1246}{2357} \arrow[rr, dotted, dash] \arrow[dr, dash] & & \frac{3468}{1457} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{2568}{1367} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{2478}{1358} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{1246}{2357} \arrow[dl, dash] \\
& \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 &
\end{tikzcd}$$

6.2.7. *Tubes with modules with filtration* $\{i, i+1, i+4, i+6\}|\{i, i+3, i+5, i+7\}$ *and* $\{i, i+2, i+4, i+5\}|\{i-1, i+1, i+3, i+4\}$. There are two rank 4 tubes of the form

$$\begin{tikzcd}[column sep=tiny]
\frac{1257}{1468} \arrow[rr, dotted, dash] \arrow[dr, dash] & & \frac{3578}{2467} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{1356}{2458} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{1347}{2368} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{1257}{1468} \arrow[dl, dash] \\
& \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 &
\end{tikzcd}$$

6.2.8. *Tubes with modules with filtration* $\{i, i+1, i+4, i+6\}|\{i+1, i+2, i+5, i+7\}$ *and* $\{i, i+2, i+3, i+6\}|\{i+1, i+3, i+4, i+7\}$. There are two rank 4 tubes of the form

$$\begin{tikzcd}[column sep=tiny]
\frac{1257}{2368} \arrow[rr, dotted, dash] \arrow[dr, dash] & & \frac{1347}{2458} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{1356}{2467} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{3578}{1468} \arrow[rr, dotted, dash] \arrow[dr, dash] \arrow[dl, dash] & & \frac{1257}{2368} \arrow[dl, dash] \\
& \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 \arrow[rr, dotted, dash] & & \mathrm{rk}\,4 &
\end{tikzcd}$$

6.3. **Summing up.** To sum up, the number of rigid indecomposable modules of rank 1, 2 and 3 in all tubes containing rank 2 modules, listed by rank is:

rank	1	2	3
#	70	120	82

Since there are 70 rank 1 modules for $(4, 8)$, we have covered all tubes containing such modules. Overall, there are 120 rank 2 rigid indecomposable modules. Among these, there are eight modules that do not correspond to the real roots. These are the modules with profile of the form 1246|3578. So there are 112 rigid indecomposable modules of rank 2 that correspond to real roots of $J_{4,8}$. Since there are 56 roots of degree 2, we have shown that the number of rigid indecomposable rank 2 modules corresponding to real roots is twice the number of real roots of degree 2 for $J_{4,8}$. Also, for every rank 2 rigid indecomposable module whose root is real, and whose filtration is $L_I|L_J$, there exist a rigid indecomposable module with filtration $L_J|L_I$. Therefore, both Theorem 2 and Conjecture 4.17 hold in this case.

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