INFINITE DIMENSIONAL REPRESENTATIONS

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ABSTRACT. The work of Aihara-Iyama and Adachi-Iyama-Reiten shows that mutation in cluster theory can be studied in terms of the notion of a silting complex. In this lecture series we will consider non-compact silting complexes over an arbitrary ring. As in the compact case, such complexes are in bijection with certain t-structures and co-t-structures in the derived module category. We will focus on silting modules, the module theoretic counterparts of 2-term silting complexes. They generalise tilting modules over an arbitrary ring, as well as support τ-tilting modules over a finite dimensional algebra. We will discuss their role in approximation theory and localisation theory. For example, for hereditary rings or finite dimensional algebras of finite representation type, silting modules parametrize universal localisations and wide subcategories of finitely presented modules. As a consequence, we will see that the homological ring epimorphisms of a finite dimensional hereditary algebra form a lattice which completes the poset of noncrossing partitions. We will also discuss some constructions of infinite dimensional representations over finite dimensional hereditary that lead to classification results for silting, tilting, or pure-injective modules.

CONTENTS

Introduction

Section I is devoted to (not necessarily compact) silting complexes over a ring R. Then we focus on silting modules, the modules that occur as zero cohomologies of 2-term silting complexes. Silting modules generate torsion classes in the category of R-modules, which we call silting classes.

In section II we investigate approximation properties of silting modules. To this end, we first recall some notions and results from infinite dimensional tilting theory. This is then used to characterise silting classes in terms of the existence of certain approximations.

Section III explores the dual concept of a cosilting module. This notion is better behaved with respect to the existence of approximations. In particular, the cosilting classes are precisely the definable torsionfree classes in the category of R-modules. The dual result holds true over noetherian rings, but fails in general, as illustrated by an example.

In the section IV we turn to the concept of a ring epimorphism and discuss some homological conditions on ring epimorphisms. We present two important constructions: universal localisations and silting ring epimorphisms. Then we see that homological ring epimorphisms of hereditary rings are parametrised by silting modules.

Section V deals with the case of a finite dimensional algebra. We discuss in detail the example of the Kronecker algebra, describing the lattice of homological ring epimorphisms and explaining the classification of silting and cosilting modules and of the definable torsion classes.

For the material in Sections III, IV, V we also refer to [5], where more details and precise references are provided.

Background on triangulated categories can be found in
CH.A. WEIBEL, An introduction to homological algebra, Cambridge University Press 1994.
M. KASHIWARA,P. SCHAPIRA, Sheaves on Manifolds, Grundlehren der mathematischen Wissenschaften Vol. 292, Springer-Verlag 1990.

I. Silting complexes

Lecture 1

1. **Introduction.**

History:

- Silting complexes were first introduced by Keller-Vossieck, [27], to study t-structures in the bounded derived category of representations of Dynkin quivers. They are a generalisation of tilting complexes.
- Hoshino-Kato-Miyachi [22]: worked on 2-term silting complexes and their associated t-structures and torsion pairs.
  Assem-Souto Salorio-Trepode [13]: related silting complexes with Ext-projectivity.
- Aihara-Iyama [2]: introduced silting mutation. In contrast to tilting, silting complexes form a class where mutation can always be performed.
  Adachi-Iyama-Reiten [1]: introduced τ-tilting modules, the module-theoretic counterpart of 2-term silting complexes. $\rightsquigarrow$ τ-tilting theory developed.
- Keller-Nicolás [26], König-Yang [28], Iyama-Jørgensen-Yang [24],
  Mendoza-Saenz-Santiago-Souto Salorio [32]: established correspondences relating silting complexes, t-structures and co-t-structures.

- Wei [40]: studied semi-tilting complexes, i. e. silting complexes which are not necessarily compact. These are the silting complexes we are going to discuss below.

Notation. Throughout, R is a (unitary) ring, $\mathrm{Mod}(R)$ the category of right R-modules, and $\mathrm{Proj}(R)$ (respectively, $\mathrm{proj}(R)$) its subcategory of (finitely generated) projective modules, $\mathrm{mod}(R)$ will be the category of modules M_R with projective resolutions $\cdots P_1 \to P_0 \to M_R \to 0$ where all the P_i are in $\mathrm{proj}(R)$. Modules will always be right R-modules. We will also be considering algebras Λ over an algebraically closed field k. Note that if $R = \Lambda$, $\mathrm{mod}(\Lambda)$ is just the finitely generated Λ-modules. The unbounded derived (respectively, homotopy) category of $\mathrm{Mod}(R)$ will be denoted by $\mathcal{D}(R)$ (respectively, $K(\mathrm{Proj}(R))$). If we restrict ourselves to bounded, we use the usual superscript b.

For $\mathcal{X} \subset \mathcal{D}(R)$ a class of objects and $I \subset \mathbb{Z}$ we also introduce notation for the following orthogonal classes.

$$\mathcal{X}^{\perp_I} := \{Y \in \mathcal{D}(R) \mid \mathrm{Hom}_{\mathcal{D}(R)}(X, Y[i]) = 0 \;\forall\, X \in \mathcal{X} \;\forall\, i \in I\}$$

$$^{\perp_I}\mathcal{X} := \{Y \in \mathcal{D}(R) \mid \mathrm{Hom}_{\mathcal{D}(R)}(Y, X[i]) = 0 \;\forall\, X \in \mathcal{X} \;\forall\, i \in I\}$$

We will abbreviate this often, e.g. to $\perp_n$ if $I = \{n\}$ or $\perp_{0,1}$ for $I = \{0,1\}$ or to $\perp_{\geq n}$ if $I = \{i \in \mathbb{Z} \mid i \geq n\}$, analogously, $\perp_{\leq n}$ is defined.

Similarly, for $\mathcal{M} \subset \mathrm{Mod}(R)$, $I \subset \mathbb{N}$ we set

$$\mathcal{M}^{\perp_I} := \{N \in \mathrm{Mod}(R) \mid \mathrm{Ext}^i(M, N) = 0 \;\forall\, M \in \mathcal{M} \;\forall\, i \in I\}$$

and

$$^{\perp_I}\mathcal{M} := \{N \in \mathrm{Mod}(R) \mid \mathrm{Ext}^i(N, M) = 0 \;\forall\, M \in \mathcal{M} \;\forall\, i \in I\}$$

We also define additive and multiplicative closures in the derived and in the module category: For $\mathcal{X} \subset \mathcal{D}(R)$ or $\mathcal{X} \subset \mathrm{Mod}(R)$ we write

$$\mathrm{Add}\,\mathcal{X} := \{Y \mid Y \text{ is isomorphic to a direct summand of } \textstyle\bigoplus(\text{obj. in } \mathcal{X})\}$$

$$\mathrm{Prod}\,\mathcal{X} := \{Y \mid Y \text{ is isomorphic to a direct summand of } \textstyle\prod(\text{obj. in } \mathcal{X})\}$$

Note that these notions depend on the context, i.e. on whether we are in $\mathcal{D}(R)$ or in $\mathrm{Mod}(R)$.

The coproduct $\bigoplus_I X$ of copies of an object X indexed over a set I is denoted by $X^{(I)}$, and the product $\prod_I X$ is denoted by X^I.

Definition ([40])**.** A complex $\sigma \in K^b(\mathrm{Proj}\,R)$ is **silting** if
(S1) $\mathrm{Hom}_{\mathcal{D}(R)}(\sigma, \sigma^{(I)}[i]) = 0 \;\forall$ sets $I \;\forall i > 0$
(S2) $K^b(\mathrm{Proj}\,R)$ coincides with $\mathrm{tria}(\mathrm{Add}\,\sigma)$, the smallest triangulated subcategory of $\mathcal{D}(R)$ containing $\mathrm{Add}\,\sigma$.

Two remarks on the definition:
Regarding $\sigma^{(I)}$: Since σ is not compact (see below for compactness) in general, we have to allow coproducts of σ with itself, indexed by I.

- (S2) is equivalent to asking that $K^b(\mathrm{Proj}\,R)$ coincides with $\mathrm{thick}(\mathrm{Add}\,\sigma)$, the smallest triangulated subcategory of $\mathcal{D}(R)$ closed under direct summands containing $\mathrm{Add}\,\sigma$.
  Moreover, (S2) implies that σ is a **generator**, i.e. if $\mathrm{Hom}_{\mathcal{D}(R)}(\sigma, X[i]) = 0$ for all $i \in \mathbb{Z}$, then $X = 0$.

- If σ is compact, that is, $\sigma \in K^b(\operatorname{proj} R)$, then we have the following equivalences
  (S2) $\Longleftrightarrow$ σ generator
  (S1) $\Longleftrightarrow$ $\operatorname{Hom}(\sigma, \sigma[i]) = 0$ for all $i > 0$.
  Indeed, (S1) simplifies because the functor $\operatorname{Hom}(\sigma, -)$ commutes with coproducts when σ is compact. The other statements are discussed in the exercises.

2. **Torsion pairs in triangulated categories.**
   Let $\mathcal{T}$ be triangulated, $[\,]$ be the shift functor on $\mathcal{T}$.

Definition. Let $\mathcal{V}$, $\mathcal{W}$ be classes in $\mathcal{T}$ closed under direct summands.
The pair $(\mathcal{V}, \mathcal{W})$ is a **torsion pair** if the following holds
(i) $\operatorname{Hom}_{\mathcal{T}}(\mathcal{V}, \mathcal{W}) = 0$
(ii) for all $X \in \mathcal{T}$ $\exists$ $V \to X \to W \to V[1]$ with $V \in \mathcal{V}, W \in \mathcal{W}$.
A torsion pair $(\mathcal{V}, \mathcal{W})$ is called a ***t*-structure** if in addition the following holds
(iii) $\mathcal{V}[1] \subset \mathcal{V}$.
The torsion pair $(\mathcal{V}, \mathcal{W})$ is called a **co-*t*-structure**, if it satisfies (iii'):
(iii') $\mathcal{V}[-1] \subset \mathcal{V}$.
The class $\mathcal{V}$ is called the **aisle** of $(\mathcal{V}, \mathcal{W})$ and $\mathcal{W}$ the **co-aisle**.

Note that the notion of a t-structure goes back to Beilinson-Bernstein-Deligne, [16]. The definition here is a slight deviation of the original definition. The notion of a co-t-structure goes back to Pauksztello, [35], and to Bondarko, [17].

We point out a few nice properties of t-structures in triangulated categories:
If $(\mathcal{V}, \mathcal{W})$ is a t-structure, then the triangles in (ii) are functorial:

$$v(X) \to X \to w(X) \to v(X)[1]$$

for the adjoint functors v, w defined as follows

$$\mathcal{V} \underset{v}{\overset{\text{incl}}{\rightleftarrows}} \mathcal{T} \underset{\text{incl}}{\overset{w}{\rightleftarrows}} \mathcal{W}$$

v and w are called **truncation functors**. The category $\mathcal{H} := \mathcal{V} \cap \mathcal{W}[1]$ is called the **heart** of the t-structure. It is an abelian category. Truncation induces a cohomological functor (turning triangles into exact sequences), $H^0 : \mathcal{T} \to \mathcal{H}$.

If $(\mathcal{V}, \mathcal{W})$ is a co-t-structure, then the category $\mathcal{V}[1] \cap \mathcal{W}$ is called the **coheart** of $(\mathcal{V}, \mathcal{W})$. In general it is not abelian.

Example. Let $\mathcal{T} = \mathcal{D}(R)$.

(1) For $n \in \mathbb{Z}$ set

$$\mathcal{D}^{\leq n} := \{X \in \mathcal{D}(R) \mid H^i(X) = 0 \ \forall\ i > n\}$$

$$\mathcal{D}^{>n} := \{X \in \mathcal{D}(R) \mid H^i(X) = 0 \ \forall\ i \leq n\}.$$

Then the pair $(\mathcal{D}^{\leq n}, \mathcal{D}^{>n})$ is a t-structure. In case $n = 0$, it is called the **standard *t*-structure**. In this case, $\mathcal{H}$ is just the module category $\operatorname{Mod}(R)$.

(2) A t-structure $(\mathcal{V}, \mathcal{W})$ is called **intermediate** if there exists $n \leq m$ such that $\mathcal{D}^{\leq n} \subset \mathcal{V} \subset \mathcal{D}^{\leq m}$.

(3) If Λ is a finite dimensinal algebra over a field $\mathbb{K}$, then the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0})$ restricts to a t-structure $(\mathcal{V}, \mathcal{W})$ in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ which is **bounded**, i.e.

$$\mathcal{D}^b(\operatorname{mod}\Lambda) = \cup_{n\in\mathbb{Z}} \mathcal{V}[n] = \cup_{n\in\mathbb{Z}} \mathcal{W}[n].$$

(4) (Happel-Reiten-Smalø [23]): Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{Mod}(R)$. Define $\mathcal{V} := \{X \in \mathcal{D}^{\leq 0} \mid H^0(X) \in \mathcal{F}\}$ and $\mathcal{W} := \{X \in \mathcal{D}^{\geq 0} \mid H^0(X) \in \mathcal{F}\}$. Then $(\mathcal{V}, \mathcal{W})$ is a t-structure. This t-structure will be used later, it will be called **HRS-*t*-structure**.

(5) (Alonso-Jeremías Lopéz-Souto Salorio [3]): Every $\sigma \in \mathcal{D}(R)$ generates a t-structure $(\mathcal{V}, \mathcal{W})$ in $\mathcal{D}(R)$ as follows: Set $\mathcal{V} := \operatorname{aisle}\langle\sigma\rangle$ to be the smallest subcategory of $\mathcal{D}(R)$ which is closed under extensions, positive shifts and coproducts. $\mathcal{W} := \sigma^{\perp_{\leq 0}}$.

3. **The bijections.**

Proposition. *Let $\sigma : \cdots 0 \to P_{-n} \to \cdots \to P_{-1} \to P_0 \to 0 \to \cdots$ be a silting complex. Define $\mathcal{V} = aisle\langle\sigma\rangle$ and $\mathcal{U} := {}^{\perp_0}\mathcal{V}$. Then the following hold:*

- *$\mathcal{V} = \sigma^{\perp_{>0}}$*
- *$\mathcal{D}^{\leq -n} \subset \mathcal{V} \subset \mathcal{D}^{\leq 0}$, so the t-structure is intermediate.*
- *$(\mathcal{U}, \mathcal{V})$ is a co-t-structure whose coheart $\mathcal{U}[1] \cap \mathcal{V}$ is equal to* $\operatorname{Add}\sigma$.

Note that $\mathcal{U}[1] = {}^{\perp_1}\mathcal{V}$, so $\operatorname{Add}\sigma = {}^{\perp_1}\mathcal{V} \cap \mathcal{V}$ consists of the "ext-projective" objects in $\mathcal{V}$. This also shows that the aisle $\mathcal{V} = \sigma^{\perp_{>0}}$ determines σ up to additive closure.

Definition. Two silting complexes σ, σ' are said to be **equivalent** if $\sigma^{\perp_{>0}} = \sigma'^{\perp_{>0}}$. (equivalently, $\operatorname{Add}(\sigma) = \operatorname{Add}(\sigma')$).

Now we are ready to characterize equivalence classes of silting complexes.

Theorem ([7])**.** *There are bijections between*

(1) *equivalence classes of silting complexes*
(2) *intermediate t-structures $(\mathcal{V}, \mathcal{W})$ in $\mathcal{D}(R)$ such that*

$$\text{there exists a generator } \sigma \in \mathcal{D}(R) \text{ with } \operatorname{Add}\sigma = {}^{\perp_1}\mathcal{V} \cap \mathcal{V} \qquad (*)$$

(3) *triples $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ of classes in $\mathcal{D}(R)$ where $(\mathcal{U}, \mathcal{V})$ is a co-t-structure and $(\mathcal{V}, \mathcal{W})$ is an intermediate t-structure.*

Note that the above theorem restricts to König-Yang's characterization in the case of finite dimensional algebras:

Theorem (König-Yang bijections, [28])**.** *If Λ is a finite dimensional algebra, there are bijections between*

(1) *equivalence classes of compact silting complexes*
(2) *bounded t-structures in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ such that the heart is a length category*
(3) *bounded co-t-structures in $K^b(\operatorname{proj}\Lambda)$.*

Here a **length category** is an abelian category in which every object has finite length, i.e. it admits a finite filtration by simple objects.

Remark. We have seen that every silting complex $\sigma \in K^b(\operatorname{Proj} R)$ induces a t-structure $(\sigma^{\perp_{>0}}, \sigma^{\perp_{\leq 0}})$ whose aisle contains σ. In fact, this property characterizes silting complexes, and it leads to a general notion of *silting object* in a triangulated category. For work in this direction we refer to [38, 33, 9, 31].

4. **Silting modules.**

Lecture 2

Let $P_{-1} \stackrel{\sigma}{\to} P_0 \in K^b(\operatorname{Proj} R)$ be a 2-term complex, $X \in \mathcal{D}^{\leq 0}$. Then we have

$$X \in \sigma^{\perp_{>0}} \quad \iff \quad \begin{array}{ccccccccc} & 0 & \longrightarrow & P_{-1} & \xrightarrow{\sigma} & P_0 & \longrightarrow & 0 & \longrightarrow \dots \\ & & \swarrow^{s_{-1}} & \downarrow f & \swarrow^{s_0} & & & & \\ \longrightarrow & X_{-1} & \xrightarrow{d_{-1}} & X_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\ & & & \downarrow \pi & & & & & \\ & & & H^0(X) & & & & & \end{array}$$

(that is: all maps of complexes $\sigma \to X[1]$ are null-homotopic)

$\iff$ πf factors through σ

$$\iff \quad \begin{array}{ccc} P_{-1} & \xrightarrow{\sigma} & P_0 \\ {\scriptstyle \forall h}\downarrow & \swarrow_{\exists} & \\ H^0(X) & & \end{array}$$

$\iff$ $\operatorname{Hom}(\sigma, H^0(X))$ is surjective.

Set

$$\mathcal{D}_\sigma := \{M \in \operatorname{Mod} R \mid \operatorname{Hom}(\sigma, M) \text{ is surjective}\}.$$

Then the above is equivalent to $H^0(X) \in \mathcal{D}_\sigma$. Thus $\mathcal{D}_\sigma = \sigma^{\perp_{>0}} \cap \operatorname{Mod} R$.

Set $T := H^0(\sigma)$. We have $P_{-1} \xrightarrow{\sigma} P_0 \longrightarrow T \longrightarrow 0$.

Lemma. (1) $\mathcal{D}_\sigma \subset T^{\perp_1}$ with equality if σ is an injective map.

(2) If σ is a silting complex, then $\mathcal{D}_\sigma$ is a torsion class, and $\operatorname{Gen} T := \{M \in \operatorname{Mod} R \mid \exists \text{ epi } T^{(I)} \twoheadrightarrow M \text{ for some set } I\}$ coincides with $\mathcal{D}_\sigma$.

Proposition. *The following are equivalent*

(1) σ *is a silting complex.*

(2) $\operatorname{Gen} T = \mathcal{D}_\sigma$.

(3) $(\mathcal{D}_\sigma, T^{\perp_0})$ *is a torsion pair in* $\operatorname{Mod} R$.

Remark. (1) ${}^{\perp_1}\operatorname{Gen} T \cap \operatorname{Gen} T = \operatorname{Add} T$. Here again, $\operatorname{Add} T$ consists of the "ext-projective" objects in $\operatorname{Gen} T$, and $\operatorname{Gen} T$ determines T up to additive closure.

(2) The HRS t-structure associated to $(\mathcal{D}_\sigma, T^{\perp_0})$ is $(\sigma^{\perp_{>0}}, \sigma^{\perp_{\leq 0}})$.

Definition. (1) A module T is **partial silting** if it admits a projective presentation $P_{-1} \xrightarrow{\sigma} P_0$ such that $\mathcal{D}_\sigma$ is a torsion class containing T.
If, moreover, $\mathcal{D} = \operatorname{Gen} T$, then T is **silting**, and $\mathcal{D}_\sigma$ is a **silting class**.

(2) Two silting modules T and T' are **equivalent** if $\operatorname{Gen} T = \operatorname{Gen} T'$ ($\iff \operatorname{Add} T = \operatorname{Add} T'$).

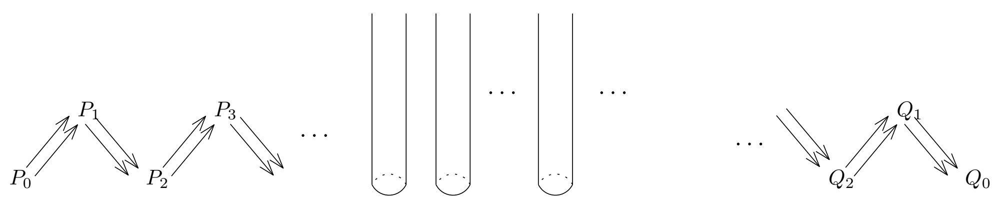
Main example 1. T is a silting module with respect to $P_{-1} \xhookrightarrow{\sigma} P_0$ (injective) $\iff$ $\operatorname{Gen} T = T^{\perp_1}$.

These are precisely the (large) tilting modules of projective dimension at most one:

Definition. T is a **(1-) tilting** module if $\operatorname{Gen} T = T^{\perp_1}$.

For the general notion of a (large) tilting module we refer to [4] and the references therein.

Example. Let $\Lambda = k(\bullet \rightrightarrows \bullet)$ be the Kronecker algebra.



$P_n \oplus P_{n+1}$ and $Q_n \oplus Q_{n+1}$ are tilting modules. Are there more tilting (silting) modules? (see last lecture).

Main example 2. If Λ is a finite-dimensional algebra and $T \in \operatorname{mod}\Lambda$, then
T is silting $\iff$ T is support τ-tilting in the sense of Adachi-Iyama-Reiten, [1].

Example. In the hereditary case, silting modules are precisely the support tilting.

Example. $\Lambda = k(\bullet \rightrightarrows \bullet)$. The simple modules S_1, S_2 are (support τ-tilting - factor out e_2 or e_1 respectively. - and therefore) silting.

Theorem ([7]). *There are bijections*

$$\begin{array}{ccccc}
\{\textit{2-silting complexes}\}/\sim & \xleftrightarrow{\;1-1\;} & \{\textit{t-structures } (\mathcal{V},\mathcal{W}) \textit{ with } (*) \textit{ and } \mathcal{D}^{\leq -1} \subseteq \mathcal{V} \subseteq \mathcal{D}^{\leq 0}\} & & \ni (\mathcal{V},\mathcal{W}) \\
{\scriptstyle 1-1}\downarrow{\scriptstyle H^0} & & {\scriptstyle 1-1}\downarrow \quad \uparrow{\scriptstyle HRS} & & \downarrow \\
\{\textit{silting modules}\}/\sim & \xrightarrow[1-1]{\operatorname{Gen}} & \{\textit{silting classes}\} & & \ni (\mathcal{V} \cap \operatorname{Mod} R)
\end{array}$$

In fact, the lower right entry, {silting classes}, is a tautology - we would like to get an intrinsic description of the torsion classes occurring as $\operatorname{Gen} T$ for some silting module T, something like {torsion classes such that ?}.

For finite-dimensional algebras, there is a bijection AIR [1]:

$$\{\text{fin. dim. silting modules}\}/\sim \;\xleftrightarrow{\;1-1\;}\; \{\text{funct. finite torsion classes in } \operatorname{mod}\Lambda\}$$

Can we imitate this result for $\operatorname{Mod} R$?

II. Silting approximations

We want to investigate approximation properties of silting classes. For the notions of precover (also called right approximation) and cover (or minimal right approximation), preenvelope (or left approximation) and envelope (or minimal left approximation), and for the corresponding definitions of (pre)covering, (pre)enveloping, and functorially finite subcategory we refer to Peter Jørgensen's lectures.

First of all, observe that every silting class $\mathcal{B}$, being a torsion class, is covering: the cover of a module M is given by the inclusion $t(M) \hookrightarrow M$ of the largest torsion submodule $t(M)$ in M. Here $t(M) = \tau_{\mathcal{B}}(M) = \Sigma\{\operatorname{Im} f \mid f \in \operatorname{Hom}_R(B,M), B \in \mathcal{B}\}$ is the trace of $\mathcal{B}$ in M. In the next section we will see that $\mathcal{B}$ is also preenveloping.

5. **Complements.**

Theorem ([7])**.** *Let T be partial silting with respect to $P_{-1} \xrightarrow{\sigma} P_0$.*

(1) *Every module $M \in \operatorname{Mod} R$ has a $\mathcal{D}_\sigma$ pre-envelope f_M, $M \xrightarrow{f_M} B_m \longrightarrow T^{(I)} \longrightarrow 0$*

(2) *There is a silting module $\overline{T}$ such that $T \overset{\oplus}{\hookrightarrow} \overline{T}$ and* $\operatorname{Gen}\overline{T} = \mathcal{D}_\sigma$.

(note that this is the "large version" of known constructions of complements).

Proof. (1) Set $I = \operatorname{Hom}(P_{-1}, M)$. We have the following pushout diagram:

$$\begin{array}{ccccccc}
{P_{-1}}^{(I)} & \xrightarrow{\sigma^{(I)}} & {P_0}^{(I)} & \longrightarrow & T^{(I)} & \longrightarrow & 0 \\
{\scriptstyle u}\downarrow & & \downarrow & & \| & & \\
M & \xrightarrow{f_M} & B_M & \longrightarrow & T^{(I)} & \longrightarrow & 0
\end{array}$$

where $u : {P_{-1}}^{(I)} \to M$ is the universal map defined as follows: given an element $x \in {P_{-1}}^{(I)}$, which has the form $x = (x_h)_{h \in I}$ where h runs over all maps $h \in I$ and only finitely many entries are non-zero, we set $u(x) = \sum h(x_h)$.

(2) Now, in case $M = R$, the diagram above yields a projective presentation

$$P_{-1}^{(I)} \xrightarrow{\gamma} P_0^{(I)} \oplus R \longrightarrow B_R \longrightarrow 0$$

and $\gamma \oplus \sigma$ is a projective presentation of $\overline{T} = B_R \oplus T$ such that $\operatorname{Gen}\overline{T} = \mathcal{D}_{\gamma\oplus\sigma} = \mathcal{D}_\sigma$.

□

Remark. So silting classes are functorially finite torsion classes. But we will see at the end of Section III that the converse fails in general.

Observe that, in the situation of [1], "functorially finite" actually means more than just the existence of preenvelopes and precovers, because one always has *minimal* approximations, and this entails orthogonality conditions on the kernel or cokernel of the approximation by a result known as Wakamatsu's Lemma. In order to formalize these properties we need the following notions.

6. **Tilting cotorsion pairs.**

Definition. Let $\mathcal{A}$, $\mathcal{B}$ be to classes in $\operatorname{Mod} R$.

1. $(\mathcal{A}, \mathcal{B})$ is a **cotorsion pair** if $\mathcal{A}^{\perp_1} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp_1}\mathcal{B}$.
2. It is a **complete cotorsion pair** if for all $M \in \operatorname{Mod} R$ there exist s.e.s.'s

$$0 \longrightarrow M \xrightarrow{f} B \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow B' \longrightarrow A' \xrightarrow{g} M \longrightarrow 0$$

   with $A, A' \in \mathcal{A}$, $B, B' \in \mathcal{B}$. (here, f has to be a $\mathcal{B}$-preenvelope.
3. A cotorsion pair is **hereditary** if

$$\operatorname{Ext}^i(\mathcal{A}, \mathcal{B}) = 0 \;\forall\, i > 0.$$

Remark. (a) In condition (2), f is a $\mathcal{B}$-preenvelope and g an $\mathcal{A}$-precover. Notice further that it is enough to require the existence of one set of s.e.s.'s (just the upper one, or just the lower one), because one can recover the other by an argument known as Salce's Lemma.

(b)

$(\mathcal{A}, \mathcal{B})$ is hereditary	$\iff$	$\mathcal{A}$ is resolving, i.e. it is closed under extensions, closed under kernels of epimorphisms and $R \in \mathcal{A}$.
	$\iff$	$\mathcal{B}$ is coresolving, i.e. it is closed under extensions, closed under cokernels of monomorphisms, and contains all injective modules.

Example. (1) $(\operatorname{Proj} R, \operatorname{Mod} R)$ and $(\operatorname{Mod} R, \operatorname{Inj} R)$ are complete hereditary cotorsion pairs.

(2) (Eklof-Trlifaj [21]). Every *set of modules* (as opposed to class) $\mathcal{S} \subset \operatorname{Mod} R$ generates a complete cotorsion pair: $({}^{\perp_1}(\mathcal{S}^{\perp_1}), \mathcal{S}^{\perp_1})$.

(3) If T is tilting, then $({}^{\perp_1}(T^{\perp_1}), T^{\perp_1})$ is a complete hereditary cotorsion pair and $({}^{\perp_1}(T^{\perp_1}) \cap T^{\perp_1}) = \operatorname{Add} T$.

Lecture 3

The following result (and its generalization to tilting modules of projective dimension n which we don't discuss here) is a "large version" of a well-known theorem of Auslander-Reiten from 1991.

Theorem ([12])**.** *Let $\mathcal{B}$ be a torsion class and $\mathcal{A} = {}^{\perp_1}\mathcal{B}$. Then the following are equivalent*

1. *$\mathcal{B}$ is a tilting class.*
2. *$(\mathcal{A}, \mathcal{B})$ is a complete (hereditary) cotorsion pair.*
3. *$\exists\, 0 \to R \to B \to A \to 0$ with $B \in \mathcal{B}$, $A \in \mathcal{A}$.*

"Proof". (3) $\Longrightarrow$ (1): $T = B \oplus A$ is tilting with $\operatorname{Gen} T = \mathcal{B}$. □

Corollary. There is a bijection between

- equivalence classes of tilting modules;
- triples $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in $\operatorname{Mod} R$, where $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair and $(\mathcal{B}, \mathcal{C})$ is a torsion pair (cf. with a similar statement for silting complexes at the end of §3).

Theorem (Bazzoni-Herbera [15] ("Finite type"))**.** *Every tilting class is of the form $\mathcal{B} = \mathcal{S}^{\perp_1}$ for some set $\mathcal{S} \subset \operatorname{mod} R$. More precisely, $\mathcal{S} = {}^{\perp_1}\mathcal{B} \cap \operatorname{mod} R$.*

This result shows that tilting modules, even the large ones, are determined by finitely presented modules. The same holds true for n-tilting modules.

Corollary. (1) There exists a bijection

$$\left\{\begin{array}{c}\text{resolving subcat.}\\ \mathcal{S}\subset \operatorname{mod}R \text{ with}\\ \operatorname{pd}\mathcal{S}\le 1\end{array}\right\} \overset{1-1}{\longleftrightarrow} \quad \{\text{tilting modules}\}/_{\sim}$$

$$\mathcal{S} \qquad\qquad \longmapsto \quad T \text{ such that } \operatorname{Gen}T=\mathcal{S}^{\perp_1}$$

(2) Every tilting class is **definable**, i. e. closed under direct products, direct limits and **pure submodules** (see below for the definition of pure submodule).

Definition. (1) An exact sequence $0\to X\to Y\to Z\to 0$ in $\operatorname{Mod}R$ is **pure-exact** if $\forall\ M\in \operatorname{Mod}R^{op}$, the sequence

$$0\to X\otimes_R M\to Y\otimes_R M\to Z\otimes_R M\to 0$$

is exact.

(2) X is a **pure submodule of** Y if the sequence $0\to X\hookrightarrow Y\to Y/X\to 0$ is pure-exact.

7. **Back to silting.**

The statements in §6 can be translated into statements on silting modules. To this end we regard silting classes as tilting classes in a new category.

The morphism category $\operatorname{Mor}R$ is defined by:

- objects: R-module homomorphisms $M\overset{g}{\to}N$
- morphisms: commutative squares $\begin{array}{ccc} M & \xrightarrow{g} & N \\ \phi\downarrow & & \downarrow\psi \\ M' & \xrightarrow{g'} & N'\end{array}$, $\psi g=g'\phi$.

Idea [30]: silting class $\mathcal{B}$ in $\operatorname{Mod}R \rightsquigarrow$ tilting class in $\operatorname{Mor}R$.

$\mathcal{B}$ gives then rise to a tilting cotorsion pair in $\operatorname{Mor}R$. Its "shadow" in $\operatorname{Mod}R$ yields a pair of classes $(\mathcal{A},\mathcal{B})$ where $\mathcal{A}$ consists of the modules having a projective presentation $P_{-1}\overset{w}{\to}P_0$ with the following factorization property: $\begin{array}{ccc} P_{-1} & \xrightarrow{w} & P_0 \\ \downarrow & \swarrow & \\ B\in\mathcal{B} & & \end{array}$, in other words, $\mathcal{B}\subset\mathcal{D}_w$.

The characterization of tilting classes in §6 then translates into the following characterization of silting classes, which was first proven directly by Breaz-Zemlicka.

Theorem. [19] *Let $\mathcal{B}$ be a torsion class, and $\mathcal{A}=\{A\in\operatorname{Mod}R\mid \exists\, proj.\ pres.\ P_{-1}\xrightarrow{w}P_0 \text{ of } A \text{ s.t. } \mathcal{B}\subset\mathcal{D}_w\}$. T.F.A.E.*

(1) *$\mathcal{B}$ is a silting class.*

(2) $\forall\, M \in \operatorname{Mod} R$*, there exist s.e.s.'s*

$$M \xrightarrow{f} B \longrightarrow A \longrightarrow 0 \qquad (**)$$

$$0 \longrightarrow B' \longrightarrow A' \xrightarrow{g} M \longrightarrow 0$$

where f is a $\mathcal{B}$-preenvelope, g is an $\mathcal{A}$-precover, and $A \in \mathcal{A}$, $B' \in \mathcal{B}$.

(3) $\exists\; R \xrightarrow{f} B \longrightarrow A \longrightarrow 0$ *where f is a $\mathcal{B}$-preenvelope and $A \in \mathcal{A}$.*

Note that the injectivity of the $\mathcal{B}$-preenvelope is lost. This is due to the fact that silting modules are not faithful in general. Indeed, they are faithful iff they are tilting.

We are now ready to improve the theorem at the end of Section I: we get bijections

$$\begin{array}{ccc}
\{\text{2-term silt. cplexes}\}/_\sim & \xleftrightarrow{\;1-1\;} & \{t\text{-str. } (\mathcal{V},\mathcal{W}) \text{ with } (*),\ \mathcal{D}^{\le -1} \subseteq \mathcal{V} \subseteq \mathcal{D}^{\le 0}\} \\
{\scriptstyle 1-1}\updownarrow{\scriptstyle H_0} & & \updownarrow{\scriptstyle 1-1} \\
\{\text{silt. mod.'s}\}/_\sim & \xleftrightarrow{\;1-1\;} & \{\text{torsion classes with } (**)\}
\end{array}$$

We will see in the next section that for a left noetherian ring the theorem can be improved further. In fact, in that case we have $\{\text{torsion classes with } (**)\} = \{\text{definable torsion classes}\}$.

Now we turn to the translation of the "finite type" result from [15].

Theorem (Marks-Stovicek [30, 6] ("Finite type"))**.** *Every silting class is of the form $\mathcal{D} = \bigcap_{\sigma \in \Sigma} \mathcal{D}_\sigma$ for some set $\Sigma \subset \operatorname{Mor}(\operatorname{proj} R)$. In particular, $\mathcal{D}$ is definable.*

Remark: there is an analogous result for silting complexes, see [31].

III. COSILTING MODULES

8. **Cosilting approximations.**

Let us now consider a dual notion. Given a morphism $E_0 \xrightarrow{w} E_1 \in \operatorname{Mor}(\operatorname{Inj} R)$, consider

$$\mathcal{C}_w := \{M \in \operatorname{Mod} R \mid \operatorname{Hom}_R(M, w) \text{ is surjective}\}.$$

Definition. (1) A module C is **cosilting** if it has an injective copresentation $C \to E_0 \xrightarrow{w} E_1$ such that $\mathcal{C}_w = \operatorname{Cogen} C := \{M \in \operatorname{Mod} R \mid \exists\, M \hookrightarrow C^I \text{ for some set } I\}$. Then $\operatorname{Cogen} C$ is said to be a **cosilting class**.

(2) Two cosilting modules C and C' are **equivalent** if $\operatorname{Cogen} C = \operatorname{Cogen} C'$ (which amounts to $\operatorname{Prod} C = \operatorname{Prod} C'$).

(3) C is a **(1-)cotilting** module if $\operatorname{Cogen} C = {}^{\perp_1}C$. Then $\operatorname{Cogen} C$ is said to be a **cotilting class**.

Remark. (1) C is cotilting $\Longleftrightarrow$ C is cosilting with respect to $E_0 \overset{w}{\twoheadrightarrow} E_1$ surjective.

(2) C is cosilting $\Longrightarrow$ C is cotilting over $R/\operatorname{ann}(C)$.

Again, we refer to [4] for the general notion of a (large) cotilting module of injective dimension n.

Definition. A module I is **pure-injective** if every pure exact sequence $0 \to I \to Y \to Z \to 0$ splits.

Theorem (Bazzoni [14], Breaz-Pop [18]). *Every cosilting module is pure-injective. Every cosilting class is definable.*

The result above can be regarded as the dual version of "finite type" for tilting/silting classes.

Theorem (Breaz-Zemlicka [19], Zhang-Wei [41]). *Let $\mathcal{F} \subset \operatorname{Mod} R$ be a torsion-free class. TFAE:*

(1) *$\mathcal{F}$ is a cosilting class.*
(2) *$\mathcal{F}$ is (preenveloping and) covering.*
(3) *$\forall\, M \in \operatorname{Mod} R$, there exists $0 \to K \to F \to M$ with $K \in \mathcal{F}^{\perp_1}$, $F \to M$ a $\mathcal{F}$-precover.*

Since definable classes are always covering, we obtain

Corollary. There is a bijection

$$\{\text{cosilting modules}\}/_\sim \xleftrightarrow{1-1} \{\text{definable torsion-free classes}\}$$

$$C \longmapsto \operatorname{Cogen} C$$

9. **Duality.**

Let K be a commutative ring such that R is a K-algebra. Let I be an injective cogenerator of $\operatorname{Mod} K$. Define $(-)^+ := \operatorname{Hom}_K(-,I)$. Then M^+ is a left R-module. Note that $M^{++} \neq M$ in general).

• if R is a fin.dim. algebra, $(-)^+$ is the usual duality of vector spaces.
• We can always take $K = \mathbb{Z}$. Then $(-)^+ = \operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$ is the character dual.

Fact. T silting w.r.t. $P_{-1} \overset{\sigma}{\to} P_0$. Then T^+ is cosilting w.r.t. $P_0^+ \overset{\sigma^+}{\to} P_{-1}^+$ and $\operatorname{Cogen} T^+ = \mathcal{C}_{\sigma^+}$.

Lecture 4

Theorem (Auslander, Gruson-Jensen, Herzog; Bazzoni; Angeleri Hügel-Hrbek). *We have two bijections which induce an injection Φ as follows:*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\textit{definable subcat's}\\ \textit{of } \operatorname{Mod} R\end{array}\right\} & \overset{1-1}{\longleftrightarrow} & \left\{\begin{array}{c}\textit{definable subcat's}\\ \textit{of } R\operatorname{Mod}\end{array}\right\} \\
\cup\!\! | & & \cup\!\! | \\
\left\{\begin{array}{c}\textit{definable torsion}\\ \textit{classes in } \operatorname{Mod} R\end{array}\right\} & \overset{1-1}{\longleftrightarrow} & \left\{\begin{array}{c}\textit{definable torsion-free}\\ \textit{classes in } R\operatorname{Mod}\end{array}\right\} \\
\cup\!\! | & & \| \\
\{\textit{silting classes in } \operatorname{Mod} R\} & \overset{\Phi}{\hookrightarrow} & \{\textit{cosilting classes in } R\operatorname{Mod}\} \\
\operatorname{Gen} T & \longmapsto & \operatorname{Cogen} T^+
\end{array}$$

If R is left noetherian, then Φ is bijective.

In fact, in the noetherian case every torsion pair $(\mathcal{T},\mathcal{F})$ in $R\operatorname{Mod}$ with definable torsion-free class $\mathcal{F}$ is determined by a set of finitely generated modules $\mathcal{U} = \mathcal{T} \cap R\operatorname{mod}$. This allows to find a set of maps $\Sigma \subset \operatorname{Mor}(\operatorname{proj} R)$ yielding a silting class $\mathcal{D} = \bigcap_{\sigma\in\Sigma} \mathcal{D}_\sigma$ such that $\Phi(\mathcal{D}) = \mathcal{F}$.

Corollary. If R is left noetherian, then we have a bijection

$$\begin{array}{ccc} \{\text{silting mod's in } \mathrm{Mod}\, R\}/\sim & \overset{1-1}{\longleftrightarrow} & \left\{ \begin{array}{c} \text{definable torsion classes} \\ \text{in } R\,\mathrm{Mod} \end{array} \right\} \\ T & \longmapsto & \mathrm{Gen}\, T \end{array}$$

For commutative noetherian rings it follows that silting modules are parametrized by specialization closed subsets of the Zariski spectrum Spec$R$. In general, if R is a commutative ring, then one can describe the image of the map Φ above: it consists of all torsion-free classes occurring in hereditary torsion pairs of finite type. As a consequence, silting modules over commutative rings are in bijection with the Gabriel localizations of Mod$R$. For details we refer to [6] or [5].

Example. (1) Let S_R be simple such that $\mathrm{Ext}^1(S,S) = 0$ and ${}_{\mathrm{End}(S)}S$ is fin. gen. Then $\mathrm{Gen}\, S = \mathrm{Add}\, S = \mathrm{Prod}\, S = \mathrm{Cogen}\, S$.
- this category is definable
- this category is a cosilting class
- $\mathrm{Add}\, S = \mathrm{Prod}\, S \Longrightarrow \mathrm{Add}\, S$ is an enveloping (torsion) class.

(2) Let $(R, \mathfrak{m})$ be a local commutative ring, with $\mathfrak{m} = \mathfrak{m}^2 \neq 0$ (e.g. the ring of Puiseux series in n variables). R is **not** noetherian, by Nakayama's Lemma. Let $S = R/\mathfrak{m}$. Then S satisfies the conditions in (1), and the cosilting class

$$\mathrm{Cogen}\, S = \{M \in \mathrm{Mod}\, R \mid M \cdot \mathfrak{m} = 0\} \notin \mathrm{Im}\, \Phi.$$

Moreover, $\mathrm{Gen}\, S$ is a functorially finite torsion class, but it is not a silting class.

IV. Silting and ring epimorphisms

10. **Ring epimorphisms.**

Definition. A ring homomorphism $\lambda : R \to S$ is a **ring epimorphism** if it is an epimorphism in the category of rings.

Equivalently: the embedding $\mathrm{Mod}\, S \overset{\lambda_*}{\hookrightarrow} \mathrm{Mod}\, R$ is full.

λ is a **homological ring epimorphism** if it is an epi. such that $\mathrm{Tor}_i^R(S,S) = 0 \ \forall\, i > 0$.

Equivalently: $\mathcal{D}(S) \overset{\lambda_*}{\hookrightarrow} \mathcal{D}(R)$ is full.

Example.
- $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a homomogical ring epimorphism.
- classical localization at a multiplicative set always yields a homological ring epi.

Definition. Two ring epimorphisms $\lambda : R \to S$, $\lambda' : R \to S'$ are **equivalent** if $\begin{array}{ccc} R & \xrightarrow{\lambda} & S \\ {\scriptstyle\lambda'}\downarrow & \swarrow {\scriptstyle\exists\cong} & \\ S' & & \end{array}$.

The equivalence classes of ring epimorphisms are called **epiclasses**.

Note. If $\lambda : R \to S$ is a ring epi, then there are adjunctions $\mathrm{Mod}\, S \overset{\lambda_*}{\hookrightarrow} \mathrm{Mod}\, R$, with $- \otimes_R S$ and $\mathrm{Hom}_R(S, -)$ from $\mathrm{Mod}\, R$ to $\mathrm{Mod}\, S$.

A full subcategory $\mathcal{X} \subset \operatorname{Mod} R$ is called **bireflective** if for the inclusion $i : \mathcal{X} \hookrightarrow \operatorname{Mod} R$ there exist adjunctions:

$$\mathcal{X} \underset{\exists r}{\overset{\exists l}{\leftleftarrows}} \xrightarrow{\;i\;} \operatorname{Mod} R \quad \text{with } (l,i) \text{ and } (i,r) \text{ adjoint pairs.}$$

Let $\mathcal{X} \subset \operatorname{Mod} R$ be bireflective. Then for any $M \in \operatorname{Mod} R$ the unit $M \xrightarrow{\eta_M} il(M)$ is a $\mathcal{X}$**-reflection**, i.e. $\operatorname{Hom}(\eta_M, X)$ is an isomorphism for all $X \in \mathcal{X}$:

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & il(M) \\ & \searrow & \swarrow_{\exists !} \\ & X & \end{array}$$

In particular, for $M = R$, this gives rise to a ring epimorphism $\lambda : R \to \operatorname{End}(il(R))$ such that $\mathcal{X}$ is the essential image of λ_*.

Theorem. *Let R be a ring.*

(1) *(Gabriel - de la Peña). There is a bijection*

$$\begin{array}{ccc} \{\textit{epiclasses of ring epis } \lambda : R \to S\} & \overset{1-1}{\longleftrightarrow} & \{\textit{bireflective subcat's } \mathcal{X} \subset \operatorname{Mod} R\} \\ \lambda : R \to S & \longmapsto & \operatorname{Im} \lambda_* \\ R \to \operatorname{End}(il(R)) & \longleftarrow & \mathcal{X} \end{array}$$

(2) *(Bergman-Dicks, Schofield)*

$$\left\{ \begin{array}{c} \textit{epiclasses of ring epis} \\ \lambda : R \to S \textit{ s.t. } \operatorname{Tor}_1^R(S,S) = 0 \end{array} \right\} \overset{1-1}{\longleftrightarrow} \{\textit{bireflective extension-closed } \mathcal{X} \subset \operatorname{Mod} R\}$$

(3) *(Geigle-Lenzing, Iyama). If Λ is a fin. dim. algebra then*

$$\left\{ \begin{array}{c} \textit{epiclasses of ring epis} \\ \lambda : \Lambda \to \Gamma \\ \textit{with } \Gamma \textit{ fin.dim.,} \\ Tor_1^{\Lambda}(\Gamma,\Gamma) = 0 \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{functorially finite wide} \\ \textit{subcat's } \mathcal{W} \subset \operatorname{Mod} R \end{array} \right\}$$

*(**wide**: abelian, extension-closed)*

11. Two constructions of ring epimorphisms.

First construction:

Theorem (Cohn, Schofield [39])**.** $\Sigma \subset \operatorname{Mor}(\operatorname{proj} R) \Longrightarrow \exists$ *ring epi* $\lambda_\Sigma : R \to R_\Sigma$ *such that*

(i) λ_Σ *is* **Σ-inverting**, *i.e.* $\forall\, \sigma \in \Sigma$, $\sigma \otimes_R R_\Sigma$ *is an isomorphism.*

(ii) *Universal property:*

$$\begin{array}{ccc} R & \xrightarrow{\lambda_\Sigma} & R_\Sigma \\ {\scriptstyle \Sigma-inverting}\downarrow & \swarrow_{\exists !} & \\ S & & \end{array}$$

In this situation, λ_Σ *is called a* **universal localization of R at Σ**

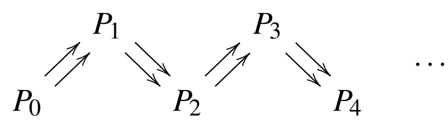
Remark. (1) $\operatorname{Im}(\lambda_\Sigma)_* = \{M \in \operatorname{Mod} R \mid \operatorname{Hom}(\sigma, M) \text{ is bijective } \forall\, \sigma \in \Sigma\} =: \mathcal{X}_\Sigma$.

(2) $\operatorname{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$.
(3) (Krause-Stovicek 2010). If R is hereditary, then the universal localizations are precisely the homological ring epimorphisms.

Example. (1) Given $P \in \operatorname{proj} R$, the trace ideal $I = \tau_P(R) = \Sigma\{\operatorname{Im} f \mid f \in \operatorname{Hom}_R(P,R)\}$ is idempotent, i.e. $I = I^2$, and the canonical epimorphism $\lambda : R \to R/I$ is the universal localization at $\Sigma = \{0 \to P\}$.
Indeed, $\mathcal{X}_\Sigma = P^{\perp_0}$, and

$$\begin{aligned} \operatorname{Mod} R/I &= \{M \in \operatorname{Mod} R \mid MI = 0\} \\ &= \{M \in \operatorname{Mod} R \mid M\tau_P(R) = 0\} \\ &= \{M \in \operatorname{Mod} R \mid \tau_P(M) = 0\} \\ &= P^{\perp_0} \end{aligned}$$

(2) $\Lambda = k(\bullet \rightrightarrows \bullet)$. Consider the preprojective component



The almost split sequences $0 \to P_0 \xrightarrow{\sigma} P_1^2 \to P_2 \to 0$ is a minimal projective resolution of P_2.
We want to compute λ_Σ for $\Sigma = \{\sigma\}$.
One checks that $\operatorname{Im}(\lambda_\Sigma)_* = P_2^{\perp_{0,1}} = \operatorname{Add}(P_1) = \mathcal{X}_\Sigma$.
The $\mathcal{X}_\Sigma$-reflection of Λ is $0 \to \Lambda \to P_1^2 \oplus P_1 \to P_2 \to 0$, so
$\lambda_\Sigma : \Lambda \to \operatorname{End}({P_1}^3) \cong M_{3\times 3}(k)$.

Lecture 5

Second construction:

Theorem ([8]). *Let T_1 be a partial silting module w.r.t. $P_{-1} \xrightarrow{\sigma} P_0$ in $\operatorname{Mor}(\operatorname{Proj} R)$, let T be a silting module s.t. $T_1 \overset{\oplus}{\hookrightarrow} T$ and $\operatorname{Gen} T = \mathcal{D}_\sigma$. Then*
(1) *$\mathcal{X}_\sigma = \mathcal{D}_\sigma \cap T_1^{\perp_0}$ bireflective, extension-closed.*
(2) *The corresponding ring epi $\lambda : R \to S$ satisfies $S = \operatorname{End} T/I$ for some $I = I^2 \leq \operatorname{End} T$. $\lambda = \lambda_{T_1}$ is a* **silting ring epimorphism**.

Remark. The theorem above extends results proved in the context support τ-tilting modules ([25]).
(1) If $\sigma \in \operatorname{Mor}(\operatorname{proj} R)$ then $\lambda_{T_1} \sim \lambda_{\{\sigma\}}$ univ. localization.
(2) For any ring epim. $\lambda : R \to S$, we have the following implications
λ universal localization $\overset{[30]}{\Longrightarrow}$ λ is a silting ring epi $\Longrightarrow \operatorname{Tor}_1^R(S,S) = 0$.
These implications are proper. In fact, the converse of the first implication holds true in some cases, e.g. for commutative noetherian rings which are regular or have Krull dimension at most one, but fails already in Krull dimension two. The converse of the second implication holds true for commutative noetherian rings, but fails e.g. for $R \to R/\mathfrak{m}$ where $(R, \mathfrak{m})$ is as in the Example at the end of Section III. For details we refer to [10, 5].

Example. If $\lambda : R \to S$ is a ring epimorphism with $\operatorname{Tor}_1^R(S,S) = 0$ and $\operatorname{pdim} S_R \leq 1$, then

- $T = S \oplus \operatorname{Coker}\lambda$ is a silting module
- $T_1 = \operatorname{Coker}\lambda$ is a partial silting module
- $\lambda = \lambda_{T_1}$.

12. **Minimal silting modules.**

Idea: we want to single out the silting modules that can be obtained by the construction in the example above.

Definition. A silting module T is **minimal** if the $\operatorname{Gen} T$-preenvelope of R can be chosen minimal, i.e. there exists

$$R \xrightarrow[\operatorname{Gen} T\text{-envelope}]{f} T_0 \longrightarrow T_1 \longrightarrow 0 \qquad (***)$$

with $T_0, T_1 \in \operatorname{Add} T$.

Example. (1) $T \in \operatorname{mod}\Lambda$ silting over a fin. dim. algebra Λ.
(2) $T = S \oplus \operatorname{Coker}\lambda$ with λ as in the example above.

Qu. If T is minimal silting and T_1 as in $(***)$: is T_1 partial silting? The answer is yes, if R is hereditary, or $R = \Lambda$, fin. dim. algebra.

In these cases we can then assign a ring epimorphism λ_{T_1} to every minimal tilting module T.

Theorem ([8])**.** *For R hereditary, there is a bijection*

$$\left\{ \begin{array}{c} \textit{minimal} \\ \textit{silting mod's} \end{array} \right\} /\sim \quad \overset{1-1}{\longleftrightarrow} \quad \left\{ \begin{array}{l} \textit{epiclasses of homological} \\ \textit{ring epis } \lambda : R \to S \end{array} \right\}$$

$$\begin{array}{ccc} T & \longmapsto & \lambda_{T_1} \\ T = S \oplus \operatorname{Coker}\lambda & \longleftarrow & \lambda : R \to S \end{array}$$

$$\begin{array}{ccc} \cup\!\mid & & \cup\!\mid \end{array}$$

$$\textit{Also,} \quad \left\{ \begin{array}{c} \textit{minimal} \\ \textit{tilting mod's} \end{array} \right\} /\sim \quad \overset{1-1}{\longleftrightarrow} \quad \left\{ \begin{array}{l} \textit{epiclasses of inj. homological} \\ \textit{ring epis } \lambda : R \to S \end{array} \right\}$$

One of the main steps in the proof consists in showing that for hereditary rings, the kernel of f in $(***)$ is an idempotent ideal $\rightsquigarrow$ we can work over the kernel, where T is even a tilting module.

V. Silting over finite dimensional algebras

Let Λ be a fin. dim. algebra.

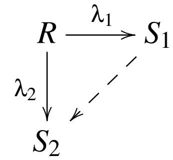
Corollary (Ingalls-Thomas, Marks [29])**.** Λ hereditary. Then there exist bijections

$$\left\{ \begin{array}{c} \text{silting mod's} \\ \text{in } \operatorname{mod}\Lambda \end{array} \right\} /\sim \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{epiclasses} \\ \text{of homological} \\ \text{ring epis } \lambda : \Lambda \to \Gamma \\ \Gamma \text{ fin. dim.} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{funct. finite} \\ \text{wide subcategories} \\ \text{of } \operatorname{mod}\Lambda \end{array} \right\}$$

Note that the last set is a poset, but in general it is not a lattice. However, we obtain a lattice if we relax the assumption "Γ finite dimensional".

13. **The lattice of ring epimorphisms.**

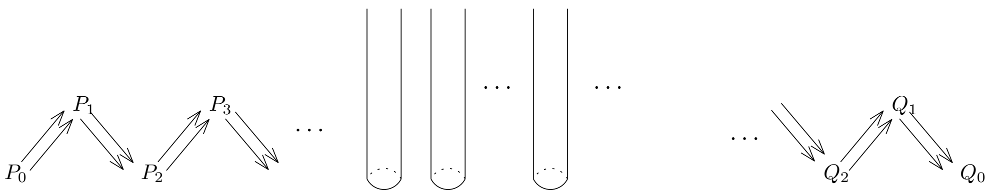
Given a ring R, $\lambda_i : R \to S_i$ ring epis, set $\lambda_1 \geq \lambda_2$ if



($\Longleftrightarrow \mathcal{X}_1 \supseteq \mathcal{X}_2$ for the corresponding bireflective subcategories $\mathcal{X}_1, \mathcal{X}_2$).

With this, we obtain a lattice structure on the ring epis.

Example. $\Lambda = k(\bullet \rightrightarrows \bullet)$



P_0 and P_1 are the two projective, Q_1 and Q_0 the two injective Λ-modules. We write **p** and **q** for the preprojective and preinjective component, respectively.
Every tube has indecomposables starting from a simple regular:

$$S = S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookrightarrow \cdots$$

in a chain of irreducible maps. AR-sequences in the tubes: $0 \to S_1 \to S_2 \to S_1$ and, for $n > 1$, $0 \to S_n \to S_{n+1} \oplus S_{n-1} \to S_n \to 0$.

We want to understand some of the inf. dim. modules. Define

$$S_\infty := \varinjlim S_n \text{ a Prüfer module (associated to the tube)}$$

$$S_{-\infty} := \varprojlim S_n \text{ an adic module (associated to the tube)}$$

The indecomposable pure injective modules have been classified (Okoh [34], Prest [36]):

- indecomposables in $\operatorname{mod}\Lambda$
- Prüfer, adic modules (S_∞, $S_{-\infty}$) for any tube
- generic module G: up to iso, the unique indecomposable, inf. dim. module which is fin. dim. over $\operatorname{End} G$.

We want to find more infinite dimensional modules. For that we look at the homological ring epis. First a few comments on notation, on the table:
We write D for $\operatorname{Hom}(-,k)$. The first two ring epis are the trivial ones. Then we consider the non-injective ones. Their kernel must be an idempotent ideal, thus a trace ideal $\tau_P(R) = \Sigma\{\operatorname{Im} f \mid f \in \operatorname{Hom}_R(P,R)\}$ given by a projective module P. Note that $\tau_{P_0}(\Lambda) = P_0 \oplus \operatorname{rad} P_1$, while $\tau_{P_1}(\Lambda)$ contains P_1 but not P_0, so it is P_1.

The remaining ring epis are universal localizations at (minimal projective resolutions of) non-projective modules.

Since the tubes are indexed by $\mathbb{P}^1(k)$, by choosing a subset $\mathcal{U}$ of $\mathbb{P}^1(k)$, we choose a subset of the tubes or of the simple regular modules at their mouths. $\Lambda_{\mathcal{U}}$ then denotes the universal localization at the set of (minimal projective resolutions of) simples for $\mathcal{U}$.

Note that $\Lambda_{\mathcal{U}}$ is infinite dimensional, and $\Lambda_{\mathcal{U}}/\Lambda \cong \bigoplus_{S \in \mathcal{U}} S_\infty$.

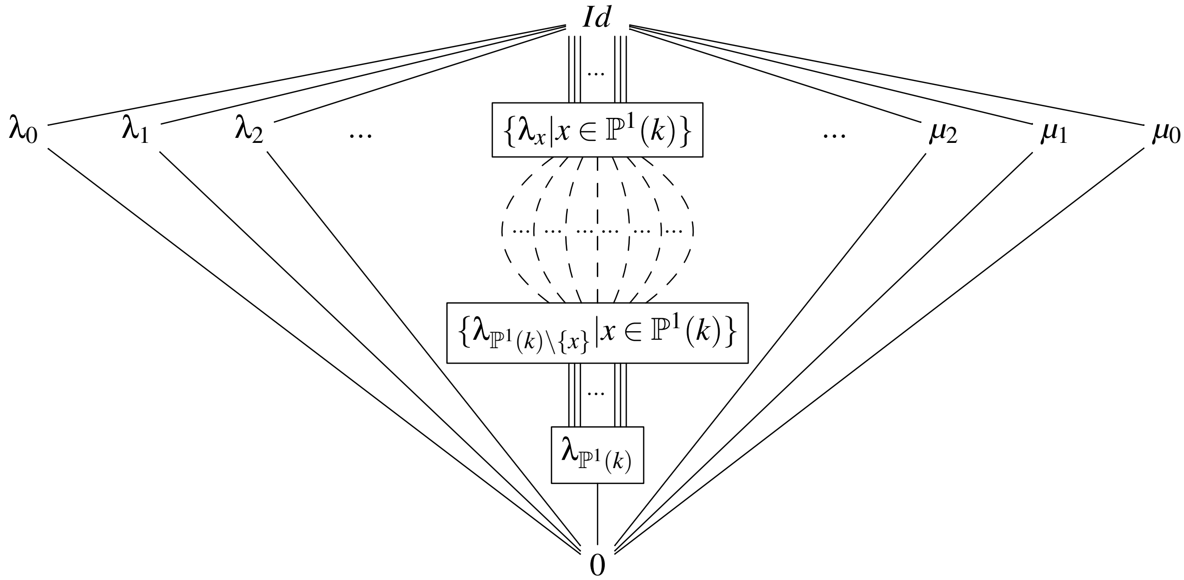


FIGURE 1. Lattice of homological ring epis of the Kronecker algebra

If $\mathcal{U} = \mathbb{P}^1(k)$, then $\Lambda_{\mathcal{U}}$ is Morita equivalent to $k(X)$, the ring of fractions, and the tilting module $T_{\mathcal{U}}$ is equivalent to $W := G \oplus \bigoplus_{\text{all } S} S_\infty$.

	homol. ring epi	birefl. subcat.	silting	cosilting
	$\Lambda \to 0$	0	0	0
	$\Lambda \to \Lambda$	$\mathrm{mod}\,\Lambda$	Λ	$D\Lambda$
	$\Lambda \overset{\mu_0}{\to} \Lambda/\tau_{P_0}(\Lambda)$	$P_0^{\perp_0} = \mathrm{Add}\,Q_0$	Q_0	(dual of fin. dim. module)
	$\Lambda \overset{\lambda_0}{\to} \Lambda/\tau_{P_1}(\Lambda)$	$P_1^{\perp_0} = \mathrm{Add}\,P_0$	P_0	(dual of fin. dim. module)
$i \geq 1$	$\Lambda \overset{\lambda_i}{\hookrightarrow} \Lambda_{P_{i+1}}$	$P_{i+1}^{\perp_{0,1}} = \mathrm{Add}\,P_i$	$P_i \oplus P_{i+1}$	(dual of fin. dim. module)
$i \geq 0$	$\Lambda \overset{\mu_{i+1}}{\hookrightarrow} \Lambda_{Q_i}$	$Q_i^{\perp_{0,1}} = \mathrm{Add}\,Q_{i+1}$	$Q_{i+1} \oplus Q_i$	(dual of fin. dim. module)
$\mathcal{U} \neq \emptyset$	$\Lambda \overset{\lambda_{\mathcal{U}}}{\hookrightarrow} \Lambda_{\mathcal{U}}$	$\mathcal{U}^{\perp_{0,1}}$	$T_{\mathcal{U}} := \Lambda_{\mathcal{U}} \oplus \Lambda_{\mathcal{U}}/\Lambda$	$DT_{\mathcal{U}} = G \oplus \oplus_{S \notin \mathcal{U}} S_\infty \oplus_{S \in \mathcal{U}} S_{-\infty}$
	–	–	L s.t. $\mathrm{Gen}\,L = \mathbf{p}^{\perp_1} = {}^{\perp_0}\mathbf{p}$	W

By the classification in [11], there is only one additional tilting module L, called the Lukas module, which is not minimal. This corresponds to the case $\mathcal{U} = \emptyset$.

We also recover the classification of cotilting modules due to [20]. Here $W = DL$ is the cotilting module corresponding to the case $\mathcal{U} = \emptyset$.

Finally we obtain a complete list of all definable torsion classes:

$0, \mathrm{Mod}\,\Lambda, \mathrm{Gen}\,Q_i (i \geq 0), \mathrm{Gen}\,P_i (i \geq 0), \mathrm{Gen}\,T_{\mathcal{U}} = \mathcal{U}^{\perp_1} = {}^{\perp_0}\mathcal{U} (\emptyset \neq \mathcal{U} \subset \mathbb{P}^1(k)), \mathrm{Gen}\,L.$

Notice that there are further torsion classes, for example $\mathrm{Add}\,\mathbf{q}$, which are not definable.

The lattice of ring epis is in Figure 1.

REFERENCES

[1] T. ADACHI, O. IYAMA, I. REITEN, *τ-tilting theory*, Compositio Math. **150** (2014), 415–452.

[2] T. AIHARA, O. IYAMA, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. **85** (2012), no. 3, 633–668.

[3] L. ALONSO TARRÍO, A. JEREMÍAS LOPÉZ, M. SOUTO SALORIO, *Constructions of t-structures and equivalences of derived categories*, Trans. Amer. Math. Soc. **355**, no. 6 (2003), 2523–2543.

[4] L. ANGELERI HÜGEL, Infinite dimensional tilting theory, Advances in Representation Theory of Algebras, Bielefeld 2012, EMS Series of Congress Reports (editors D. J. Benson, H. Krause, A. Skowronski) 2013, pp.1-37.

[5] L. ANGELERI HÜGEL, On the abundance of silting modules, preprint 2016.

[6] L. ANGELERI HÜGEL, M. HRBEK, Silting modules over commutative rings, to appear in International Mathematics Research Notices.

[7] L. ANGELERI HÜGEL, F. MARKS, J. VITÓRIA, *Silting modules.* Int. Math. Res. Not. IMRN, **4** (2016), 1251–1284.

[8] L. ANGELERI HÜGEL, F. MARKS, J. VITÓRIA, *Silting modules and ring epimorphisms*, Adv. Math.,**303**, 2016,1044–1076.

[9] L. ANGELERI HÜGEL, F. MARKS, J. VITÓRIA, Torsion pairs in silting theory, to appear in *Pacific Journal of Mathematics*.

[10] L. ANGELERI HÜGEL, F. MARKS, J. ŠŤOVÍČEK, R. TAKAHASHI, J. VITÓRIA, work in progress.

[11] L. ANGELERI HÜGEL, J. SÁNCHEZ, Tilting modules over tame hereditary algebras, *Journal für die Reine und Angewandte Mathematik* **682** (2013), 1–48.

[12] L. ANGELERI HÜGEL, A. TONOLO, J. TRLIFAJ, *Tilting preenvelopes and cotilting precovers*, Algebr. Represent. Theory **4** (2001), 155-170.

[13] I. ASSEM, M. SOUTO SALORIO, S. TREPODE, *Ext-projectives in suspended subcategories*, J. Pure Appl. Algebra **212** (2008), 423–434.

[14] S. BAZZONI, *Cotilting modules are pure-injective*, Proc. Amer. Math. Soc. **131** (2003): 3665– 3672.

[15] S. BAZZONI, D. HERBERA, *One dimensional tilting modules are of finite type,* Algebras and Repres. Theory **11** (2008), 43–61.

[16] A. BEILINSON, J. BERNSTEIN, P. DELIGNE, *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. **100** (1982), Soc. Math. France.

[17] M. BONDARKO, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory **6** (2010), no. 3, 387–504.

[18] S. BREAZ, F. POP, Cosilting modules, `http://arxiv.org/abs/1510.05098`.

[19] S. BREAZ, J. ŽEMLIČKA, Torsion classes generated by silting modules, `http://arxiv.org/abs/1601.06655`.

[20] Aslak Bakke Buan and Henning Krause. Cotilting modules over tame hereditary algebras. *Pacific J. Math.*, 211(1):41–59, 2003.

[21] P. EKLOF, J. TRLIFAJ, *How to make Ext vanish*, Bull. London Math. Soc. **33**, (2001), 41–51.

[22] M. HOSHINO, Y. KATO, J. MIYACHI, *On t-structures and torsion theories induced by compact objects*, J. Pure Appl. Algebra **167** (2002) 15–35.

[23] D. HAPPEL, I. REITEN, S. SMALØ, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. **120** (1996), no. 575, viii+ 88.

[24] O. IYAMA, P. JØRGENSEN, D. YANG, *Intermediate co-t-structures, two-term silting objects, τ-tilting modules, and torsion classes*, Algebra Number Theory **8** (2014), 2413–2431.

[25] G. JASSO, *Reduction of τ-tilting modules and torsion pairs*, Int. Math. Res. Not. IMRN (2014), doi: 10.1093/imrn/rnu163.

[26] B. KELLER, P. NICOLÁS, *Weight structures and simple dg modules for positive dg algebras*, Int. Math. Res. Not. IMRN (2013), no. 5, 1028–1078.

[27] B. KELLER, D. VOSSIECK, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A **40** (1988), no. 2, 239–253.

[28] S. KOENIG, D. YANG, *Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras*, Doc. Math. **19** (2014), 403–438.

[29] F. MARKS, Universal localisations and tilting modules for finite dimensional algebras, J. Pure Appl. Algebra **219** (2015), 3053-3088.

[30] F. MARKS, J. ŠŤOVÍČEK, Universal localisations via silting, `http://arxiv.org/abs/1605.04222`

[31] F. MARKS, J. VITÓRIA, Silting and cosilting classes in derived categories, `http://arxiv.org/abs/1704.06484`

[32] O. Mendoza Hernández, E. Sáenz Valadez, V.Santiago Vargas, M.Souto Salorio, *Auslander-Buchweitz context and co-t-structures*, Appl. Categ. Structures **21**, no.5 (2013), 417-440.
[33] P. Nicolás, M. Saorin and A. Zvonareva, *Silting theory in triangulated categories with coproducts*, http://arxiv.org/abs/1512.04700.
[34] F. Okoh, *Indecomposable pure-injective modules over hereditary artin algebras of tame type*, Comm. Algebra. **8** (1980), 1939-1941.
[35] D. Pauksztello, *Compact corigid objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. **6** (2008), no. 1, 25–42.
[36] M. Prest, Model theory and modules, London Math. Soc. Lecture Notes Ser. **130**, Cambridge University Press 1988.
[37] M. Prest, *Purity, spectra and localisation*, Cambridge University Press 2009.
[38] C. Psaroudakis, J. Vitoria, Realisation functors in Tilting Theory. http://arxiv.org/abs/1511.02677.
[39] A. H. Schofield, *Representations of rings over skew fields*, Cambridge University Press (1985).
[40] J. Wei, *Semi-tilting complexes*, Israel J. Math. **194**, no. 2 (2013), 871–893.
[41] P. Zhang, J. Wei, Cosilting complexes and AIR-cotilting modules, http://arxiv.org/abs/1601.01385.

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