Higher dimensional homological algebra

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1 Preface

These are notes on higher dimensional homological algebra for the LMS–CMI Research School "New trends in representation theory — The impact of cluster theory in representation theory", held at the University of Leicester 19–23 June 2017.

The notes take the view that the natural framework for higher dimensional homological algebra are d-abelian and $(d+2)$-angulated categories, as introduced by Jasso and Geiss–Keller–Oppermann. The theory of d-cluster tilting subcategories and higher Auslander–Reiten theory, as introduced by Iyama, are developed as means to this end.

The notes are organised as follows: Section 2 gives a list of notation. Section 3 introduces d-cluster tilting subcategories. Section 4 introduces higher Auslander–Reiten translations which, among other applications, permit the construction of objects of d-cluster tilting subcategories. Section 5 introduces d-abelian categories. The main examples are d-cluster tilting subcategories of abelian categories. Section 6 states the basics of the theory of higher Auslander–Reiten sequences. Section 7 introduces $(d+2)$-angulated categories. The main examples are d-cluster tilting subcategories of triangulated categories which are stable under Σ^d. Appendix A describes a class of examples by Vaso, and Appendix B an example by Iyama.

We do not describe any of the other examples which are known: Higher Auslander, Nakayama, and preprojective algebras, Iyama's cone construction, etc. In particular, we do not explain their rich combinatorial structure, which is one of the motivations for higher homological algebra. Several other important parts of the theory are not covered: The higher Auslander correspondence, d-exact categories, d-Frobenius categories, relative d-cluster tilting subcategories, etc.

None of the material in these notes is due to me. I have tried to give due credit for all definitions and results, but would be grateful to learn if there are missing, incomplete, or wrong attributions.

I thank the organisers of the LMS–CMI Research School, Karin Baur and Sibylle Schroll, for the opportunity to give a course, and for the chance to present these notes. I thank the participants in the School for their feedback and for spotting a number of typos.

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School, parts of which have subsequently been turned into Appendix B, and for providing several exercises.

I thank Laertis Vaso for permitting me to include the material in Appendix A.

2 Notation and Terminology

Symbol	Meaning
d	A positive integer
Φ	A finite dimensional $\mathbb{C}$-algebra
$\mathsf{mod}(\Phi)$	The category of finite dimensional right Φ-modules
$\underline{\mathsf{mod}}(\Phi)$	$\mathsf{mod}(\Phi)$ modulo morphisms which factor through a projective
$\overline{\mathsf{mod}}(\Phi)$	$\mathsf{mod}(\Phi)$ modulo morphisms which factor through an injective
$\mathsf{mod}(\Phi^{\mathrm{op}})$, $\underline{\mathsf{mod}}(\Phi^{\mathrm{op}})$, $\overline{\mathsf{mod}}(\Phi^{\mathrm{op}})$	The same for left Φ-modules
$\mathscr{A}$	An abelian category
$\mathscr{D}$	A triangulated category
$\mathscr{C}$, $\mathscr{F}$	d-cluster tilting subcategories
$\underline{\mathscr{F}}$	$\mathscr{F}$ modulo morphisms which factor through a projective
$\overline{\mathscr{F}}$	$\mathscr{F}$ modulo morphisms which factor through an injective
$\mathrm{D}(-)$	$\mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})$
$(-)^*$	$\mathrm{Hom}_{\Phi}(-,\Phi)$ if $-$ is a Φ-module
Tr_d	The d'th higher transpose
$\tau_d = \mathrm{D}\,\mathrm{Tr}_d$	The d-Auslander–Reiten translation
$\tau_d^- = \mathrm{Tr}_d\,\mathrm{D}$	The inverse d-Auslander–Reiten translation

3 d-cluster tilting subcategories

Definition 3.1. Let $\mathscr{A}$ be a category, $\mathscr{F} \subseteq \mathscr{A}$ a full subcategory. Then $\mathscr{F}$ is called:

(i) Generating if each $a \in \mathscr{A}$ permits an epimorphism $f \twoheadrightarrow a$ with $f \in \mathscr{F}$,

(ii) Cogenerating if each $a \in \mathscr{A}$ permits a monomorphism $a \hookrightarrow f$ with $f \in \mathscr{F}$.

Definition 3.2 (Enochs [3])**.** Let $\mathscr{A}$ be a category, $\mathscr{F} \subseteq \mathscr{A}$ a full subcategory. A morphism $a \xrightarrow{\alpha} f$ in $\mathscr{A}$ with $f \in \mathscr{F}$ is called:

- An *$\mathscr{F}$-preenvelope of a* if it has the extension property

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & f \arrow[dl, dotted, "\exists"] \\ f' & \end{tikzcd}$$

  for each morphism $a \to f'$ with $f' \in \mathscr{F}$.

- An *$\mathscr{F}$-envelope of a* if it is an $\mathscr{F}$-preenvelope which is *left minimal*, that is, satisfies that each morphism $f \xrightarrow{\varphi} f$ with $\varphi\alpha = \alpha$ is an automorphism.

We say that $\mathscr{F}$ is *preenveloping* (resp. *enveloping*) in $\mathscr{A}$ if each $a \in \mathscr{A}$ has an $\mathscr{F}$-preenvelope (resp. envelope).

The dual notion of preenvelope is *precover*, and $\mathscr{F}$ is called *functorially finite* if it is preenveloping and precovering. $\square$

The following definition is illustrated by Example A.4 and Remark B.4.

Definition 3.3 (Iyama [6, def. 2.2], Jasso [8, def. 3.14])**.** Let $\mathscr{A}$ be an abelian or triangulated category. A *d-cluster tilting subcategory $\mathscr{F}$ of $\mathscr{A}$* is a full subcategory which satisfies the following.

(i) $$\begin{aligned}\mathscr{F} &= \{a \in \mathscr{A} \mid \operatorname{Ext}^{1..d-1}_{\mathscr{A}}(\mathscr{F}, a) = 0\} \\ &= \{a \in \mathscr{A} \mid \operatorname{Ext}^{1..d-1}_{\mathscr{A}}(a, \mathscr{F}) = 0\}.\end{aligned}$$

(ii) $\mathscr{F}$ is functorially finite.

(iii) If $\mathscr{A}$ is abelian then $\mathscr{F}$ is generating and cogenerating.

A *d-cluster tilting object f of $\mathscr{A}$* is an object f such that $\mathscr{F} = \operatorname{add}(f)$ is a d-cluster tilting subcategory.

Remark 3.4. If $d = 1$ then $\mathscr{F} = \mathscr{A}$ is the unique choice of $\mathscr{F}$. This reflects that if $d = 1$, then higher dimensional homological algebra becomes classic homological algebra.

Remark 3.5. If $p \in \mathscr{A}$ is projective, then 3.3(i) implies $p \in \mathscr{F}$. Similarly, if $i \in \mathscr{A}$ is injective, then $i \in \mathscr{F}$. Hence if $\mathscr{A}$ has enough projectives and injectives, then 3.3(i) implies 3.3(iii).

The following definition is again illustrated by Example A.4.

Definition 3.6. Let $\mathscr{A}$ be an abelian category, $\mathscr{F}$ a full subcategory. An *augmented left $\mathscr{F}$-resolution* of $a \in \mathscr{A}$ is a sequence

$$\cdots \to f_2 \to f_1 \to f_0 \to a \to 0$$

with $f_i \in \mathscr{F}$ for each i, which becomes exact under $\mathscr{A}(f, -)$ for each $f \in \mathscr{F}$. Then

$$\cdots \to f_2 \to f_1 \to f_0 \to 0 \to \cdots$$

is called a *left $\mathscr{F}$-resolution* of a.

The resolutions are said to have *length* $\leqslant n$ if $f_j = 0$ for $j \geqslant n + 1$.

(Augmented) right resolutions are defined dually. $\square$

Proposition 3.7 (Iyama [6, prop. 2.2.2])**.** *Let $\mathscr{A}$ be an abelian category, $\mathscr{F} \subseteq \mathscr{A}$ a d-cluster tilting subcategory. Then each*

$a \in \mathscr{A}$ has (augmented) left and right $\mathscr{F}$-resolutions of length $\leqslant d-1$.

Proof. We give a proof for left resolutions in the special case $d = 2$.

There is an $\mathscr{F}$-precover $f \xrightarrow{\varphi} a$ by Definition 3.3(ii), and φ is an epimorphism by Exercise 3.8. Hence there is a short exact sequence $0 \longrightarrow k \longrightarrow f \xrightarrow{\varphi} a \to 0$. If $\widetilde{f} \in \mathscr{F}$ then there is a long exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{\mathscr{A}}(\widetilde{f}, k) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(\widetilde{f}, f) \xrightarrow{\varphi_*} \operatorname{Hom}_{\mathscr{A}}(\widetilde{f}, a) \qquad (1) \\ \longrightarrow \operatorname{Ext}^1_{\mathscr{A}}(\widetilde{f}, k) \longrightarrow \operatorname{Ext}^1_{\mathscr{A}}(\widetilde{f}, f). \end{aligned}$$

Here φ_* is surjective since φ is an $\mathscr{F}$-precover, and $\operatorname{Ext}^1_{\mathscr{A}}(\widetilde{f}, f) = 0$ by Definition 3.3(i) whence $\operatorname{Ext}^1_{\mathscr{A}}(\widetilde{f}, k) = 0$. This shows $k \in \mathscr{F}$. It also shows that

$$0 \to \operatorname{Hom}_{\mathscr{A}}(\widetilde{f}, k) \to \operatorname{Hom}_{\mathscr{A}}(\widetilde{f}, f) \to \operatorname{Hom}_{\mathscr{A}}(\widetilde{f}, a) \to 0$$

is exact for each $\widetilde{f} \in \mathscr{F}$ whence

$$\cdots \to 0 \to k \to f \to a \to 0$$

is an augmented left $\mathscr{F}$-resolution of a. $\square$

Exercise 3.8. Let $\mathscr{A}$ be an abelian category, $\mathscr{F}$ a full subcategory.

(i) Show that if $\mathscr{F}$ is generating, then each $\mathscr{F}$-precover is an epimorphism.

(ii) Show that if $\mathscr{F}$ is cogenerating, then each $\mathscr{F}$-preenvelope is a monomorphism.

Exercise 3.9. Let R be a commutative ring, $\mathscr{A}$ an R-linear Hom-finite Krull–Schmidt category, $\mathscr{F}$ a full subcategory closed under sums and summands. Show that if $\mathscr{F}$ has only finitely many isomorphism classes of indecomposable objects, then it is functorially finite.

Exercise 3.10. Let $\mathscr{A}$ be an abelian category, $\mathscr{F}$ a full subcategory. Show that if $\mathscr{F}$ is precovering, then each $a \in \mathscr{A}$ has an augmented left $\mathscr{F}$-resolution which can be obtained as follows:

Set $k_{-1} = a$. Suppose k_i has been defined. Pick an $\mathscr{F}$-precover $f_{j+1} \to k_j$ and complete to a left exact sequence $0 \to k_{j+1} \to f_{j+1} \to k_j$.

Exercise 3.11. Prove Proposition 3.7 in general.

4 Higher Auslander–Reiten translations

The following definition is illustrated by Proposition A.9 and Remark B.4.

Definition 4.1 (Iyama [6, 1.4.1])**.** Let $m \in \mathsf{mod}(\Phi)$ have the augmented projective resolution

$$\cdots \to p_2 \to p_1 \to p_0 \to m \to 0.$$

The *d'th higher transpose* of m is

$$\mathrm{Tr}_d(m) = \mathrm{Coker}(p_{d-1}^* \to p_d^*).$$

The *d-Auslander–Reiten translation of* m and the *inverse d-Auslander–Reiten translation of* m are

$$\tau_d(m) = \mathrm{D}\,\mathrm{Tr}_d(m) \ \ , \ \ \tau_d^-(m) = \mathrm{Tr}_d\,\mathrm{D}(m).$$

Remark 4.2. Note that $\mathrm{Tr}_d(m) = \mathrm{Tr}_1\,\Omega^{d-1}(m)$, where Tr_1 is the classic Auslander–Bridger transpose and Ω^{d-1} is the $(d-1)$st syzygy. Hence

$$\tau_d(m) = \mathrm{D}\,\mathrm{Tr}_d(m) = \mathrm{D}\,\mathrm{Tr}_1\,\Omega^{d-1}(m) = \tau_1\Omega^{d-1}(m),$$

where τ_1 is the classic Auslander–Reiten translation.

Proposition 4.3. *There are functors*

$$\underline{\mathsf{mod}}(\Phi) \xrightarrow{\mathrm{Tr}_d} \underline{\mathsf{mod}}(\Phi^{\mathrm{op}}),$$

and

$$\underline{\mathsf{mod}}(\Phi) \underset{\tau_d^-}{\overset{\tau_d}{\rightleftarrows}} \overline{\mathsf{mod}}(\Phi).$$

Lemma 4.4. *Let $\mathscr{F} \subseteq \mathsf{mod}(\Phi)$ be a d-cluster tilting subcategory. Let $f \in \mathscr{F}$ have the augmented projective resolution*

$$\cdots \to p_2 \to p_1 \to p_0 \to f \to 0.$$

Then $\mathrm{Tr}_d(f)$ has an augmented projective resolution which begins

$$\cdots \to p_0^* \to p_1^* \to \cdots \to p_{d-1}^* \to p_d^* \to \mathrm{Tr}_d(f) \to 0, \qquad (2)$$

and $\tau_d(f)$ has an augmented injective resolution which begins

$$0 \to \tau_d(f) \to \mathrm{D}(p_d^*) \to \mathrm{D}(p_{d-1}^*) \to \cdots \to \mathrm{D}(p_1^*) \to \mathrm{D}(p_0^*) \to \cdots .$$

Proof. Each module $p_j^* = \mathrm{Hom}_\Phi(p_j, \Phi)$ is projective.

The last part of the augmented projective resolution,

$$p_{d-1}^* \to p_d^* \to \mathrm{Tr}_d(f) \to 0,$$

is exact by the definition of $\mathrm{Tr}_d(f)$.

If $1 \leqslant j \leqslant d-1$ then the homology of (2) at p_j^* is $\operatorname{Ext}^j_\Phi(f,\Phi)$ by definition, and this Ext is 0 since $f, \Phi \in \mathscr{F}$.

The augmented injective resolution is obtained by dualising the augmented projective resolution. □

Lemma 4.5. *For $m, p \in \mathsf{mod}(\Phi)$ with p projective, there is a natural isomorphism*

$$\mathrm{Hom}_\Phi\big(m, \mathrm{D}(p^*)\big) \cong \mathrm{DHom}_\Phi(p, m).$$

Proof. We can compute as follows, using tensor-Hom adjointness (a) and an evaluation morphism (b).

$$\begin{aligned}
\mathrm{Hom}_\Phi\big(m, \mathrm{D}(p^*)\big) &= \mathrm{Hom}_\Phi\Big(m, \mathrm{Hom}_\mathbb{C}\big(\,\mathrm{Hom}_\Phi(p,\Phi), \mathbb{C}\big)\Big) \\
&\overset{(a)}{\cong} \mathrm{Hom}_\mathbb{C}\big(m \underset{\Phi}{\otimes} \mathrm{Hom}_\Phi(p,\Phi), \mathbb{C}\big) \\
&\overset{(b)}{\cong} \mathrm{Hom}_\mathbb{C}\big(\,\mathrm{Hom}_\Phi(p, m \underset{\Phi}{\otimes} \Phi), \mathbb{C}\big) \\
&\cong \mathrm{Hom}_\mathbb{C}\big(\,\mathrm{Hom}_\Phi(p,m), \mathbb{C}\big) \\
&= \mathrm{DHom}_\Phi(p,m).
\end{aligned}$$

□

Theorem 4.6 (Iyama [6, thm. 2.3])**.** *Let $\mathscr{F} \subseteq \mathsf{mod}(\Phi)$ be a d-cluster tilting subcategory. The τ_d and τ_d^- map $\mathscr{F}$ to itself.*

Proof. We show the statement for τ_d. Let $f, f' \in \mathscr{F}$ and $1 \leqslant j \leqslant d-1$ be given. Lemma 4.4 gives an injective resolution of $\tau_d(f)$. Applying $\mathrm{Hom}_\Phi(f', -)$ gives a complex

$$\cdots \to 0 \to \mathrm{Hom}_\Phi\big(f', \mathrm{D}(p_d^*)\big) \to \cdots \to \mathrm{Hom}_\Phi\big(f', \mathrm{D}(p_0^*)\big) \to \cdots .$$

The homology at $\mathrm{Hom}_\Phi\big(f', \mathrm{D}(p_j^*)\big)$ is $\mathrm{Ext}_\Phi^{d-j}\big(f', \tau_d(f)\big)$. By Lemma 4.5, the complex is isomorphic to the following.

$$\cdots \to 0 \to \mathrm{DHom}_\Phi(p_d, f') \to \cdots \to \mathrm{DHom}_\Phi(p_0, f') \to \cdots$$

The homology at $\mathrm{DHom}_\Phi(p_j, f')$ is $\mathrm{DExt}_\Phi^j(f, f')$, so

$$\mathrm{Ext}_\Phi^{d-j}\big(f', \tau_d(f)\big) \cong \mathrm{DExt}_\Phi^j(f, f').$$

The right hand side is zero since $f, f' \in \mathscr{F}$, so $\mathrm{Ext}_\Phi^{d-j}\big(f', \tau_d(f)\big) = 0$ whence $\tau_d(f) \in \mathscr{F}$ since $f' \in \mathscr{F}$ and $1 \leqslant j \leqslant d-1$ are arbitrary. $\square$

Theorem 4.7 (Iyama [6, thm. 2.3]). *Let $\mathscr{F} \subseteq \mathsf{mod}(\Phi)$ be a d-cluster tilting subcategory. There are quasi-inverse equivalences* $\underline{\mathscr{F}} \underset{\tau_d^-}{\overset{\tau_d}{\rightleftarrows}} \overline{\mathscr{F}}$.

Proof. Combine Proposition 4.3 and Theorem 4.6 to get the existence of the functors. It is a separate argument to show that they are quasi-inverse to each other. $\square$

Definition 4.8 (Iyama–Oppermann [7, def. 2.2])**.** The algebra Φ is called *d-representation finite* if $\operatorname{gldim}(\Phi) \leqslant d$ and Φ has a d-cluster tilting object.

Examples of d-representation finite algebras ase given in Appendices A and B.

Theorem 4.9 (Iyama)**.** *If Φ is d-representation finite, then $\mathsf{mod}(\Phi)$ has the unique d-cluster tilting subcategory*

$$\mathscr{F} = \operatorname{add}\{\, \tau_d^j(i) \mid i \textit{ injective in } \mathsf{mod}(\Phi) \textit{ and } 0 \leqslant j \,\}.$$

Exercise 4.10. Prove Proposition 4.3.

Exercise 4.11. Show that a d-representation finite algebra has global dimension 0 or d.

Exercise 4.12. Let Φ be the path algebra of the following quiver with the indicated relation.

$$\begin{tikzcd} & \circ \arrow[dr] & \\ \circ \arrow[ur] \arrow[rr, dotted, no head] & & \circ \end{tikzcd}$$

Show that Φ is 2-representation finite.

5 d-abelian categories

The following definition is illustrated by Example A.4.

Definition 5.1 (Jasso [8, defs. 2.2 and 2.4])**.** Let $\mathscr{F}$ be an additive category.

(i) A diagram $f_{d+1} \to \cdots \to f_2 \to f_1$ is a *d-kernel* of a morphism $f_1 \to f_0$ if

$$0 \to f_{d+1} \to \cdots \to f_2 \to f_1 \to f_0$$

becomes an exact sequence under $\mathscr{F}(\widetilde{f}, -)$ for each $\widetilde{f} \in \mathscr{F}$.

(ii) A diagram $f_d \to f_{d-1} \to \cdots \to f_0$ is a *d-cokernel* of a morphism $f_{d+1} \to f_d$ if

$$f_{d+1} \to f_d \to f_{d-1} \to \cdots \to f_0 \to 0$$

becomes an exact sequence under $\mathscr{F}(-, \widetilde{f})$ for each $\widetilde{f} \in \mathscr{F}$.

(iii) A *d-exact sequence* is a diagram

$$f_{d+1} \to f_d \to f_{d-1} \to \cdots \to f_2 \to f_1 \to f_0$$

such that $f_{d+1} \to \cdots \to f_2 \to f_1$ is a d-kernel of $f_1 \to f_0$ and $f_d \to f_{d-1} \to \cdots \to f_0$ is a d-cokernel of $f_{d+1} \to f_d$.

Remark 5.2. A d-exact sequence is often written

$$0 \to f_{d+1} \to f_d \to f_{d-1} \to \cdots \to f_2 \to f_1 \to f_0 \to 0. \qquad (3)$$

This is motived by Exercise 5.7.

The following definition is illustrated by Theorem A.3, Example A.4, and Remark B.4.

Definition 5.3 (Jasso [8, def. 3.1])**.** An additive category $\mathscr{F}$ is called *d-abelian* if it satisfies the following:

(A0) $\mathscr{F}$ has split idempotents.

(A1) Each morphism in $\mathscr{F}$ has a d-kernel and a d-cokernel.

(A2) If $f_{d+1} \to f_d$ is a monomorphism which has a d-cokernel $f_d \to \cdots \to f_0$, then

$$0 \to f_{d+1} \to f_d \to \cdots \to f_0 \to 0$$

is a d-exact sequence.

($\mathrm{A2}^{\mathrm{op}}$) The dual of (A2).

Conditions (A2) and ($\mathrm{A2}^{\mathrm{op}}$) can be replaced with:

(A2') If $f_{d+1} \to f_d$ is a monomorphism, then there exists a d-exact sequence $0 \to f_{d+1} \to f_d \to \cdots \to f_0 \to 0$.

(A2'$^{\text{op}}$) The dual of (A2'). $\square$

Remark 5.4. The notions of 1-kernel, 1-cokernel, 1-exact sequence, and 1-abelian category coincide with the notions of kernel, cokernel, short exact sequence, and abelian category.

Theorem 5.5 (Jasso [8, thm. 3.16])**.** *If $\mathscr{F}$ is a d-cluster tilting subcategory of an abelian category $\mathscr{A}$, then $\mathscr{F}$ is d-abelian.*

Remark 5.6. We will not prove Theorem 5.5, but will explain how to obtain d-kernels and d-cokernels in $\mathscr{F}$:

Let $f_1 \xrightarrow{\varphi_1} f_0$ be a morphism in $\mathscr{F}$. Viewed as a morphism in $\mathscr{A}$, is has a kernel $k \xrightarrow{\kappa} f_1$. By Proposition 3.7 there is an augmented left $\mathscr{F}$-resolution

$$\cdots \longrightarrow 0 \longrightarrow f_{d+1} \longrightarrow \cdots \longrightarrow f_2 \xrightarrow{\varepsilon} k \longrightarrow 0.$$

Then

$$f_{d+1} \longrightarrow \cdots \longrightarrow f_2 \xrightarrow{\kappa\varepsilon} f_1$$

is a d-kernel of φ_1.

Let $f_{d+1} \xrightarrow{\varphi_{d+1}} f_d$ be a morphism in $\mathscr{F}$. Viewed as a morphism in $\mathscr{A}$, is has a cokernel $f_d \xrightarrow{\varphi} c$. By Proposition 3.7 there is an augmented right $\mathscr{F}$-resolution

$$0 \longrightarrow c \xrightarrow{\gamma} f_{d-1} \longrightarrow \cdots \longrightarrow f_0 \longrightarrow 0 \longrightarrow \cdots .$$

Then

$$f_d \xrightarrow{\gamma\varphi} f_{d-1} \longrightarrow \cdots \longrightarrow f_0$$

is a d-cokernel of φ_{d+1}.

Exercise 5.7. (i) Let $f_{d+1} \to \cdots \to f_2 \to f_1$ be a d-kernel of $f_1 \to f_0$. Show that $f_{d+1} \to f_d$ is a monomorphism.

(ii) Let $f_d \to f_{d-1} \to \cdots \to f_0$ be a d-cokernel of $f_{d+1} \to f_d$. Show that $f_1 \to f_0$ is an epimorphism.

Exercise 5.8. (i) Show that 1-kernel means kernel.

(ii) Show that 1-cokernel means cokernel.

(iii) Show that 1-exact sequence means short exact sequence.

(iv) Show that 1-abelian category means abelian category.

Exercise 5.9. Show that the d-kernels and d-cokernels constructed in Remark 5.6 have the properties required by Definition 5.1.

Exercise 5.10. (i) Show that kernels and cokernels (=1-kernels and 1-cokernels) are unique up to unique isomorphism.

(ii) Give an example to show that if $d \geqslant 2$, then d-kernels and d-cokernels are not unique up to isomorphism.

Exercise 5.11. Let

$$\delta \;=\; 0 \longrightarrow f_3 \xrightarrow{\varphi_3} f_2 \longrightarrow f_1 \xrightarrow{\varphi_1} f_0 \longrightarrow 0$$

be a 2-exact sequence in an additive category. Show that the following conditions are equivalent.

(i) φ_3 is a split monomorphism (see Definition 6.1).

(ii) φ_1 is a split epimorphism (see Definition 6.1).

(iii) Viewed as a complex, δ is null homotopic.

6 Higher Auslander–Reiten sequences

Definition 6.1. Let $\mathscr{F}$ be a category.

(i) A morphism $f_1 \xrightarrow{\varphi_1} f_0$ is a *split epimorphism* if it has a right inverse, that is, if there is a morphism $f_0 \xrightarrow{\varphi_0} f_1$ such that $\varphi_1\varphi_0 = \mathrm{id}_{f_0}$.

(ii) A morphism $f_1 \xrightarrow{\varphi_1} f_0$ is *right almost split* if it is not a split epimorphism and has the lifting property

$$\begin{tikzcd} & f \arrow[d] \arrow[dl, dotted, "\exists"'] \\ f_1 \arrow[r, "\varphi_1"'] & f_0 \end{tikzcd}$$

for each morphism $f \rightarrow f_0$ which is not a split epimorphism.

(iii) A morphism $f_1 \xrightarrow{\varphi_1} f_0$ is *right minimal* if each morphism $f_1 \xrightarrow{\varphi} f_1$ with $\varphi_1\varphi = \varphi_1$ is an isomorphism.

There are dual notions of *split monomorphism*, *left almost split morphism*, and *left minimal morphism*.

The notion of d-Auslander–Reiten sequences is due to Iyama, see

[5, def. 2.1]. The definition is illustrated by Proposition A.10 and Remark B.5.

Definition 6.2 ([5, def. 2.1])**.** A *d-Auslander–Reiten sequence* in a d-abelian category $\mathscr{F}$ is a d-exact sequence

$$0 \longrightarrow f_{d+1} \xrightarrow{\varphi_{d+1}} f_d \xrightarrow{\varphi_d} f_{d-1} \xrightarrow{\varphi_{d-1}} \cdots \xrightarrow{\varphi_3} f_2 \xrightarrow{\varphi_2} f_1 \xrightarrow{\varphi_1} f_0 \longrightarrow 0 \qquad (4)$$

such that φ_{d+1} if left almost split and left minimal, φ_1 is right almost split and right minimal, and $\varphi_d, \ldots, \varphi_2$ are in the radical of $\mathscr{F}$.

Theorem 6.3 ([6, thm. 3.3.1])**.** *Let $\mathscr{F} \subseteq \mathsf{mod}(\Phi)$ be a d-cluster tilting subcategory.*

(i) *For each non-projective indecomposable object $f_0 \in \mathscr{F}$, there exists a d-Auslander–Reiten sequence* (4) *in $\mathscr{F}$.*

(ii) *For each non-injective indecomposable object $f_{d+1} \in \mathscr{F}$, there exists a d-Auslander–Reiten sequence* (4) *in $\mathscr{F}$.*

(iii) *If* (4) *is a d-Auslander–Reiten sequence in $\mathscr{F}$, then $f_{d+1} = \tau_d(f_0)$ and $f_0 = \tau_d^-(f_{d+1})$.*

7 $(d+2)$-angulated categories

Definition 7.1 (Geiss–Keller–Oppermann [4, def. 2.1])**.** Let $\mathscr{C}$ be an additive category with an automorphism Σ^d. The inverse is denoted Σ^{-d}, but Σ^d is not assumed to be a power of another functor. A Σ^d*-sequence in* $\mathscr{C}$ is a diagram of the form

$$c^0 \xrightarrow{\gamma^0} c^1 \longrightarrow c^2 \longrightarrow \cdots \longrightarrow c^d \longrightarrow c^{d+1} \xrightarrow{\gamma^{d+1}} \Sigma^d(c^0). \tag{5}$$

Definition 7.2 (Geiss–Keller–Oppermann [4, def. 2.1])**.** A $(d+2)$-*angulated category* is a triple $(\mathscr{C}, \Sigma^d, \pentagon)$ where $\pentagon$ is a class of Σ^d-sequences, called $(d+2)$-*angles*, satisfying the following conditions.

(N1) $\pentagon$ is closed under sums and summands and contains

$$c \xrightarrow{\mathrm{id}_c} c \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Sigma^d(c)$$

for each $c \in \mathscr{C}$. For each morphism $c^0 \xrightarrow{\gamma^0} c^1$ in $\mathscr{C}$, the class $\pentagon$ contains a Σ^d-sequence of the form (5).

(N2) The Σ^d-sequence (5) is in $\pentagon$ if and only if so is its *left rotation*

$$c^1 \longrightarrow c^2 \longrightarrow \cdots \longrightarrow c^{d+1} \xrightarrow{\gamma^{d+1}} \Sigma^d(c^0) \xrightarrow{(-1)^d \Sigma^d(\gamma^0)} \Sigma^d(c^1).$$

(N3) A commutative diagram with rows in $\pentagon$ has the following extension property.

$$
\begin{tikzcd}
b^0 \arrow[r] \arrow[d, "\beta_0"'] & b^1 \arrow[r] \arrow[d] & b^2 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & b^{d-1} \arrow[r] \arrow[d, dotted] & b^d \arrow[r] \arrow[d, dotted] & b^{d+1} \arrow[r] \arrow[d, dotted] & \Sigma^d(b^0) \arrow[d, "\Sigma^d(\beta_0)"] \\
c^0 \arrow[r] & c^1 \arrow[r] & c^2 \arrow[r] & \cdots \arrow[r] & c^{d-1} \arrow[r] & c^d \arrow[r] & c^{d+1} \arrow[r] & \Sigma^d(c^0)
\end{tikzcd}
$$

(N4) "The octahedral axiom" (the following version of the axiom is equivalent to the one in [4] by [2, thm. 4.4]): A commutative diagram with rows in $\pentagon$ has the following extension property,

$$
\begin{tikzcd}
x \arrow[r, "\xi"] \arrow[d, equal] & y \arrow[r, "\zeta"] \arrow[d, "\upsilon"'] & a^2 \arrow[r] \arrow[d, dotted] & a^3 \arrow[r] \arrow[d, dotted] \arrow[ddl, dotted] & \cdots \arrow[r] & a^d \arrow[r] \arrow[d, dotted] & a^{d+1} \arrow[r] \arrow[d, dotted] \arrow[ddl, dotted] & \Sigma^d(x) \arrow[d, equal] \\
x \arrow[r] \arrow[d, "\xi"'] & z \arrow[r] \arrow[d, equal] & b^2 \arrow[r] \arrow[d, dotted] & b^3 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & b^d \arrow[r] \arrow[d, dotted] & b^{d+1} \arrow[r] \arrow[d, dotted] & \Sigma^d(x) \arrow[d, "\Sigma^d(\xi)"] \\
y \arrow[r, "\upsilon"'] & z \arrow[r] & c^2 \arrow[r] & c^3 \arrow[r] & \cdots \arrow[r] & c^d \arrow[r] & c^{d+1} \arrow[r, "\delta"'] & \Sigma^d(y),
\end{tikzcd}
$$

where each upright square is commutative and the totalisation

$$
\begin{aligned}
&a^2 \to a^3 \oplus b^2 \to a^4 \oplus b^3 \oplus c^2 \to \cdots \\
&\quad \to a^{d+1} \oplus b^d \oplus c^{d-1} \to b^{d+1} \oplus c^d \to c^{d+1} \xrightarrow{\Sigma^d(\zeta)\delta} \Sigma^d(a^2)
\end{aligned}
$$

is a $(d+2)$-angle.

The following is a sample of what can be concluded from the definition.

Proposition 7.3 (Geiss–Keller–Oppermann [4, prop. 2.5(a)])**.** *Let $(\mathscr{C}, \Sigma^d, \pentagon)$ be a $(d+2)$-angulated category, $c \in \mathscr{C}$ an object. Each $(d+2)$-angle* (5) *induces long exact sequences*

$$\cdots \to \mathscr{C}\big(c, \Sigma^{-d}(c^{d+1})\big) \to \mathscr{C}(c, c^0) \to \cdots \to \mathscr{C}(c, c^{d+1}) \to \mathscr{C}\big(c, \Sigma^d(c^0)\big) \to \cdots$$

and

$$\cdots \to \mathscr{C}\big(\Sigma^d(c^0), c\big) \to \mathscr{C}(c^{d+1}, c) \to \cdots \to \mathscr{C}(c^0, c) \to \mathscr{C}\big(\Sigma^{-d}(c^{d+1}), c\big) \to \cdots.$$

We would like to be able to obtain $(d+2)$-angulated categories from d-abelian categories:

Question 7.4 (The higher derived category problem)**.** Let $\mathscr{F}$ be a suitable d-abelian category. Does there exist a d-derived category of $\mathscr{F}$, in the sense that there is a $(d+2)$-angulated category $\mathscr{C}$ with the following properties?

(i) $\mathscr{C}$ contains $\mathscr{F}$ as an additive subcategory.

(ii) Each d-exact sequence (3) in $\mathscr{F}$ induces a $(d+2)$-angle (5) in $\mathscr{C}$.

(iii) We have $\operatorname{Ext}^{dj}_{\mathscr{F}}(f'', f') \cong \mathscr{C}\big(f'', (\Sigma^d)^j(f')\big)$ when $j \geqslant 0$ is an integer and $f', f'' \in \mathscr{F}$. The Ext groups are defined by means of Yoneda extensions.

(iv) Each d-exact sequence in $\mathscr{F}$ induces two long exact sequences of Ext-groups (note that this is only true for suitable $\mathscr{F}$). Each $(d+2)$-angle in $\mathscr{C}$ induces the long exact sequences in Proposition 7.3. The long exact sequences are identified under the isomorphisms in (iii).

The answer to Question 7.4 is unknown. Here is a method we do have for obtaining $(d+2)$-angulated categories:

Theorem 7.5 (Geiss–Keller–Oppermann [4, thm. 1])**.** *Let $\mathscr{D}$ be a triangulated category with suspension functor Σ. Let $\mathscr{C}$ be a d-cluster tilting subcategory of $\mathscr{D}$ which satisfies $\Sigma^d(\mathscr{C}) \subseteq \mathscr{C}$.*

Then $(\mathscr{C}, \Sigma^d, \pentagon)$ is a $(d+2)$-angulated category, where $\pentagon$ consists of all Σ^d-sequences (5) *which can be obtained from diagrams in $\mathscr{D}$ of the form*

$$\begin{tikzcd}
 & c^1 \arrow[rr] \arrow[dr] & & c^2 \arrow[rr] \arrow[dr] & & \cdots \arrow[rr] & & c^{d-1} \arrow[rr] \arrow[dr] & & c^d \arrow[dr] & \\
c^0 \arrow[ur] & & x^{1.5} \arrow[ll, squiggly] \arrow[ur] & & x^{2.5} \arrow[ll, squiggly] \arrow[ur] & & \cdots \arrow[ll, squiggly] & & x^{d-0.5} \arrow[ll, squiggly] \arrow[ur] & & c^{d+1} \arrow[ll, squiggly]
\end{tikzcd}.$$

A wavy arrow $x \rightsquigarrow y$ signifies a morphism $x \to \Sigma(y)$, and the composition of the wavy arrows is $c^{d+1} \xrightarrow{\gamma^{d+1}} \Sigma^d(c^0)$ in (5)*. Each oriented triangle is a triangle in $\mathscr{D}$, and each non-oriented triangle is commutative.*

This permits answering Question 7.4 in a special case:

Theorem 7.6 (Iyama)**.** *Let Φ be a d-representation finite algebra, and let $\mathscr{F}$ be the unique d-cluster tilting subcategory of* $\mathsf{mod}(\Phi)$*, cf. Theorem 4.9. Then $\mathscr{F}$ has a d-derived category $\mathscr{C}$ in the sense of Question 7.4.*

Remark 7.7. What Iyama in fact proved is that the subcategory

$$\mathscr{C} = \mathrm{add}\{\, \Sigma^{dj} f \mid j \in \mathbb{Z}, f \in \mathscr{F} \,\} \subseteq \mathscr{D}^{\mathrm{b}}(\mathsf{mod}\,\Phi)$$

is d-cluster tilting, see [5, thm. 1.21]. It clearly satisfies $\Sigma^{d}(\mathscr{C}) \subseteq \mathscr{C}$, so is $(d+2)$-angulated by Theorem 7.5. It is also clear that $\mathscr{C}$ has property (i) in Question 7.4, and it can be proved that it satisfies (ii)–(iv) as well.

Exercise 7.8. Prove Proposition 7.3.

A A class of examples by Vaso

We follow the conventions of [1, chps. II and III] for quivers and quiver representations.

All the results in this appendix are due to Vaso, see [9, sec. 4].

Definition A.1. Let $d \geqslant 2$, $\ell \geqslant 2$, $m \geqslant 3$ be integers with d even such that

$$\frac{m-1}{\ell} = \frac{d}{2}.$$

Let Q be the quiver

$$m \to m-1 \to \cdots \to 2 \to 1.$$

Set $\Gamma = \mathbb{C}Q/(\operatorname{rad}\mathbb{C}Q)^{\ell}$.

Equivalently, Γ is the path algebra of the quiver Q with the relations that ℓ consecutive arrows compose to zero.

Remark A.2. The indecomposable projective and injective modules in $\mathsf{mod}(\Gamma)$, shown as representations of Q, are the following, where each morphism $\mathbb{C} \to \mathbb{C}$ is the identity.

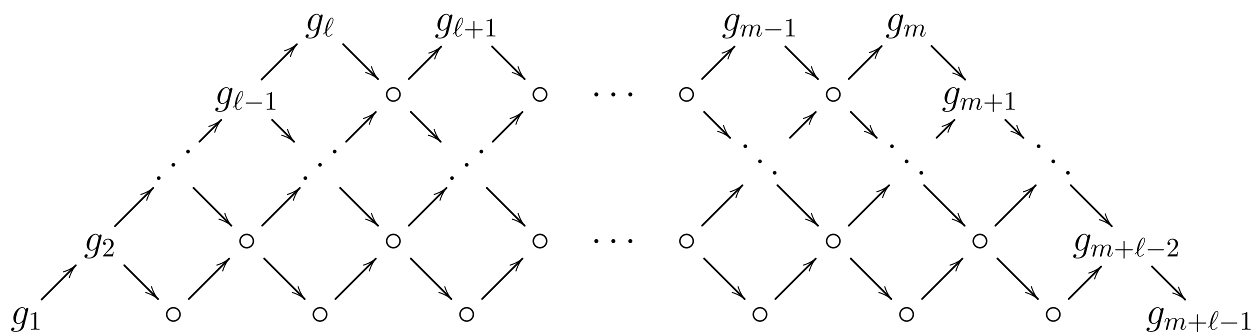
$$
\begin{array}{l}
\left.\begin{array}{ll}
g_1 & = 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to \mathbb{C} \\
g_2 & = 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to \mathbb{C} \to \mathbb{C} \\
\vdots & \\
g_{\ell-1} & = 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to \underbrace{\mathbb{C} \to \cdots \to \mathbb{C}}_{\ell-1}
\end{array}\right\} \text{projective} \\
\left.\begin{array}{ll}
g_\ell & = 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to \underbrace{\mathbb{C} \to \mathbb{C} \to \cdots \to \mathbb{C}}_{\ell} \\
g_{\ell+1} & = 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to \underbrace{\mathbb{C} \to \mathbb{C} \to \cdots \to \mathbb{C}}_{\ell} \to 0 \\
\vdots & \\
g_{m-1} & = 0 \to \underbrace{\mathbb{C} \to \mathbb{C} \to \cdots \to \mathbb{C}}_{\ell} \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \\
g_m & = \underbrace{\mathbb{C} \to \mathbb{C} \to \cdots \to \mathbb{C}}_{\ell} \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0
\end{array}\right\} \text{projective-injective} \\
\left.\begin{array}{ll}
g_{m+1} & = \underbrace{\mathbb{C} \to \cdots \to \mathbb{C}}_{\ell-1} \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \\
\vdots & \\
g_{m+\ell-2} & = \mathbb{C} \to \mathbb{C} \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \\
g_{m+\ell-1} & = \mathbb{C} \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0
\end{array}\right\} \text{injective}
\end{array}
$$

Among them, the indecomposable projective modules are

$$g_i = e_i \Gamma \text{ for } 1 \leqslant i \leqslant m$$

where e_i is the idempotent corresponding to the vertex i in Q. The

g_i appear in the Auslander–Reiten quiver of $\mathsf{mod}(\Gamma)$ as follows.



It will be convenient to set $g_i = 0$ if $i \leqslant 0$ or $i \geqslant m + \ell$.

A.i The d-abelian category of Γ

The following result shows that there is a canonical d-abelian category $\mathscr{G}$ associated to Γ, cf. Theorem 4.9.

Theorem A.3. *The algebra Γ is d-representation finite. The unique d-cluster tilting subcategory of* $\mathsf{mod}(\Gamma)$, *which is d-abelian by Theorem 5.5 is*

$$\mathscr{G} = \operatorname{add}(\Gamma \oplus \mathrm{D}\Gamma) = \operatorname{add}\{\, g_i \,|\, 1 \leqslant i \leqslant m + \ell - 1 \,\}.$$

Example A.4. Consider the algebra Γ of Definition A.1 in the case $d = 4$, $\ell = 4$, $m = 9$. The Auslander–Reiten quiver of $\mathsf{mod}(\Gamma)$ is the following.

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & & g_4 \arrow[dr] & & g_5 \arrow[dr] & & g_6 \arrow[dr] & & g_7 \arrow[dr] & & g_8 \arrow[dr] & & g_9 \arrow[dr] & & & \\
& & g_3 \arrow[ur] \arrow[dr] & & a \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & c \arrow[ur] \arrow[dr] & & g_{10} \arrow[dr] & & \\
& g_2 \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & g_{11} \arrow[dr] & \\
g_1 \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & b \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & g_{12}
\end{tikzcd}$$

The unique 4-cluster tilting subcategory of $\mathsf{mod}(\Gamma)$ is

$$\mathscr{G} = \mathrm{add}\{g_1, \ldots, g_{12}\};$$

it is a 4-abelian category. There is a short exact sequence

$$\begin{array}{l}
0 \\
\downarrow \\
g_1 = 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \mathbb{C} \\
\downarrow \\
g_4 = 0 \to 0 \to 0 \to 0 \to 0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \\
\downarrow \\
a = 0 \to 0 \to 0 \to 0 \to 0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to 0 \\
\downarrow \\
0,
\end{array}$$

and similarly

$$\begin{array}{ccccccccc}
0 & \to & a & \to & g_5 & \to & b & \to & 0, \\
0 & \to & b & \to & g_8 & \to & c & \to & 0, \\
0 & \to & c & \to & g_9 & \to & g_{12} & \to & 0.
\end{array}$$

The short exact sequences can be spliced to give augmented $\mathscr{G}$-resolutions of a, b, and c in the sense of Definition 3.6.

$$\begin{aligned} \cdots \to 0 \to g_1 \to g_4 \to a \to 0, \\ \cdots \to 0 \to g_1 \to g_4 \to g_5 \to b \to 0, \\ \cdots \to 0 \to g_1 \to g_4 \to g_5 \to g_8 \to c \to 0. \end{aligned}$$

The short exact sequences can also be spliced to an exact sequence

$$0 \to g_1 \to g_4 \to g_5 \to g_8 \to g_9 \to g_{12} \to 0$$

which is a 4-exact sequence in $\mathscr{G}$, see Definition 5.1. In particular, several 4-kernels and 4-cokernels can be read off. Observe that a 4-kernel does not necessarily have 4 non-zero objects; for instance, $0 \to 0 \to g_1 \to g_4 \to g_5$ is a 4-kernel of $g_5 \to g_8$.

Proposition A.5. (i) *We have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\Gamma}(g_i, g_j) = \begin{cases} 1 & \text{if } 0 \leqslant j - i \leqslant \ell - 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $i \leqslant q \leqslant j$ then each morphism $g_i \to g_j$ factors as $g_i \to g_q \to g_j$.*

(iii) *The quiver of $\mathscr{G}$ is*

$$g_1 \to g_2 \to g_3 \to \cdots \to g_{m+\ell-3} \to g_{m+\ell-2} \to g_{m+\ell-1}$$

where the composition of ℓ consecutive arrows is zero.

(iv) *If* $0 \leqslant j - i \leqslant \ell - 1$ *then each non-zero morphism* $g_i \to g_j$ *fits into an exact sequence*

$$g_{j-\ell} \to g_i \to g_j \to g_{i+\ell}.$$

Remark A.6. Part (iii) is immediate from parts (i) and (ii).

Part (iv) implies that if $j \leqslant \ell$, then $g_i \to g_j$ is an injection because $g_{j-\ell} = 0$.

Similarly, if $i \geqslant m$ then $g_i \to g_j$ is a surjection because $g_{i+\ell} = 0$.

Proposition A.7. *If* $0 \leqslant j - i \leqslant \ell - 1$ *then there is an exact sequence*

$$\cdots \to g_{i-\ell} \to g_{j-\ell} \to g_i \to g_j \to g_{i+\ell} \to g_{j+\ell} \to \cdots .$$

Proof. Iterate Proposition A.5(iv). □

Remark A.8. We can follow the sequence in Proposition A.7 left or right until we reach a zero module. This gives a d-kernel or a d-cokernel of $g_i \to g_j$.

Proposition A.9. *We have*

$$\tau_d(g_j) = \begin{cases} 0 & \textit{if } 1 \leqslant j \leqslant m, \\ g_{j-m} & \textit{if } m+1 \leqslant j \leqslant m+\ell-1. \end{cases}$$

Proof. If $1 \leqslant j \leqslant m$ then g_j is projective whence $\tau_d(g_j) = 0$.

Suppose $m+1 \leqslant j \leqslant m+\ell-1$. There is a surjection $g_m \to g_j$ which fits into an exact sequence from Proposition A.7:

$$\cdots \to \underbrace{g_{m-2\ell} \to g_{j-2\ell}}_{2} \to \underbrace{g_{m-\ell} \to g_{j-\ell}}_{1} \to \underbrace{g_m \to g_j}_{0} \to 0.$$

The numbered brackets are useful for bookkeeping. Since $\frac{m-1}{\ell} = \frac{d}{2}$ we have $m - \frac{d}{2}\ell = 1$, so on the left, the exact sequence contains

$$\cdots \to 0 \to \underbrace{g_1 \to g_{j-\frac{d}{2}\ell}}_{\frac{d}{2}} \to \cdots$$

which can also be written

$$\cdots \to 0 \to \underbrace{g_1 \to g_{j+1-m}}_{\frac{d}{2}} \to \cdots . \tag{6}$$

The exact sequence is an augmented projective resolution of g_j because each term except g_j is projective. The non-zero terms in Equation (6) are $p_d \to p_{d-1}$. We can write them

$$e_1\Gamma \to e_{j+1-m}\Gamma.$$

Applying $(-)^*$ and augmenting with the cokernel gives an exact sequence

$$\Gamma e_{j+1-m} \to \Gamma e_1 \to \mathrm{Tr}_d(g_j) \to 0.$$

These left modules can be viewed as representations of the quiver

$$Q^{\mathrm{op}} = m \leftarrow m-1 \leftarrow \cdots \leftarrow 2 \leftarrow 1.$$

Then Γe_{j+1-m} is supported on the vertex $j+1-m$ and at most ℓ higher vertices, while Γe_1 is supported on the vertices 1 through ℓ. Hence $\mathrm{Tr}_d(g_j)$ is supported on the vertices 1 through $j-m$. This finally means that $\tau_d(g_j) = \mathrm{D}\mathrm{Tr}_d(g_j)$, viewed as a representation of Q, is supported on the same vertices, 1 through $j-m$, and hence

$$\tau_d(g_j) = e_{j-m}\Gamma = g_{j-m}.$$

$\square$

Proposition A.10. *If $m+1 \leqslant j \leqslant m+\ell-1$ then the d-Auslander–Reiten sequence ending in g_j is*

$$0 \to g_{j-m} \to g_{j+1-m} \to \cdots \to g_{j-1-\ell} \to g_{j-\ell} \to g_{j-1} \to g_j \to 0.$$

A.ii The $(d+2)$-angulated category of Γ

Combining Remark 7.7 and Theorem A.3 shows that

$$\mathscr{C} = \mathrm{add}\{\, \Sigma^{dj} g \mid j \in \mathbb{Z}, g \in \mathscr{G} \,\} \subset \mathscr{D}^{\mathrm{b}}(\mathsf{mod}\,\Gamma)$$

is a $(d+2)$-angulated category which can be viewed as the d-derived category of the d-abelian category $\mathscr{G}$ from Subsection A.i.

Proposition A.11. (i) *Up to isomorphism, the indecomposable objects of $\mathscr{C}$ are the $\Sigma^{dj} g_i$ where i, j are integers with $1 \leqslant i \leqslant m+\ell-1$.*

(ii) *The quiver of $\mathscr{C}$ is*

$$\cdots \to \Sigma^{-d} g_1 \to \cdots \to \Sigma^{-d} g_{m+\ell-1} \to g_1 \to \cdots \to g_{m+\ell-1} \to \Sigma^{d} g_1 \to \cdots \to \Sigma^{d} g_{m+\ell-1} \to \cdots$$

where the composition of ℓ consecutive arrows is zero.

Exercise A.12. Prove Proposition A.5.

Exercise A.13. Show that the sequence in Proposition A.10 is indeed a d-Auslander–Reiten sequence.

Exercise A.14. Prove Proposition A.11.

B An example by Iyama

We follow the conventions of [1, chps. II and III] for quivers and quiver representations.

All the results in this appendix are due to Iyama.

Definition B.1. Let Λ be the algebra defined by the following quiver with relations.

$$\begin{tikzcd}[column sep=small]
 & & 6 \arrow[dr, "\varepsilon"] & & \\
 & 5 \arrow[ur, "\zeta"] \arrow[dr, "\gamma"'] \arrow[rr, dotted, no head] & & 4 \arrow[dr, "\alpha"] & \\
3 \arrow[ur, "\delta"] \arrow[rr, dotted, no head] & & 2 \arrow[ur, "\beta"'] \arrow[rr, dotted, no head] & & 1
\end{tikzcd}$$

The relations are $\delta\gamma = \beta\alpha = \gamma\beta + \zeta\varepsilon = 0$.

Observe that the quiver with relations is the Auslander–Reiten quiver of the algebra $\Phi = \mathbb{C}(3 \longrightarrow 2 \longrightarrow 1)$, and that hence Λ is the Auslander algebra of Φ.

Theorem B.2 (Iyama [5, thm. 1.18])**.** *The algebra Λ is 2-representation finite.*

Remark B.3. We will denote indecomposable modules in $\mathsf{mod}(\Lambda)$ by their radical series. For instance, ${}_{1}$ is the simple module given by the representation

$$
\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & \\
 & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & \mathbb{C}
\end{tikzcd}
$$

while $\begin{smallmatrix}6\\4\\1\end{smallmatrix}$ is its indecomposable projective cover given by the representation

$$
\begin{tikzcd}[column sep=small]
 & & \mathbb{C} \arrow[dr] & & \\
 & 0 \arrow[ur] \arrow[dr] & & \mathbb{C} \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & \mathbb{C}
\end{tikzcd}
$$

where each morphism $\mathbb{C} \to \mathbb{C}$ is the identity.

The Auslander–Reiten quiver of $\mathsf{mod}(\Lambda)$ is the following, where red means projective (but not injective), green means injective (but not projective), and blue means projective-injective.

$$
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & \color{blue}\begin{smallmatrix}6\\4\\1\end{smallmatrix} \arrow[dr] & & & & & & \color{blue}\begin{smallmatrix}3\\5\\6\end{smallmatrix} \arrow[dr] & & \\
 & \color{red}\begin{smallmatrix}4\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}6\\4\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}2\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}5\\6\end{smallmatrix} \arrow[ur] \arrow[dr] & & \color{green}\begin{smallmatrix}3\\5\end{smallmatrix} \arrow[dr] & \\
\color{red}\begin{smallmatrix}1\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}4\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}26\\4\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \color{blue}\begin{smallmatrix}5\\26\\4\end{smallmatrix} \arrow[r] & \begin{smallmatrix}5\\26\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}5\end{smallmatrix} \arrow[ur] & & \color{green}\begin{smallmatrix}3\end{smallmatrix} \\
 & & & \color{red}\begin{smallmatrix}2\\4\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}6\end{smallmatrix} \arrow[ur] & & \color{green}\begin{smallmatrix}5\\2\end{smallmatrix} \arrow[ur] & & &
\end{tikzcd}
$$

The classic Auslander–Reiten translation τ_1 is given by moving one step to the left on the quiver. For instance, $\tau_1\left(\begin{smallmatrix}6\\4\end{smallmatrix}\right) = \begin{smallmatrix}4\\1\end{smallmatrix}$, while $\tau_1\left(\begin{smallmatrix}4\\1\end{smallmatrix}\right) = 0$ (there is no module one step left of $\begin{smallmatrix}4\\1\end{smallmatrix}$).

Remark B.4. By Theorems 4.9 and B.2 the unique 2-cluster tilting subcategory of $\mathsf{mod}(\Lambda)$ is

$$\mathscr{L} = \operatorname{add}\{\, \tau_2^j(i) \mid i \text{ injective in } \mathsf{mod}(\Phi) \text{ and } 0 \leqslant j \,\}.$$

It is 2-abelian by Theorem 5.5.

The injective modules $\begin{smallmatrix}6\\4\\1\end{smallmatrix}, \begin{smallmatrix}5\\26\\4\end{smallmatrix}, \begin{smallmatrix}3\\5\\6\end{smallmatrix}$ are projective, so the 2-Auslander–Reiten translation τ_2 satisfies

$$\tau_2\left(\begin{smallmatrix}6\\4\\1\end{smallmatrix}\right) = \tau_2\left(\begin{smallmatrix}5\\26\\4\end{smallmatrix}\right) = \tau_2\left(\begin{smallmatrix}3\\5\\6\end{smallmatrix}\right) = 0.$$

The injective module $\begin{smallmatrix}3\end{smallmatrix}$ permits a short exact sequence

$$0 \to \begin{smallmatrix}5\\6\end{smallmatrix} \to \begin{smallmatrix}3\\5\\6\end{smallmatrix} \to \begin{smallmatrix}3\end{smallmatrix} \to 0$$

whence $\Omega\left(\begin{smallmatrix}3\end{smallmatrix}\right) = \begin{smallmatrix}5\\6\end{smallmatrix}$. Remark 4.2 gives

$$\tau_2\left(\begin{smallmatrix}3\end{smallmatrix}\right) = \tau_1\left(\begin{smallmatrix}5\\6\end{smallmatrix}\right) = \begin{smallmatrix}2\end{smallmatrix}.$$

In turn, the module $\begin{smallmatrix}2\end{smallmatrix}$ permits a short exact sequence

$$0 \to \begin{smallmatrix}4\end{smallmatrix} \to \begin{smallmatrix}2\\4\end{smallmatrix} \to \begin{smallmatrix}2\end{smallmatrix} \to 0$$

whence $\Omega\left(\,{\scriptstyle 2}\,\right) = {\scriptstyle 4}$. Remark 4.2 gives

$$\tau_2\left(\,{\scriptstyle 2}\,\right) = \tau_1\left(\,{\scriptstyle 4}\,\right) = {\scriptstyle 1}\,.$$

This module is projective, so applying τ_2 again gives zero.

The injective modules ${\scriptstyle \substack{3\\5}}$, ${\scriptstyle \substack{5\\2}}$ can be handled by the same method to give

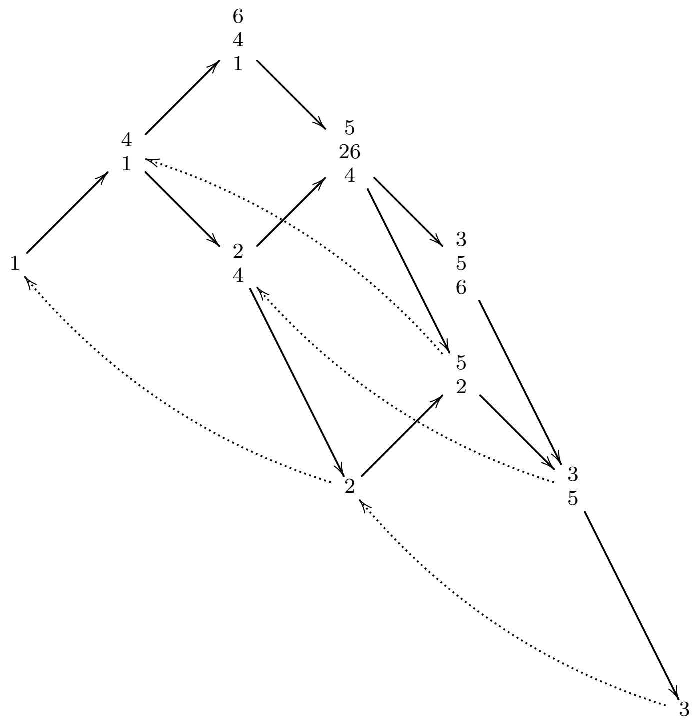
$$\tau_2\left(\,{\scriptstyle \substack{3\\5}}\,\right) = {\scriptstyle \substack{2\\4}} \quad , \quad \tau_2\left(\,{\scriptstyle \substack{5\\2}}\,\right) = {\scriptstyle \substack{4\\1}}\,.$$

These modules are projective, so applying τ_2 again gives zero.

It follows that

$$\mathscr{L} = \operatorname{add}\{\,{\scriptstyle 1}\,,\,{\scriptstyle \substack{4\\1}}\,,\,{\scriptstyle \substack{6\\4\\1}}\,,\,{\scriptstyle \substack{2\\4}}\,,\,{\scriptstyle 2}\,,\,{\scriptstyle \substack{5\\26\\4}}\,,\,{\scriptstyle \substack{5\\2}}\,,\,{\scriptstyle \substack{3\\5\\6}}\,,\,{\scriptstyle \substack{3\\5}}\,,\,{\scriptstyle 3}\,\}.$$

Remark B.5. Here is the AR quiver of $\mathscr{L}$. The dotted arrows show the action of the 2-Auslander–Reiten translation τ_2.



Note that the quiver is 3-dimensional, consisting of 3-dimensional analogues of the meshes found in classic Auslander–Reiten quivers. This is reflected in the 2-Auslander–Reiten sequences which can be read off, such as

$$0 \to \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \to \begin{smallmatrix} 5 \\ 26 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \end{smallmatrix} \to \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 2 \end{smallmatrix} \to \begin{smallmatrix} 3 \\ 5 \end{smallmatrix} \to 0.$$

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