Higher dimensional homological algebra

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Contents

1 Preface 3

2 Notation and Terminology 6

3 $d$-cluster tilting subcategories 7

4 Higher Auslander–Reiten translations 12

5 $d$-abelian categories 17
6 Higher Auslander–Reiten sequences

7 (d + 2)-angulated categories

A A class of examples by Vaso
   A.i The d-abelian category of Γ . . . . . . . . . . . . . . . . . 31
   A.ii The (d + 2)-angulated category of Γ . . . . . . . . . . . . . 36

B An example by Iyama

2
1 Preface

These are notes on higher dimensional homological algebra for the LMS–CMI Research School “New trends in representation theory — The impact of cluster theory in representation theory”, held at the University of Leicester 19–23 June 2017.

The notes take the view that the natural framework for higher dimensional homological algebra are $d$-abelian and $(d + 2)$-angulated categories, as introduced by Jasso and Geiss–Keller–Oppermann. The theory of $d$-cluster tilting subcategories and higher Auslander–Reiten theory, as introduced by Iyama, are developed as means to this end.

The notes are organised as follows: Section 2 gives a list of notation. Section 3 introduces $d$-cluster tilting subcategories. Section 4 introduces higher Auslander–Reiten translations which, among other applications, permit the construction of objects of $d$-cluster tilting subcategories. Section 5 introduces $d$-abelian categories. The main examples are $d$-cluster tilting subcategories of abelian categories. Section 6 states the basics of the theory of higher Auslander–Reiten sequences. Section 7 introduces $(d + 2)$-angulated categories. The main examples are $d$-cluster tilting subcategories of triangulated ca-
categories which are stable under $\Sigma^d$. Appendix A describes a class of examples by Vaso, and Appendix B an example by Iyama.

We do not describe any of the other examples which are known: Higher Auslander, Nakayama, and preprojective algebras, Iyama’s cone construction, etc. In particular, we do not explain their rich combinatorial structure, which is one of the motivations for higher homological algebra. Several other important parts of the theory are not covered: The higher Auslander correspondence, $d$-exact categories, $d$-Frobenius categories, relative $d$-cluster tilting subcategories, etc.

None of the material in these notes is due to me. I have tried to give due credit for all definitions and results, but would be grateful to learn if there are missing, incomplete, or wrong attributions.

I thank the organisers of the LMS–CMI Research School, Karin Baur and Sibylle Schroll, for the opportunity to give a course, and for the chance to present these notes. I thank the participants in the School for their feedback and for spotting a number of typos.

I thank Gustavo Jasso for providing feedback on a preliminary version, for conducting a number of useful tutorials during the Research
School, parts of which have subsequently been turned into Appendix B, and for providing several exercises.

I thank Laertis Vaso for permitting me to include the material in Appendix A.
## 2 Notation and Terminology

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
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<tbody>
<tr>
<td>$d$</td>
<td>A positive integer</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>A finite dimensional $\mathbb{C}$-algebra</td>
</tr>
<tr>
<td>$\text{mod}(\Phi)$</td>
<td>The category of finite dimensional right $\Phi$-modules</td>
</tr>
<tr>
<td>$\text{mod}(\Phi)$</td>
<td>$\text{mod}(\Phi)$ modulo morphisms which factor through a projective</td>
</tr>
<tr>
<td>$\text{mod}(\Phi)$</td>
<td>$\text{mod}(\Phi)$ modulo morphisms which factor through an injective</td>
</tr>
<tr>
<td>$\text{mod}(\Phi^{\text{op}})$, $\text{mod}(\Phi^{\text{op}})$, $\text{mod}(\Phi^{\text{op}})$</td>
<td>The same for left $\Phi$-modules</td>
</tr>
<tr>
<td>$\mathcal{A}$</td>
<td>An abelian category</td>
</tr>
<tr>
<td>$\mathcal{D}$</td>
<td>A triangulated category</td>
</tr>
<tr>
<td>$\mathcal{C}$, $\mathcal{F}$</td>
<td>$d$-cluster tilting subcategories</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>$\mathcal{F}$ modulo morphisms which factor through a projective</td>
</tr>
<tr>
<td>$\overline{\mathcal{F}}$</td>
<td>$\mathcal{F}$ modulo morphisms which factor through an injective</td>
</tr>
<tr>
<td>$\mathcal{D}(-)$</td>
<td>$\text{Hom}_{\mathbb{C}}(-, \mathbb{C})$</td>
</tr>
<tr>
<td>$(-)^*$</td>
<td>$\text{Hom}_{\Phi}(-, \Phi)$ if $-$ is a $\Phi$-module</td>
</tr>
<tr>
<td>$\text{Tr}_d$</td>
<td>The $d$'th higher transpose</td>
</tr>
<tr>
<td>$\tau_d = \mathcal{D} \text{Tr}_d$</td>
<td>The $d$-Auslander–Reiten translation</td>
</tr>
<tr>
<td>$\tau_d^{-} = \text{Tr}_d \mathcal{D}$</td>
<td>The inverse $d$-Auslander–Reiten translation</td>
</tr>
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</table>
3 $d$-cluster tilting subcategories

**Definition 3.1.** Let $\mathcal{A}$ be a category, $\mathcal{F} \subseteq \mathcal{A}$ a full subcategory. Then $\mathcal{F}$ is called:

(i) Generating if each $a \in \mathcal{A}$ permits an epimorphism $f \twoheadrightarrow a$ with $f \in \mathcal{F}$,

(ii) Cogenerating if each $a \in \mathcal{A}$ permits a monomorphism $a \hookrightarrow f$ with $f \in \mathcal{F}$.

**Definition 3.2** (Enochs [3]). Let $\mathcal{A}$ be a category, $\mathcal{F} \subseteq \mathcal{A}$ a full subcategory. A morphism $a \xrightarrow{\alpha} f$ in $\mathcal{A}$ with $f \in \mathcal{F}$ is called:

- An $\mathcal{F}$-prenvelope of $a$ if it has the extension property
  \[
  \begin{array}{ccc}
  a & \xrightarrow{\alpha} & f \\
  \downarrow & & \downarrow \exists \\
  f' & \xrightarrow{\exists} & \end{array}
  \]
  for each morphism $a \rightarrow f'$ with $f' \in \mathcal{F}$.

- An $\mathcal{F}$-envelope of $a$ if it is an $\mathcal{F}$-prenvelope which is left minimal, that is, satisfies that each morphism $f \xrightarrow{\varphi} f$ with $\varphi \alpha = \alpha$ is an automorphism.
We say that $\mathcal{F}$ is preenveloping (resp. enveloping) in $\mathcal{A}$ if each $a \in \mathcal{A}$ has an $\mathcal{F}$-preenvelope (resp. envelope).

The dual notion of preenvelope is precover, and $\mathcal{F}$ is called functorially finite if it is preenveloping and precovering.

The following definition is illustrated by Example A.4 and Remark B.4.

**Definition 3.3** (Iyama [6, def. 2.2], Jasso [8, def. 3.14]). Let $\mathcal{A}$ be an abelian or triangulated category. A $d$-cluster tilting subcategory $\mathcal{F}$ of $\mathcal{A}$ is a full subcategory which satisfies the following.

(i) $\mathcal{F} = \{ a \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^{1, \ldots, d-1}(\mathcal{F}, a) = 0 \}$

$= \{ a \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^{1, \ldots, d-1}(a, \mathcal{F}) = 0 \}$.

(ii) $\mathcal{F}$ is functorially finite.

(iii) If $\mathcal{A}$ is abelian then $\mathcal{F}$ is generating and cogenerating.

A $d$-cluster tilting object $f$ of $\mathcal{A}$ is an object $f$ such that $\mathcal{F} = \text{add}(f)$ is a $d$-cluster tilting subcategory.

**Remark 3.4.** If $d = 1$ then $\mathcal{F} = \mathcal{A}$ is the unique choice of $\mathcal{F}$. This
reflects that if \( d = 1 \), then higher dimensional homological algebra becomes classic homological algebra.

**Remark 3.5.** If \( p \in \mathcal{A} \) is projective, then 3.3(i) implies \( p \in \mathcal{F} \). Similarly, if \( i \in \mathcal{A} \) is injective, then \( i \in \mathcal{F} \). Hence if \( \mathcal{A} \) has enough projectives and injectives, then 3.3(i) implies 3.3(iii).

The following definition is again illustrated by Example A.4.

**Definition 3.6.** Let \( \mathcal{A} \) be an abelian category, \( \mathcal{F} \) a full subcategory. An **augmented left \( \mathcal{F} \)-resolution** of \( a \in \mathcal{A} \) is a sequence

\[
\cdots \rightarrow f_2 \rightarrow f_1 \rightarrow f_0 \rightarrow a \rightarrow 0
\]

with \( f_i \in \mathcal{F} \) for each \( i \), which becomes exact under \( \mathcal{A}(f,-) \) for each \( f \in \mathcal{F} \). Then

\[
\cdots \rightarrow f_2 \rightarrow f_1 \rightarrow f_0 \rightarrow 0 \rightarrow \cdots
\]

is called a **left \( \mathcal{F} \)-resolution** of \( a \).

The resolutions are said to have **length** \( \leq n \) if \( f_j = 0 \) for \( j \geq n + 1 \).

(Augmented) right resolutions are defined dually. \( \square \)

**Proposition 3.7** (Iyama [6, prop. 2.2.2]). Let \( \mathcal{A} \) be an abelian category, \( \mathcal{F} \subseteq \mathcal{A} \) a \( d \)-cluster tilting subcategory. Then each
\( a \in \mathcal{A} \) has (augmented) left and right \( \mathcal{F} \)-resolutions of length \( \leq d - 1 \).

**Proof.** We give a proof for left resolutions in the special case \( d = 2 \).

There is an \( \mathcal{F} \)-precover \( f \xrightarrow{\varphi} a \) by Definition 3.3(ii), and \( \varphi \) is an epimorphism by Exercise 3.8. Hence there is a short exact sequence

\[
0 \rightarrow k \rightarrow f \xrightarrow{\varphi} a \rightarrow 0.
\]

If \( \tilde{f} \in \mathcal{F} \) then there is a long exact sequence

\[
0 \rightarrow \text{Hom}_{\mathcal{A}}(\tilde{f}, k) \rightarrow \text{Hom}_{\mathcal{A}}(\tilde{f}, f) \xrightarrow{\varphi_*} \text{Hom}_{\mathcal{A}}(\tilde{f}, a) \rightarrow \text{Ext}^1_{\mathcal{A}}(\tilde{f}, k) \rightarrow \text{Ext}^1_{\mathcal{A}}(\tilde{f}, f).
\]

Here \( \varphi_* \) is surjective since \( \varphi \) is an \( \mathcal{F} \)-precover, and \( \text{Ext}^1_{\mathcal{A}}(\tilde{f}, f) = 0 \) by Definition 3.3(i) whence \( \text{Ext}^1_{\mathcal{A}}(\tilde{f}, k) = 0 \). This shows \( k \in \mathcal{F} \). It also shows that

\[
0 \rightarrow \text{Hom}_{\mathcal{A}}(\tilde{f}, k) \rightarrow \text{Hom}_{\mathcal{A}}(\tilde{f}, f) \rightarrow \text{Hom}_{\mathcal{A}}(\tilde{f}, a) \rightarrow 0
\]

is exact for each \( \tilde{f} \in \mathcal{F} \) whence

\[
\cdots \rightarrow 0 \rightarrow k \rightarrow f \rightarrow a \rightarrow 0
\]

is an augmented left \( \mathcal{F} \)-resolution of \( a \).

**Exercise 3.8.** Let \( \mathcal{A} \) be an abelian category, \( \mathcal{F} \) a full subcategory.
(i) Show that if $\mathcal{F}$ is generating, then each $\mathcal{F}$-precover is an epimorphism.

(ii) Show that if $\mathcal{F}$ is cogenerating, then each $\mathcal{F}$-preenvelope is a monomorphism.

**Exercise 3.9.** Let $R$ be a commutative ring, $\mathcal{A}$ an $R$-linear Hom-finite Krull–Schmidt category, $\mathcal{F}$ a full subcategory closed under sums and summands. Show that if $\mathcal{F}$ has only finitely many isomorphism classes of indecomposable objects, then it is functorially finite.

**Exercise 3.10.** Let $\mathcal{A}$ be an abelian category, $\mathcal{F}$ a full subcategory. Show that if $\mathcal{F}$ is precovering, then each $a \in \mathcal{A}$ has an augmented left $\mathcal{F}$-resolution which can be obtained as follows:

Set $k_{-1} = a$. Suppose $k_i$ has been defined. Pick an $\mathcal{F}$-precover $f_{j+1} \rightarrow k_j$ and complete to a left exact sequence $0 \rightarrow k_{j+1} \rightarrow f_{j+1} \rightarrow k_j$.

**Exercise 3.11.** Prove Proposition 3.7 in general.
4 Higher Auslander–Reiten translations

The following definition is illustrated by Proposition A.9 and Remark B.4.

**Definition 4.1** (Iyama [6, 1.4.1]). Let \( m \in \text{mod}(\Phi) \) have the augmented projective resolution
\[
\cdots \to p_2 \to p_1 \to p_0 \to m \to 0.
\]
The \( d \)'th higher transpose of \( m \) is
\[
\text{Tr}_d(m) = \text{Coker}(p_{d-1}^* \to p_d^*).
\]
The \( d \)-Auslander–Reiten translation of \( m \) and the inverse \( d \)-Auslander–Reiten translation of \( m \) are
\[
\tau_d(m) = D \text{Tr}_d(m), \quad \tau_d^{-1}(m) = \text{Tr}_d D(m).
\]

**Remark 4.2.** Note that \( \text{Tr}_d(m) = \text{Tr}_1 \Omega^{d-1}(m) \), where \( \text{Tr}_1 \) is the classic Auslander–Bridger transpose and \( \Omega^{d-1} \) is the \((d-1)\)st syzygy. Hence
\[
\tau_d(m) = D \text{Tr}_d(m) = D \text{Tr}_1 \Omega^{d-1}(m) = \tau_1 \Omega^{d-1}(m),
\]
where \( \tau_1 \) is the classic Auslander–Reiten translation.
Proposition 4.3. There are functors

\[
\text{mod}(\Phi) \xrightarrow{\text{Tr}_d} \text{mod}(\Phi^{\text{op}}),
\]

and

\[
\text{mod}(\Phi) \xleftarrow{\tau_d} \text{mod}(\Phi) \xrightarrow{\tau_d^{-1}} \text{mod}(\Phi).
\]

Lemma 4.4. Let \( \mathcal{F} \subseteq \text{mod}(\Phi) \) be a \( d \)-cluster tilting subcategory. Let \( f \in \mathcal{F} \) have the augmented projective resolution

\[
\cdots \to p_2 \to p_1 \to p_0 \to f \to 0.
\]

Then \( \text{Tr}_d(f) \) has an augmented projective resolution which begins

\[
\cdots \to p_0^* \to p_1^* \to \cdots \to p_{d-1}^* \to p_d^* \to \text{Tr}_d(f) \to 0, \quad (2)
\]

and \( \tau_d(f) \) has an augmented injective resolution which begins

\[
0 \to \tau_d(f) \to D(p_d^*) \to D(p_{d-1}^*) \to \cdots \to D(p_1^*) \to D(p_0^*) \to \cdots .
\]

Proof. Each module \( p_j^* = \text{Hom}_\Phi(p_j, \Phi) \) is projective.

The last part of the augmented projective resolution,

\[
p_{d-1}^* \to p_d^* \to \text{Tr}_d(f) \to 0,
\]

is exact by the definition of \( \text{Tr}_d(f) \).
If $1 \leq j \leq d - 1$ then the homology of (2) at $p_j^*$ is $\text{Ext}_\Phi^j(f, \Phi)$ by definition, and this Ext is 0 since $f, \Phi \in \mathcal{F}$.

The augmented injective resolution is obtained by dualising the augmented projective resolution. □

**Lemma 4.5.** For $m, p \in \text{mod}(\Phi)$ with $p$ projective, there is a natural isomorphism

$$\text{Hom}_\Phi(m, D(p^*)) \cong \text{DHom}_\Phi(p, m).$$

**Proof.** We can compute as follows, using tensor-Hom adjointness (a) and an evaluation morphism (b).

$$\text{Hom}_\Phi(m, D(p^*)) = \text{Hom}_\Phi\left(m, \text{Hom}_\mathbb{C}\left(\text{Hom}_\Phi(p, \Phi), \mathbb{C}\right)\right)$$

$$\overset{(a)}{=} \text{Hom}_\mathbb{C}\left(m \otimes \text{Hom}_\Phi(p, \Phi), \mathbb{C}\right)$$

$$\overset{(b)}{=} \text{Hom}_\mathbb{C}\left(\text{Hom}_\Phi(p, m \otimes \Phi), \mathbb{C}\right)$$

$$\cong \text{Hom}_\mathbb{C}\left(\text{Hom}_\Phi(p, m), \mathbb{C}\right)$$

$$= \text{DHom}_\Phi(p, m).$$

□

**Theorem 4.6** (Iyama [6, thm. 2.3]). Let $\mathcal{F} \subseteq \text{mod}(\Phi)$ be a $d$-cluster tilting subcategory. The $\tau_d$ and $\tau_d^-$ map $\mathcal{F}$ to itself.
Proof. We show the statement for $\tau_d$. Let $f, f' \in \mathcal{F}$ and $1 \leq j \leq d - 1$ be given. Lemma 4.4 gives an injective resolution of $\tau_d(f)$. Applying $\text{Hom}_\Phi(f', -)$ gives a complex

$$
\cdots \to 0 \to \text{Hom}_\Phi(f', D(p^*_d)) \to \cdots \to \text{Hom}_\Phi(f', D(p^*_0)) \to \cdots.
$$

The homology at $\text{Hom}_\Phi(f', D(p^*_j))$ is $\text{Ext}^{d-j}_\Phi(f', \tau_d(f))$. By Lemma 4.5, the complex is isomorphic to the following.

$$
\cdots \to 0 \to \text{DHom}_\Phi(p_d, f') \to \cdots \to \text{DHom}_\Phi(p_0, f') \to \cdots
$$

The homology at $\text{DHom}_\Phi(p_j, f')$ is $\text{DExt}^j_\Phi(f, f')$, so

$$
\text{Ext}^{d-j}_\Phi(f', \tau_d(f)) \cong \text{DExt}^j_\Phi(f, f').
$$

The right hand side is zero since $f, f' \in \mathcal{F}$, so $\text{Ext}^{d-j}_\Phi(f', \tau_d(f)) = 0$ whence $\tau_d(f) \in \mathcal{F}$ since $f' \in \mathcal{F}$ and $1 \leq j \leq d-1$ are arbitrary. \qed

**Theorem 4.7** (Iyama [6, thm. 2.3]). Let $\mathcal{F} \subseteq \text{mod}(\Phi)$ be a $d$-cluster tilting subcategory. There are quasi-inverse equivalences

$$
\mathcal{F} \xleftarrow{\tau_d} \xrightarrow{\tau_d^{-1}} \mathcal{F}.
$$

**Proof.** Combine Proposition 4.3 and Theorem 4.6 to get the existence of the functors. It is a separate argument to show that they are quasi-inverse to each other. \qed
**Definition 4.8** (Iyama–Oppermann [7, def. 2.2]). The algebra Φ is called *d-representation finite* if gldim(Φ) ≤ d and Φ has a d-cluster tilting object.

Examples of *d*-representation finite algebras are given in Appendices A and B.

**Theorem 4.9** (Iyama). *If Φ is d-representation finite, then mod(Φ) has the unique d-cluster tilting subcategory*

\[ \mathcal{F} = \text{add}\{ \tau_d^j(i) \mid i \text{ injective in } \text{mod}(Φ) \text{ and } 0 \leq j \}. \]

**Exercise 4.10.** Prove Proposition 4.3.

**Exercise 4.11.** Show that a *d*-representation finite algebra has global dimension 0 or d.

**Exercise 4.12.** Let Φ be the path algebra of the following quiver with the indicated relation.

\[
\begin{array}{ccc}
\circ & \nwarrow & \circ \\
\circ & \nearrow & \circ \\
\circ & \searrow & \circ \\
\circ & \cdots & \circ \\
\end{array}
\]

Show that Φ is 2-representation finite.
5 \textit{d-abelian categories}

The following definition is illustrated by Example A.4.

\textbf{Definition 5.1} (Jasso [8, defs. 2.2 and 2.4]). Let $\mathcal{F}$ be an additive category.

(i) A diagram $f_{d+1} \to \cdots \to f_2 \to f_1$ is a \textit{d-kernel} of a morphism $f_1 \to f_0$ if

$$0 \to f_{d+1} \to \cdots \to f_2 \to f_1 \to f_0$$

becomes an exact sequence under $\mathcal{F}(\tilde{f}, -)$ for each $\tilde{f} \in \mathcal{F}$.

(ii) A diagram $f_d \to f_{d-1} \to \cdots \to f_0$ is a \textit{d-cokernel} of a morphism $f_{d+1} \to f_d$ if

$$f_{d+1} \to f_d \to f_{d-1} \to \cdots \to f_0 \to 0$$

becomes an exact sequence under $\mathcal{F}(-, \tilde{f})$ for each $\tilde{f} \in \mathcal{F}$.

(iii) A \textit{d-exact sequence} is a diagram

$$f_{d+1} \to f_d \to f_{d-1} \to \cdots \to f_2 \to f_1 \to f_0$$

such that $f_{d+1} \to \cdots \to f_2 \to f_1$ is a d-kernel of $f_1 \to f_0$ and $f_d \to f_{d-1} \to \cdots \to f_0$ is a d-cokernel of $f_{d+1} \to f_d$. 

17
Remark 5.2. A $d$-exact sequence is often written

$$0 \to f_{d+1} \to f_d \to f_{d-1} \to \cdots \to f_2 \to f_1 \to f_0 \to 0. \quad (3)$$

This is motivated by Exercise 5.7.

The following definition is illustrated by Theorem A.3, Example A.4, and Remark B.4.

Definition 5.3 (Jasso [8, def. 3.1]). An additive category $\mathcal{F}$ is called $d$-abelian if it satisfies the following:

1. (A0) $\mathcal{F}$ has split idempotents.
2. (A1) Each morphism in $\mathcal{F}$ has a $d$-kernel and a $d$-cokernel.
3. (A2) If $f_{d+1} \to f_d$ is a monomorphism which has a $d$-cokernel $f_d \to \cdots \to f_0$, then

$$0 \to f_{d+1} \to f_d \to \cdots \to f_0 \to 0$$

is a $d$-exact sequence.
4. (A2$^{\text{op}}$) The dual of (A2).

Conditions (A2) and (A2$^{\text{op}}$) can be replaced with:
(A2’) If \( f_{d+1} \rightarrow f_d \) is a monomorphism, then there exists a \( d \)-exact sequence \( 0 \rightarrow f_{d+1} \rightarrow f_d \rightarrow \cdots \rightarrow f_0 \rightarrow 0 \).

\( (A2'^{\text{op}}) \) The dual of (A2’). □

**Remark 5.4.** The notions of 1-kernel, 1-cokernel, 1-exact sequence, and 1-abelian category coincide with the notions of kernel, cokernel, short exact sequence, and abelian category.

**Theorem 5.5** (Jasso [8, thm. 3.16]). If \( \mathcal{F} \) is a \( d \)-cluster tilting subcategory of an abelian category \( \mathcal{A} \), then \( \mathcal{F} \) is \( d \)-abelian.

**Remark 5.6.** We will not prove Theorem 5.5, but will explain how to obtain \( d \)-kernels and \( d \)-cokernels in \( \mathcal{F} \):

Let \( f_1 \xrightarrow{\varphi_1} f_0 \) be a morphism in \( \mathcal{F} \). Viewed as a morphism in \( \mathcal{A} \), is has a kernel \( k \xrightarrow{\kappa} f_1 \). By Proposition 3.7 there is an augmented left \( \mathcal{F} \)-resolution

\[
\cdots \rightarrow 0 \rightarrow f_{d+1} \rightarrow \cdots \rightarrow f_2 \xrightarrow{\varepsilon} k \rightarrow 0.
\]

Then

\[
f_{d+1} \rightarrow \cdots \rightarrow f_2 \xrightarrow{\kappa \varepsilon} f_1
\]

is a \( d \)-kernel of \( \varphi_1 \).
Let $f_{d+1} \xrightarrow{\varphi_{d+1}} f_d$ be a morphism in $\mathcal{F}$. Viewed as a morphism in $\mathcal{A}$, is has a cokernel $f_d \xrightarrow{\varphi} c$. By Proposition 3.7 there is an augmented right $\mathcal{F}$-resolution

$$0 \rightarrow c \xrightarrow{\gamma} f_{d-1} \rightarrow \cdots \rightarrow f_0 \rightarrow 0 \rightarrow \cdots.$$ 

Then

$$f_d \xrightarrow{\gamma \varphi} f_{d-1} \rightarrow \cdots \rightarrow f_0$$

is a $d$-cokernel of $\varphi_{d+1}$.

**Exercise 5.7.**

(i) Let $f_{d+1} \rightarrow \cdots \rightarrow f_2 \rightarrow f_1$ be a $d$-kernel of $f_1 \rightarrow f_0$. Show that $f_{d+1} \rightarrow f_d$ is a monomorphism.

(ii) Let $f_d \rightarrow f_{d-1} \rightarrow \cdots \rightarrow f_0$ be a $d$-cokernel of $f_{d+1} \rightarrow f_d$. Show that $f_1 \rightarrow f_0$ is an epimorphism.

**Exercise 5.8.**

(i) Show that 1-kernel means kernel.

(ii) Show that 1-cokernel means cokernel.

(iii) Show that 1-exact sequence means short exact sequence.

(iv) Show that 1-abelian category means abelian category.

**Exercise 5.9.** Show that the $d$-kernels and $d$-cokernels constructed in Remark 5.6 have the properties required by Definition 5.1.

**Exercise 5.10.**

(i) Show that kernels and cokernels (=1-kernels and 1-cokernels) are unique up to unique isomorphism.

(ii) Give an example to show that if $d \geq 2$, then $d$-kernels and $d$-cokernels are not unique up to isomorphism.

**Exercise 5.11.** Let

$$\delta = 0 \rightarrow f_3 \xrightarrow{\varphi_3} f_2 \rightarrow f_1 \xrightarrow{\varphi_1} f_0 \rightarrow 0$$

be a 2-exact sequence in an additive category. Show that the following conditions are equivalent.
(i) $\varphi_3$ is a split monomorphism (see Definition 6.1).
(ii) $\varphi_1$ is a split epimorphism (see Definition 6.1).
(iii) Viewed as a complex, $\delta$ is null homotopic.
6 Higher Auslander–Reiten sequences

Definition 6.1. Let $\mathcal{F}$ be a category.

(i) A morphism $f_1 \xrightarrow{\varphi_1} f_0$ is a split epimorphism if it has a right inverse, that is, if there is a morphism $f_0 \xrightarrow{\varphi_0} f_1$ such that $\varphi_1 \varphi_0 = \text{id}_{f_0}$.

(ii) A morphism $f_1 \xrightarrow{\varphi_1} f_0$ is right almost split if it is not a split epimorphism and has the lifting property

\[
\begin{array}{ccc}
& & f \\
\exists & \nearrow & \\
& f_1 \xrightarrow{\varphi_1} f_0 & \\
\end{array}
\]

for each morphism $f \to f_0$ which is not a split epimorphism.

(iii) A morphism $f_1 \xrightarrow{\varphi_1} f_0$ is right minimal if each morphism $f_1 \xrightarrow{\varphi} f_1$ with $\varphi_1 \varphi = \varphi_1$ is an isomorphism.

There are dual notions of split monomorphism, left almost split morphism, and left minimal morphism.

The notion of $d$-Auslander–Reiten sequences is due to Iyama, see
The definition is illustrated by Proposition A.10 and Remark B.5.

**Definition 6.2 ([5, def. 2.1]).** A $d$-Auslander–Reiten sequence in a $d$-abelian category $\mathcal{F}$ is a $d$-exact sequence

$$
0 \longrightarrow f_{d+1} \xrightarrow{\varphi_{d+1}} f_d \xrightarrow{\varphi_d} f_{d-1} \xrightarrow{\varphi_{d-1}} \cdots \xrightarrow{\varphi_3} f_2 \xrightarrow{\varphi_2} f_1 \xrightarrow{\varphi_1} f_0 \longrightarrow 0 \quad (4)
$$

such that $\varphi_{d+1}$ if left almost split and left minimal, $\varphi_1$ is right almost split and right minimal, and $\varphi_d, \ldots, \varphi_2$ are in the radical of $\mathcal{F}$.

**Theorem 6.3 ([6, thm. 3.3.1]).** Let $\mathcal{F} \subseteq \text{mod}(\Phi)$ be a $d$-cluster tilting subcategory.

(i) For each non-projective indecomposable object $f_0 \in \mathcal{F}$, there exists a $d$-Auslander–Reiten sequence (4) in $\mathcal{F}$.

(ii) For each non-injective indecomposable object $f_{d+1} \in \mathcal{F}$, there exists a $d$-Auslander–Reiten sequence (4) in $\mathcal{F}$.

(iii) If (4) is a $d$-Auslander–Reiten sequence in $\mathcal{F}$, then $f_{d+1} = \tau_d(f_0)$ and $f_0 = \tau_d^{-1}(f_{d+1})$. 23
7 (d + 2)-angulated categories

**Definition 7.1** (Geiss–Keller–Oppermann [4, def. 2.1]). Let $\mathcal{C}$ be an additive category with an automorphism $\Sigma^d$. The inverse is denoted $\Sigma^{-d}$, but $\Sigma^d$ is not assumed to be a power of another functor. A $\Sigma^d$-sequence in $\mathcal{C}$ is a diagram of the form

$$c^0 \xrightarrow{\gamma^0} c^1 \xrightarrow{\cdot} c^2 \xrightarrow{\cdot} \cdots \xrightarrow{\cdot} c^d \xrightarrow{\cdot} c^{d+1} \xrightarrow{\gamma^{d+1}} \Sigma^d(c^0).$$

**Definition 7.2** (Geiss–Keller–Oppermann [4, def. 2.1]). A $(d + 2)$-angulated category is a triple $(\mathcal{C}, \Sigma^d, \bigcirc)$ where $\bigcirc$ is a class of $\Sigma^d$-sequences, called $(d + 2)$-angles, satisfying the following conditions.

(N1) $\bigcirc$ is closed under sums and summands and contains

$$c \xrightarrow{id_c} c \xrightarrow{\cdot} 0 \xrightarrow{\cdot} \cdots \xrightarrow{\cdot} 0 \xrightarrow{\cdot} 0 \xrightarrow{\cdot} \Sigma^d(c)$$

for each $c \in \mathcal{C}$. For each morphism $c^0 \xrightarrow{\gamma^0} c^1$ in $\mathcal{C}$, the class $\bigcirc$ contains a $\Sigma^d$-sequence of the form (5).

(N2) The $\Sigma^d$-sequence (5) is in $\bigcirc$ if and only if so is its left rotation

$$c^1 \xrightarrow{\cdot} c^2 \xrightarrow{\cdot} \cdots \xrightarrow{\cdot} c^{d+1} \xrightarrow{\gamma^{d+1}} \Sigma^d(c^0) \xrightarrow{\Sigma^d(\gamma^0)} \Sigma^d(c^1).$$
(N3) A commutative diagram with rows in $\bigcirc$ has the following extension property.

\[
\begin{array}{cccccccccc}
  b^0 & \rightarrow & b^1 & \rightarrow & b^2 & \rightarrow & \cdots & \rightarrow & b^{d-1} & \rightarrow & b^d & \rightarrow & b^{d+1} & \rightarrow & \Sigma^d(b^0) \\
  \downarrow \beta_0 & & \downarrow & & \downarrow & & \ddots & & \downarrow & & \downarrow & & \downarrow & & \Sigma^d(\beta_0) \\
  c^0 & \rightarrow & c^1 & \rightarrow & c^2 & \rightarrow & \cdots & \rightarrow & c^{d-1} & \rightarrow & c^d & \rightarrow & c^{d+1} & \rightarrow & \Sigma^d(c^0) \\
\end{array}
\]

(N4) “The octahedral axiom” (the following version of the axiom is equivalent to the one in [4] by [2, thm. 4.4]): A commutative diagram with rows in $\bigcirc$ has the following extension property,

\[
\begin{array}{cccccccccc}
  x & \rightarrow & \xi & \rightarrow & y & \rightarrow & \zeta & \rightarrow & a^2 & \rightarrow & a^3 & \rightarrow & \cdots & \rightarrow & a^d & \rightarrow & a^{d+1} & \rightarrow & \Sigma^d(x) \\
  \downarrow v & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \Sigma^d(\xi) \\
  x & \rightarrow & z & \rightarrow & b^2 & \rightarrow & b^3 & \rightarrow & \cdots & \rightarrow & b^d & \rightarrow & b^{d+1} & \rightarrow & \Sigma^d(x) \\
  \downarrow \xi & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \Sigma^d(y), \\
\end{array}
\]

where each upright square is commutative and the totalisation

\[
\begin{array}{cccccccccccc}
  a^2 & \rightarrow & a^3 & \oplus & b^2 & \rightarrow & a^4 & \oplus & b^3 & \oplus & c^2 & \rightarrow & \cdots \\
  & & & & & & & & & & & \\
  & & & & & & & & & & & \\
  & & & & & & & & & & & \\
  a^{d+1} & \oplus & b^d & \oplus & c^{d-1} & \rightarrow & b^{d+1} & \oplus & c^d & \rightarrow & c^{d+1} & \rightarrow & \Sigma^d(a^2) \\
\end{array}
\]

is a $(d + 2)$-angle.
The following is a sample of what can be concluded from the definition.

**Proposition 7.3** (Geiss–Keller–Oppermann [4, prop. 2.5(a)]). Let $(\mathcal{C}, \Sigma^d, \sigma)$ be a $(d+2)$-angulated category, $c \in \mathcal{C}$ an object. Each $(d+2)$-angle (5) induces long exact sequences

\[ \cdots \rightarrow \mathcal{C}(c, \Sigma^{-d}(c^{d+1})) \rightarrow \mathcal{C}(c, c^0) \rightarrow \cdots \rightarrow \mathcal{C}(c, c^{d+1}) \rightarrow \mathcal{C}(c, \Sigma^d(c^0)) \rightarrow \cdots \]

and

\[ \cdots \rightarrow \mathcal{C}(\Sigma^d(c^0), c) \rightarrow \mathcal{C}(c^{d+1}, c) \rightarrow \cdots \rightarrow \mathcal{C}(c^0, c) \rightarrow \mathcal{C}(\Sigma^{-d}(c^{d+1}), c) \rightarrow \cdots . \]

We would like to be able to obtain $(d+2)$-angulated categories from $d$-abelian categories:

**Question 7.4** (The higher derived category problem). Let $\mathcal{F}$ be a suitable $d$-abelian category. Does there exist a $d$-derived category of $\mathcal{F}$, in the sense that there is a $(d+2)$-angulated category $\mathcal{C}$ with the following properties?

(i) $\mathcal{C}$ contains $\mathcal{F}$ as an additive subcategory.

(ii) Each $d$-exact sequence (3) in $\mathcal{F}$ induces a $(d+2)$-angle (5) in $\mathcal{C}$.

(iii) We have $\text{Ext}^d_{\mathcal{F}}(f'', f') \cong \mathcal{C}(f'', (\Sigma^d)^j(f'))$ when $j \geq 0$ is an integer and $f', f'' \in \mathcal{F}$. The Ext groups are defined by means of Yoneda extensions.
(iv) Each $d$-exact sequence in $\mathcal{F}$ induces two long exact sequences of Ext-groups (note that this is only true for suitable $\mathcal{F}$). Each $(d + 2)$-angle in $\mathcal{C}$ induces the long exact sequences in Proposition 7.3. The long exact sequences are identified under the isomorphisms in (iii).

The answer to Question 7.4 is unknown. Here is a method we do have for obtaining $(d + 2)$-angulated categories:

**Theorem 7.5** (Geiss–Keller–Oppermann [4, thm. 1]). Let $\mathcal{D}$ be a triangulated category with suspension functor $\Sigma$. Let $\mathcal{C}$ be a $d$-cluster tilting subcategory of $\mathcal{D}$ which satisfies $\Sigma^d(\mathcal{C}) \subseteq \mathcal{C}$.

Then $(\mathcal{C}, \Sigma^d, \bigtriangleup)$ is a $(d + 2)$-angulated category, where $\bigtriangleup$ consists of all $\Sigma^d$-sequences (5) which can be obtained from diagrams in $\mathcal{D}$ of the form

\[
\begin{array}{c}
\begin{array}{cccccccc}
& & & & & & \\
& & & & & & \\
& & & & & & \\
\longrightarrow & c^1 & \longrightarrow & c^2 & \longrightarrow & \cdots & \longrightarrow & c^{d-1} & \longrightarrow & c^d \\
\longrightarrow & c^0 & \longrightarrow & x^{1.5} & \longrightarrow & x^{2.5} & \longrightarrow & \cdots & \longrightarrow & x^{d-0.5} & \longrightarrow & c^{d+1}
\end{array}
\end{array}
\]

A wavy arrow $x \rightsquigarrow y$ signifies a morphism $x \to \Sigma(y)$, and the composition of the wavy arrows is $c^{d+1} \gamma^{d+1} \longrightarrow \Sigma^d(c^0)$ in (5). Each oriented triangle is a triangle in $\mathcal{D}$, and each non-oriented triangle is commutative.
This permits answering Question 7.4 in a special case:

**Theorem 7.6** (Iyama). Let \( \Phi \) be a \( d \)-representation finite algebra, and let \( \mathcal{F} \) be the unique \( d \)-cluster tilting subcategory of \( \text{mod}(\Phi) \), cf. Theorem 4.9. Then \( \mathcal{F} \) has a \( d \)-derived category \( \mathcal{C} \) in the sense of Question 7.4.

**Remark 7.7.** What Iyama in fact proved is that the subcategory

\[
\mathcal{C} = \text{add}\{ \sum^d f \mid j \in \mathbb{Z}, f \in \mathcal{F} \} \subseteq \mathbb{D}^b(\text{mod} \Phi)
\]

is \( d \)-cluster tilting, see [5, thm. 1.21]. It clearly satisfies \( \Sigma^d(\mathcal{C}) \subseteq \mathcal{C} \), so is \((d + 2)\)-angulated by Theorem 7.5. It is also clear that \( \mathcal{C} \) has property (i) in Question 7.4, and it can be proved that it satisfies (ii)–(iv) as well.

**Exercise 7.8.** Prove Proposition 7.3.
A  A class of examples by Vaso

We follow the conventions of [1, chps. II and III] for quivers and quiver representations.

All the results in this appendix are due to Vaso, see [9, sec. 4].

**Definition A.1.** Let $d \geq 2, \ell \geq 2, m \geq 3$ be integers with $d$ even such that

$$\frac{m - 1}{\ell} = \frac{d}{2}.$$ 

Let $Q$ be the quiver

$$m \to m - 1 \to \cdots \to 2 \to 1.$$ 

Set $\Gamma = \mathbb{C}Q/(\text{rad } \mathbb{C}Q)^{\ell}$.

Equivalently, $\Gamma$ is the path algebra of the quiver $Q$ with the relations that $\ell$ consecutive arrows compose to zero.

**Remark A.2.** The indecomposable projective and injective modules in $\text{mod}(\Gamma)$, shown as representations of $Q$, are the following, where
each morphism $C \to C$ is the identity.

$$g_1 = 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to C$$

$$g_2 = 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to C \to C$$

$$\vdots$$

$$g_{\ell-1} = 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to C \to \cdots \to C$$

$$g_{\ell} = 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to C \to C \to \cdots \to C$$

$$g_{\ell+1} = 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to C \to C \to \cdots \to C \to 0$$

$$\vdots$$

$$g_{m-1} = 0 \to C \to C \to \cdots \to C \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0$$

$$g_{m} = C \to C \to \cdots \to C \to 0 \to 0 \to \cdots \to 0 \to 0 \to C \to \cdots \to C \to 0$$

$$g_{m+1} = C \to \cdots \to C \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to C \to \cdots \to C \to 0$$

$$\vdots$$

$$g_{m+\ell-2} = C \to C \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0$$

$$g_{m+\ell-1} = C \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0$$

Among them, the indecomposable projective modules are

$$g_i = e_i \Gamma \text{ for } 1 \leq i \leq m$$

where $e_i$ is the idempotent corresponding to the vertex $i$ in $Q$. The
$g_i$ appear in the Auslander–Reiten quiver of $\text{mod}(\Gamma)$ as follows.

It will be convenient to set $g_i = 0$ if $i \leq 0$ or $i \geq m + \ell$.

**A.i The $d$-abelian category of $\Gamma$**

The following result shows that there is a canonical $d$-abelian category $\mathcal{G}$ associated to $\Gamma$, cf. Theorem 4.9.

**Theorem A.3.** The algebra $\Gamma$ is $d$-representation finite. The unique $d$-cluster tilting subcategory of $\text{mod}(\Gamma)$, which is $d$-abelian by Theorem 5.5 is

$$\mathcal{G} = \text{add}(\Gamma \oplus D\Gamma) = \text{add} \{ g_i \mid 1 \leq i \leq m + \ell - 1 \}.$$  

**Example A.4.** Consider the algebra $\Gamma$ of Definition A.1 in the case $d = 4, \ell = 4, m = 9$. The Auslander–Reiten quiver of $\text{mod}(\Gamma)$ is the
The unique 4-cluster tilting subcategory of $\text{mod}(\Gamma)$ is \\

\[ G = \text{add}\{g_1, \ldots, g_{12}\}; \]

it is a 4-abelian category. There is a short exact sequence

\[
\begin{array}{c}
0 \\
\downarrow \\
g_1 = 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{C} \\
\downarrow \\
g_4 = 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow \mathbb{C} \\
\downarrow \\
a = 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow 0 \\
\downarrow \\
0,
\end{array}
\]

and similarly

\[
\begin{array}{c}
0 \rightarrow a \rightarrow g_5 \rightarrow b \rightarrow 0, \\
0 \rightarrow b \rightarrow g_8 \rightarrow c \rightarrow 0, \\
0 \rightarrow c \rightarrow g_9 \rightarrow g_{12} \rightarrow 0.
\end{array}
\]
The short exact sequences can be spliced to give augmented $\mathcal{G}$-resolutions of $a$, $b$, and $c$ in the sense of Definition 3.6.

\[ \cdots \to 0 \to g_1 \to g_4 \to a \to 0, \]
\[ \cdots \to 0 \to g_1 \to g_4 \to g_5 \to b \to 0, \]
\[ \cdots \to 0 \to g_1 \to g_4 \to g_5 \to g_8 \to c \to 0. \]

The short exact sequences can also be spliced to an exact sequence

\[ 0 \to g_1 \to g_4 \to g_5 \to g_8 \to g_9 \to g_{12} \to 0 \]

which is a 4-exact sequence in $\mathcal{G}$, see Definition 5.1. In particular, several 4-kernels and 4-cokernels can be read off. Observe that a 4-kernel does not necessarily have 4 non-zero objects; for instance, \(0 \to 0 \to g_1 \to g_4 \to g_5\) is a 4-kernel of \(g_5 \to g_8\).

**Proposition A.5.** (i) We have

\[ \dim \mathbb{C} \text{Hom}_\Gamma(g_i, g_j) = \begin{cases} 1 & \text{if } 0 \leq j - i \leq \ell - 1, \\ 0 & \text{otherwise}. \end{cases} \]

(ii) If \(i \leq q \leq j\) then each morphism \(g_i \to g_j\) factors as \(g_i \to g_q \to g_j\).

(iii) The quiver of $\mathcal{G}$ is

\[ g_1 \to g_2 \to g_3 \to \cdots \to g_{m+\ell-3} \to g_{m+\ell-2} \to g_{m+\ell-1} \]

where the composition of $\ell$ consecutive arrows is zero.
(iv) If $0 \leq j - i \leq \ell - 1$ then each non-zero morphism $g_i \to g_j$ fits into an exact sequence

$$g_{j-\ell} \to g_i \to g_j \to g_{i+\ell}. $$

**Remark A.6.** Part (iii) is immediate from parts (i) and (ii).

Part (iv) implies that if $j \leq \ell$, then $g_i \to g_j$ is an injection because $g_{j-\ell} = 0$.

Similarly, if $i \geq m$ then $g_i \to g_j$ is a surjection because $g_{i+\ell} = 0$.

**Proposition A.7.** If $0 \leq j - i \leq \ell - 1$ then there is an exact sequence

$$\cdots \to g_{i-\ell} \to g_{j-\ell} \to g_i \to g_j \to g_{i+\ell} \to g_{j+\ell} \to \cdots. $$

**Proof.** Iterate Proposition A.5(iv). \qed

**Remark A.8.** We can follow the sequence in Proposition A.7 left or right until we reach a zero module. This gives a $d$-kernel or a $d$-cokernel of $g_i \to g_j$.

**Proposition A.9.** We have

$$\tau_d(g_j) = \begin{cases} 
0 & \text{if } 1 \leq j \leq m, \\
g_{j-m} & \text{if } m + 1 \leq j \leq m + \ell - 1. 
\end{cases}$$
Proof. If $1 \leq j \leq m$ then $g_j$ is projective whence $\tau_d(g_j) = 0$.

Suppose $m + 1 \leq j \leq m + \ell - 1$. There is a surjection $g_m \to g_j$ which fits into an exact sequence from Proposition A.7:

$$
\cdots \to g_{m-2\ell} \to g_{j-2\ell} \to g_{m-\ell} \to g_{j-\ell} \to g_m \to g_j \to 0.
$$

The numbered brackets are useful for bookkeeping. Since $\frac{m-1}{\ell} = \frac{d}{2}$ we have $m - \frac{d}{2}\ell = 1$, so on the left, the exact sequence contains

$$
\cdots \to 0 \to g_1 \to g_{j-\frac{d}{2}\ell} \to \cdots
$$

which can also be written

$$
\cdots \to 0 \to g_1 \to g_{j+1-m} \to \cdots. \quad (6)
$$

The exact sequence is an augmented projective resolution of $g_j$ because each term except $g_j$ is projective. The non-zero terms in Equation (6) are $p_d \to p_{d-1}$. We can write them

$$
e_1\Gamma \to e_{j+1-m}\Gamma.
$$

Applying $(-)^*$ and augmenting with the cokernel gives an exact sequence

$$
\Gamma e_{j+1-m} \to \Gamma e_1 \to \text{Tr}_d(g_j) \to 0.
$$
These left modules can be viewed as representations of the quiver

\[ Q^{\text{op}} = m \leftarrow m - 1 \leftarrow \cdots \leftarrow 2 \leftarrow 1. \]

Then \( \Gamma e_{j+1-m} \) is supported on the vertex \( j + 1 - m \) and at most \( \ell \) higher vertices, while \( \Gamma e_1 \) is supported on the vertices 1 through \( \ell \). Hence \( \text{Tr}_d(g_j) \) is supported on the vertices 1 through \( j - m \). This finally means that \( \tau_d(g_j) = D\text{Tr}_d(g_j) \), viewed as a representation of \( Q \), is supported on the same vertices, 1 through \( j - m \), and hence

\[ \tau_d(g_j) = e_{j-m}\Gamma = g_{j-m}. \]

\[ \square \]

**Proposition A.10.** If \( m + 1 \leq j \leq m + \ell - 1 \) then the \( d \)-Auslander–Reiten sequence ending in \( g_j \) is

\[ 0 \rightarrow g_{j-m} \rightarrow g_{j+1-m} \rightarrow \cdots \rightarrow g_{j-1-\ell} \rightarrow g_{j-\ell} \rightarrow g_{j-1} \rightarrow g_j \rightarrow 0. \]

**A.ii The \((d+2)\)-angulated category of \( \Gamma \)**

Combining Remark 7.7 and Theorem A.3 shows that

\[ \mathcal{C} = \text{add}\{ \Sigma^{dj} g \mid j \in \mathbb{Z}, g \in \mathcal{G} \} \subset \mathcal{D}^b(\text{mod} \, \Gamma) \]
is a $(d+2)$-angulated category which can be viewed as the $d$-derived category of the $d$-abelian category $G$ from Subsection A.i.

**Proposition A.11.** (i) Up to isomorphism, the indecomposable objects of $\mathcal{C}$ are the $\Sigma^d g_i$ where $i$, $j$ are integers with $1 \leq i \leq m + \ell - 1$.

(ii) The quiver of $\mathcal{C}$ is

$$
\cdots \rightarrow \Sigma^{-d} g_1 \rightarrow \cdots \rightarrow \Sigma^{-d} g_{m+\ell-1} \rightarrow g_1 \rightarrow \cdots \rightarrow g_{m+\ell-1} \rightarrow \Sigma^d g_1 \rightarrow \cdots \rightarrow \Sigma^d g_{m+\ell-1} $$

where the composition of $\ell$ consecutive arrows is zero.

**Exercise A.12.** Prove Proposition A.5.

**Exercise A.13.** Show that the sequence in Proposition A.10 is indeed a $d$-Auslander–Reiten sequence.

**Exercise A.14.** Prove Proposition A.11.
B  An example by Iyama

We follow the conventions of [1, chps. II and III] for quivers and quiver representations.

All the results in this appendix are due to Iyama.

**Definition B.1.** Let $\Lambda$ be the algebra defined by the following quiver with relations.

![Quiver diagram]

The relations are $\delta \gamma = \beta \alpha = \gamma \beta + \zeta \varepsilon = 0$.

Observe that the quiver with relations is the Auslander–Reiten quiver of the algebra $\Phi = \mathbb{C}(\begin{array}{c} 3 \rightarrow 2 \rightarrow 1 \end{array})$, and that hence $\Lambda$ is the Auslander algebra of $\Phi$.

**Theorem B.2** (Iyama [5, thm. 1.18]). *The algebra $\Lambda$ is 2-representation finite.*

**Remark B.3.** We will denote indecomposable modules in $\text{mod}(\Lambda)$ by their radical series. For instance, $1$ is the simple module given by
the representation

\[
\begin{array}{c}
0 \\
\downarrow & \downarrow \\
0 & 0 \\
\downarrow & \downarrow \\
0 & 0 \\
\downarrow & \downarrow \\
C \\
\end{array}
\]

while \( \frac{6}{1} \) is its indecomposable projective cover given by the representation

\[
\begin{array}{c}
C \\
\downarrow & \\
0 & 0 \\
\downarrow & \downarrow \\
C & C \\
\end{array}
\]

where each morphism \( C \to C \) is the identity.

The Auslander–Reiten quiver of \( \text{mod}(\Lambda) \) is the following, where red means projective (but not injective), green means injective (but not projective), and blue means projective-injective.
The classic Auslander–Reiten translation $\tau_1$ is given by moving one step to the left on the quiver. For instance, $\tau_1(\begin{smallmatrix} 6 \\ 4 \end{smallmatrix}) = \begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$, while $\tau_1(\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}) = 0$ (there is no module one step left of $\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$).

**Remark B.4.** By Theorems 4.9 and B.2 the unique 2-cluster tilting subcategory of $\text{mod}(\Lambda)$ is

\[ \mathcal{L} = \text{add}\{ \tau_j^i(i) \mid i \text{ injective in } \text{mod}(\Phi) \text{ and } 0 \leq j \}. \]

It is 2-abelian by Theorem 5.5.

The injective modules $\begin{smallmatrix} 6 \\ 4 \\ 1 \end{smallmatrix}$, $\begin{smallmatrix} 5 \\ 26 \\ 4 \end{smallmatrix}$, $\begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix}$ are projective, so the 2-Auslander–Reiten translation $\tau_2$ satisfies

\[ \tau_2 \left( \begin{smallmatrix} 6 \\ 4 \\ 1 \end{smallmatrix} \right) = \tau_2 \left( \begin{smallmatrix} 5 \\ 26 \\ 4 \end{smallmatrix} \right) = \tau_2 \left( \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \right) = 0. \]

The injective module $\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}$ permits a short exact sequence

\[ 0 \rightarrow \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \rightarrow \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \rightarrow \begin{smallmatrix} 3 \\ 5 \end{smallmatrix} \rightarrow 0 \]

whence $\Omega(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}) = \begin{smallmatrix} 5 \\ 6 \end{smallmatrix}$. Remark 4.2 gives

\[ \tau_2 \left( \begin{smallmatrix} 3 \\ 5 \end{smallmatrix} \right) = \tau_1 \left( \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \right) = 2. \]

In turn, the module $\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}$ permits a short exact sequence

\[ 0 \rightarrow \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \rightarrow \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \rightarrow \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \rightarrow 0 \]
whence $\Omega \left( ^2 \right) = 4$. Remark 4.2 gives

$$\tau_2 \left( ^2 \right) = \tau_1 \left( ^4 \right) = 1.$$  

This module is projective, so applying $\tau_2$ again gives zero.

The injective modules $\frac{3}{5}, \frac{5}{2}$ can be handled by the same method to give

$$\tau_2 \left( \frac{3}{5} \right) = \frac{2}{4}, \quad \tau_2 \left( \frac{5}{2} \right) = \frac{4}{1}.$$  

These modules are projective, so applying $\tau_2$ again gives zero.

It follows that

$$\mathcal{L} = \text{add}\{ ^1, \frac{4}{4}, ^2, \frac{6}{4}, ^4, ^2, ^6, ^2, ^2, \frac{5}{26}, ^5, \frac{5}{6}, ^5, ^3, ^3 \}.$$  

**Remark B.5.** Here is the AR quiver of $\mathcal{L}$. The dotted arrows
show the action of the 2-Auslander–Reiten translation $\tau_2$.

Note that the quiver is 3-dimensional, consisting of 3-dimensional analogues of the meshes found in classic Auslander–Reiten quivers. This is reflected in the 2-Auslander–Reiten sequences which can be read off, such as

$$0 \rightarrow \frac{2}{4} \rightarrow \frac{5}{26} \oplus 2 \rightarrow \frac{3}{5} \oplus \frac{5}{2} \rightarrow \frac{3}{5} \rightarrow 0.$$ 

References


