Introduction
to
Ordinary Differential Equations

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Outline

- Some examples to start
- Basic terminology for differential equations
- Existence and uniqueness of solutions of differential equations
- Linear O.D.E’s (Ordinary Differential Equations) of arbitrary order
  - Solution of linear O.D.E’s with method of characteristic polynomials
  - A model for the detection of diabetes
  - Solution method: Variation of constants
  - Solution method: Laplace transform

Referring to:
Miller, R.K., *Introduction to Differential Equations*, Prentice-Hall,New Jersey, 1987
Boyce, W.E., DiPrima, R.C., *Gewöhnliche Differentialgleichungen*, Spektrum Verlag, Heidelberg, 1995

1 Motivation

In physics, biology, medicine and most other sciences we want to create models of the world using mathematics. Often these models describe rates of change.

Example 1. A function $x(t)$ with constant derivative

$$\frac{dx(t)}{dt} = x'(t) = k, \quad k = const$$

Thus we get

$$x(t) = k \cdot t + d$$

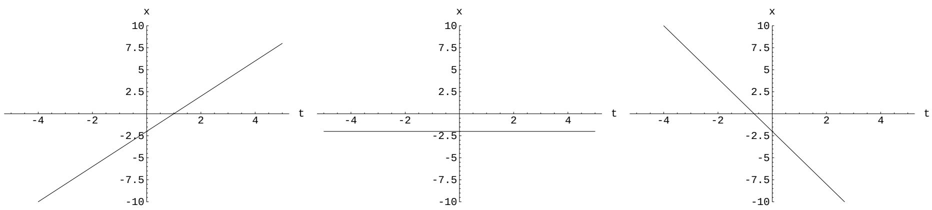

Figure 1: left: x(t)=2t-2, middle: x(t)=0t-2, right: x(t)=-3x-2

Example 2. Assume A and B are two chemical compounds. A transforms into B and vice versa.
The chemical reaction is:

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B.$$

Letting $[A]$ denote concentration of A, the differential equation for this interaction is:

$$\frac{d[A](t)}{dT} = -k_1[A](t) + k_{-1}[B](t)$$

$$\text{with } [A](t) + [B](t) = M$$

From this we conclude that

$$\begin{cases} \dfrac{d[A](t)}{dt} = -(k_1 + k_{-1})[A](t) + k_{-1}M \\ \text{assuming initial condition } [A](0) = [A]_0 \end{cases}$$

With a bit of math we get the solution:

$$\frac{d[A](t)}{dt} = -\alpha[A](t) + \beta, \quad \alpha = k_1 + k_{-1}, \ \beta = k_{-1}M$$
$$\frac{d[A](t)}{-\alpha[A](t) + \beta} = dt$$
$$\frac{\ln(-\alpha[A](t) + \beta)}{\alpha} - \frac{\ln(-\alpha[A](0) + \beta)}{\alpha} = t - 0$$
$$\ln\left(\frac{-\alpha[A](t) + \beta}{-\alpha[A]_0 + \beta}\right) = \alpha t$$
$$-\alpha[A](t) + \beta = (-\alpha[A]_0 + \beta)e^{\alpha t}$$
$$[A](t) = \frac{\beta}{\alpha} + \left([A]_0 - \frac{\beta}{\alpha}\right)e^{-\alpha t}, \quad \alpha = k_1 + k_{-1}, \ \beta = k_{-1}M.$$

Or with substitution:

$$\begin{aligned}
\frac{dx(t)}{dt} &= -\alpha x(t) + \beta \text{ substituting } y(t) = x(t) - \frac{\beta}{\alpha} \\
\frac{dy(t)}{dt} &= \frac{dx(t)}{dt} = -\alpha y(t) \\
y(t) &= y_0 e^{-\alpha t} \text{resubstituting} \\
x(t) &= \frac{\beta}{\alpha} + \left(x_0 - \frac{\beta}{\alpha}\right) e^{-\alpha t}
\end{aligned}$$

Looking at this equation we can see the long run behavior, which depends on the parameters.

There is an equilibrium point at $[A](t) = \frac{\beta}{\alpha}$.
For all initial values $[A]_0$ the solutions approach the equilibrium point.

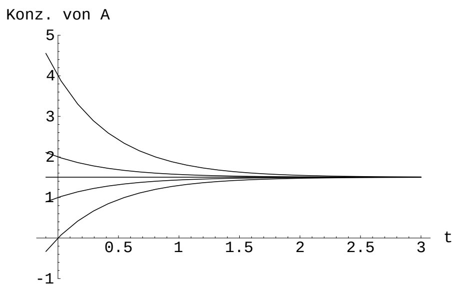

Figure 2: [A](t) with $\alpha = 2$, $\beta = 3$, different initial values

2 Terminology

Definition 1. Ordinary differential equations contain only functions and derivatives of a single variable. For example:

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + \sin(x(t)) = 0$$

In general

$$x^{(n)}(t) = f(t, x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t))$$

with

$$x^{(j)}(t) := \frac{d^j x(t)}{dt^j}$$

Definition 2. Partial differential equations contain the derivatives of a function of two or more variables

$$\frac{\partial x(t,s)}{\partial t} + \frac{\partial x(t,s)}{\partial s} \sin(x(t,s)) + x(t,s) = 0$$

Definition 3. The **order** of a differential equation is the order of the highest derivative contained in the equation.

Example 3.

(a)	$x'''(t) = 3\frac{1}{t-2}x'(t) - b(t)x(t)$	third order
(b)	$x''(t) = \sqrt{2}x(t)^3 + x'(t)$	second order

Definition 4. An **initial value problem** consists of a differential equation and **initial conditions.**

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t)) \\ x(t_0) = a_0,\ x'(t_0) = a_1, \ldots,\ x^{(n-1)}(t_0) = a_{n-1} \end{cases}$$

A differential equation may have no solution or even more than one solution, so it is interesting under what constraints a solution exists and is unique.

3 Existence and uniqueness of solutions

Theorem (Existence and Uniqueness). *Given the initial value problem*

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), x''(t), \dots, x^{(n-1)}(t)) \\ x(t_0) = a_0,\ x'(t_0) = a_1, \dots,\ x^{(n-1)}(t_0) = a_{n-1} \end{cases}$$

Suppose that f is continuous and has continuous partial derivatives

$$\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x(t)}, \frac{\partial f}{\partial x'(t)}, \dots, \frac{\partial f}{\partial x^{(n-1)}(t)}$$

in a certain region D and the initial values lie in this region.
$\Longrightarrow$ *Then there exists one and only one solution $x(t)$ that satisfies the given initial conditions.*

For existence local integrability is enough, for uniqueness we need at least Lipschitz-continuity.

Definition 5. A **well posed problem** has the following characteristics:

- There exists a solution of the differential equation
- The solution is unique
- There is continuous dependence on the initial conditions

Example 4.

$$\begin{aligned} &\text{(a)} \quad x''(t) &=& \quad 3\frac{1}{t-2}x'(t) - b(t)x(t) \\ &\text{(b)} \quad x'(t) &=& \quad 3x(t)^{2/3} \text{ with } x(0) = 0 \\ &\text{(c)} \quad x''(t) &=& \quad 3cx'(t) - x(t) + 2c^2, \quad c \text{ is constant} \end{aligned}$$

4 Linear equations of arbitrary order

Definition 6. A linear equation of nth order can be written in the form

$$x^{(n)}(t) + b_{n-1}(t)x^{(n-1)}(t) + \ldots + b_1(t)x'(t) + b_0(t)x(t) = g(t)$$

With the b_j's continuous on the basic Interval $J = [a, b]$, the uniqueness theorem implies that there is one and only one solution that satisfies the n initial conditions

$$x(t_0) = c_0,\ x'(t_0) = c_1, \dots,\ x^{(n-1)}(t_0) = c_{n-1} \text{ with } t_0 \text{ in } J$$

A general solution contains n arbitrary constants, which will be specified by the initial conditions.

Example 5.

$$\begin{aligned} \text{(a)} \quad x''(t) &= 3\frac{1}{t-2}x'(t) - b(t)x(t) \\ \text{(b)} \quad x'(t) &= 3x(t)^{2/3} \text{ with } x(0) = 0 \\ \text{(c)} \quad x''(t) &= 3cx'(t) - x(t) + 2c^2, \quad c \text{ is constant} \end{aligned}$$

(a) and (c) are linear, (b) is non-linear.

4.1 Homogeneous and non-homogeneous equations

Definition 7. We call an equation **homogeneous** if it can be written

$$\begin{aligned} x^{(n)}(t) &+ b_{n-1}(t)x^{(n-1)}(t) + \ldots \\ &+ b_1(t)x'(t) + a_0(t)x(t) = \mathbf{0} \end{aligned}$$

Definition 8. A **non-homogeneous** equation is

$$\begin{aligned} x^{(n)}(t) &+ b_{n-1}(t)x^{(n-1)}(t) + \ldots \\ &+ b_1(t)x'(t) + a_0(t)x(t) = \mathbf{g}(\mathbf{t}) \end{aligned}$$

with $g(t) \neq 0$ for at least one value of t.

Example 6.

$$\begin{aligned} \text{(a)} \quad x''(t) &= 3x'(t) - e^{t^2-2}x(t) \\ \text{(b)} \quad x''(t) &= 3x'(t) - x(t) + 2e^{2t} \end{aligned}$$

(a) is linear homogeneous, (b) is linear non-homogeneous.

4.2 The differential operator

Let

$$D^i := \frac{d^i}{dt^i}$$

be short for an operator to take the ith derivative, then we can write the non-homogeneous equation as

$$L(x(t)) = g(t) \text{ with } L = D^n + b_{n-1}(t)D^{n-1} + \ldots + b_1(t)D + b_0(t)$$

Definition 9. This operator is called a **linear operator**, because

$$L(x(t) + y(t)) = L(x(t)) + L(y(t))$$

$$L(cx(t)) = cL(x(t)), \quad c \text{ is real}$$

Note: If $x_1(t), \ldots,\ x_k(t)$ are solutions of $L(x(t)) = 0$ and if $c_1, \ldots,\ c_k$ are any real constants, then, since L is a linear operator,

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + \ldots + c_k x_k(t)$$

is also a solution of $L(x(t)) = 0$.

Example 7. Thus we can write

$$\text{(a)} \quad x''(t) = 3\frac{b(t)}{t-2}x'(t) - e^{t^2-2}x(t)$$

$$L(x(t)) = 0 \quad \text{with } L = D^2 - 3\frac{b(t)}{t-2}D - e^{t^2-2}$$

4.3 How to solve homogeneous linear equations with constant coefficients?

Consider a second-order linear differential equation, written as

$$L(x(t)) = (a\frac{d^2}{dt^2} + b\frac{d}{dt} + c)x(t) = (aD^2 + bD + c)x(t) = 0$$

Now we guess, that $x(t) = e^{\lambda t}$ is a solution, then note that

$$aD^2(e^{\lambda t}) = a\frac{d^2}{dt^2}(e^{\lambda t}) = a\lambda^2 e^{\lambda t}.$$

So that

$$L(e^{\lambda t}) = (a\lambda^2 + b\lambda + c)e^{\lambda t} = 0$$

Because $e^{\lambda t}$ is never zero, we must have the so-called **characteristic equation** equal to zero.

Definition 10.

$$p(\lambda) = a\lambda^2 + b\lambda + c = 0$$

with $p(\lambda)$ is called the **characteristic polynomial**. The roots of $p(\lambda)$ are called the **characteristic roots** or **eigenvalues**.

There are three possibilities for the eigenvalues:

- The roots of $p(\lambda)$ are real and not equal
  $\Rightarrow$ general solution is $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.
- Roots are real and equal
  $\Rightarrow$ general solution is $x(t) = c_1 t e^{\lambda t} + c_2 e^{\lambda t}$.

- If the roots are complex, then $\lambda_1 = u + iv = \overline{\lambda}_2$
  We get
  $$x_1(t) = e^{\lambda_1 t},\ x_2(t) = e^{\lambda_2 t}$$
  which have complex values. To get real-valued functions we define
  $$x_3(t) = \frac{x_1(t) + x_2(t)}{2} \text{ and } x_4(t) = \frac{x_1(t) - x_2(t)}{2i}$$
  then we get
  $$x_3(t) = e^{ut}\cos vt \text{ and } x_4(t) = e^{ut}\sin vt$$
  the solution is given by:
  $$\begin{aligned} x(t) &= c_1 e^{ut}\cos vt + c_2 e^{ut}\sin vt \\ &= c_3 e^{ut}\cos(vt - c_4) \end{aligned}$$

Note: In general, if L has n derivatives, then the **characteristic polynomial** associated with

$$L = D^n + b_{n-1}(t)D^{n-1} + \ldots + b_1(t)D + b_0(t)$$

is

$$p(\lambda) = \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0$$

Thus if the characteristic equation $p(\lambda) = 0$ has n real distinct roots then the set $\{x_1(t) = e^{\lambda_1 t}, x_2(t) = e^{\lambda_2 t}, \ldots,\ x_n(t) = e^{\lambda_n t}\}$ is a fundamental set of solutions. Can there be other solutions not involving the form $x(t) = e^{\lambda t}$? It turns out that for a finite system of linear O.D.E.'s the answer is no.

4.4 A model for detection of diabetes

The following two examples follow:
Braun, M., *Differential Equations and Their Applications*, Springer-Verlag, New York, 1993

Example 8. Ackerman described that the diagnoses depends on the overall performance of the blood glucose regulatory system, which is dominated by insulin-glucose interactions. Thus we have

$$\begin{aligned}\frac{dg(t)}{dt} &= F_1(g(t), h(t)) + j(t)\\ \frac{dh(t)}{dt} &= F_2(g(t), h(t))\end{aligned}$$

with $g(t)$ = deviation of glucose concentration, $h(t)$ = deviation of hormonal concentration, $j(t)$ = introduced glucose.
We use a model, where the glucose is not slowly external increased. We assume that it is increased at once and consider the injection in our initial conditions.

- A rise in the concentration of glucose in the bloodstream results in the liver absorbing more of the glucose, which it ocnverts and stores as glycogen; a drop in the concentration reverses the process.
- A rise in the concentration of insulin in the bloodstream enables the glucose to pass more readily through the membranes of the skeletal muscle, resulting in a greater absorption of glucose from the bloodstream.
- A rise in the concentration of glucose in the bloodstream stimulates pancreas to produce insulin at a faster rate; a drop in the glucose concentration lowers the rate of insulin production.
- Insulin, produced by the pancreas, is constantly being degraded by the liver.

Assuming the relationships are linear we get to

$$\begin{aligned}\frac{dg(t)}{dt} &= -m_1 g(t) - m_2 h(t)\\ \frac{dh(t)}{dt} &= +m_3 g(t) - m_4 h(t) \quad \text{with } m_i > 0\end{aligned}$$

By transform to second order equation...

$$\begin{aligned}\frac{d^2g(t)}{dt^2} &= -m_1\frac{dg(t)}{dt} - m_2\frac{dh(t)}{dt} \\ &= -m_1\frac{dg(t)}{dt} - m_2m_3g(t) + m_2m_4h(t) \\ &= -m_1\frac{dg(t)}{dt} - m_2m_3g(t) - m_4\frac{dg(t)}{dt} - m_4m_1g(t) \\ &= -(m_1+m_4)\frac{dg(t)}{dt} - (m_2m_3 + m_1m_4)g(t)\end{aligned}$$

... and simplifying with $2\alpha = m_1 + m_4$ and $\omega_0^2 = m_2m_3 + m_1m_4$ it yields

$$\frac{d^2g(t)}{dt^2} + 2\alpha\frac{dg(t)}{dt} + \omega_0^2 g = 0$$

and solution of the characteristic equation

$$\lambda^2 + 2\alpha\lambda + \omega_0^2 = 0$$

$$\begin{aligned}\lambda &= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \\ &= -\alpha \pm i\omega \quad \text{assuming } \omega = \omega_0^2 - \alpha^2 > 0.\end{aligned}$$

Thus we have complex eigenvalues and get a solution of the form

$$g(t) = Ae^{-\alpha t}\cos(\omega t - \delta)$$

Does this characterize all possible solutions? Yes, as we have noted before. Let $G_f$be the fasting steady state level of glucose. Then:

$$G(t) = G_f + Ae^{-\alpha t}\cos(\omega t - \delta)$$

The initial value G_0 is the injected glucose level.

$$G_0 = G(0) = G_f + A\cos(-\delta)$$

The glucose tolerance test (GTT): The patient comes to the hospital after an overnight fast and his glucose concentration is measured. During the next three to five hours at least five more measurements are made.
We try to find the parameters of above equation that the least square error is minimized. In various experiments Ackerman noticed, that the natural frequency ω_0 can be considered for detecting diabetes. Because if

$$T = \frac{2\pi}{\omega_0} > 4 \text{ hours}$$

the patient has at least mild diabetes.

Example 9. Consider the following data from two GTTs. After overnight fast the glucose concentration was 70 (75) mg glucose per 100 ml blood. Then for both patients, glucose concentrations were taken after short injection of glucose.

t	1	2	3	3.5	4	4.5	5	hours
$p1$	81	67	66	69	71	72	72	mg glucose per 100 ml blood
$p2$	85	76	73	73	74	75	75	-"-

We want to use the model we derived in the last example. Thus we have to find the parameters of the function which fit best to our given data. This is done by the method of least squares.
Thus we get the parameters

	A	α	δ	ω_0
$p1$	19.545	0.487	1.150	1.654
$p2$	20.253	0.694	1.067	1.401

$$T = \frac{2\pi}{\omega_0}$$

is the important parameter for the diagnose

$$T(p1) = 3.8188 \text{ (normal)} \quad T(p2) = 4.4833 \text{ mild diabetes}$$

We can draw the function

$$G(t) = G_f + Ae^{-\alpha t}\cos(\omega t - \delta)$$

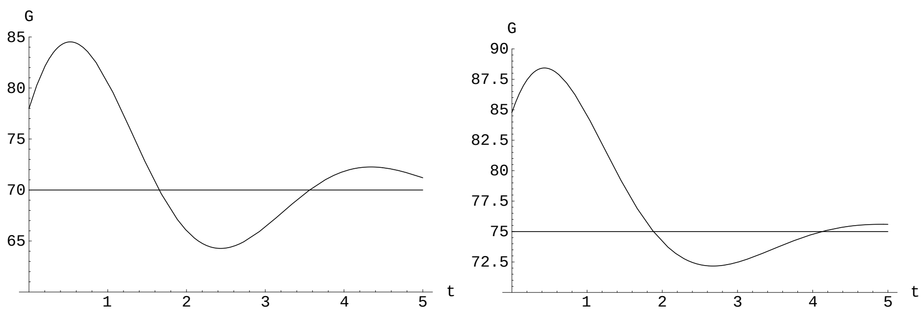

Figure 3: left: normal curve, right: patient with diabetes

4.5 Variation of constants

If we have an non-homogeneous equation or one with variable coefficients, i.e.

$$x'(t) + b(t)x(t) = g(t),$$

and assuming we know the homogeneous solution $x(t) = cx_1(t)$, we replace the constant c by an unknown function $c(t)$ and take $x(t) = c(t)x_1(t)$ as a trial solution. In general

$$x_1(t) = e^{-\int b(t)dt}.$$

Thus

$$\begin{aligned} x'(t) &= c'(t)x_1(t) + c(t)x_1'(t) \\ &= c'(t)x_1(t) + c(t)(-b(t)x_1(t)) \\ &= -b(t)(c(t)x_1(t)) + g(t) \quad \text{cause } c(t)x_1(t) \text{ is a solution} \end{aligned}$$

Comparing last two equations yields

$$c'(t) = \frac{g(t)}{x_1(t)} = g(t)e^{\int b(t)dt}$$

$$\begin{aligned} c(t) &= \int g(t)e^{\int b(t)dt}dt + c_1 \\ &\Rightarrow x(t) = c(t)x_1(t) = x_p(t) + x_c(t) \end{aligned}$$

$x_p(t)$ = particular solution, $x_c(t)$ = corresponding homogeneous solution. This method of solution is known as **variation of constants**.

The advantage of **variation of constants:**

- large class of non-homogeneous terms $g(t)$
- solving equations with non-constant coefficients.

4.6 The Laplace-transform

Laplace transform advantages:

- Transformed solutions are found algebraically without referring to the characteristic equation.
- the easiest method of solution for many non-homogeneous discontinuous or impulsive problems

Definitions and basic properties

Definition 11. Let $f(t)$ be a given real-valued function defined on the Interval $0 \leq t < \infty$. The **Laplace transform** of $f(t)$, denoted by $L\{f\} = F$, is defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t) dt$$

We assume now, that s is a real number (in general it can be complex). If $F(s)$ exists at some s_0 it can be shown, that $F(s)$ also exists for all real numbers $s > s_0$.

The Laplace transform of $f(t)$ is:

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t) dt$$

Example 10. Let $f(t) = 1$ for all $t \geq 0$. Then for $s > 0$

$$F(s) = \int_0^{\infty} e^{-st} f(t) = \lim_{T \to \infty} \int_o^T e^{-st} dt = \lim_{T \to \infty} \left[\frac{e^{-st}}{-s} \right]_0^T = \frac{1}{s}$$

Hence, $L\{1\} = 1/s$ exists for all $s > 0$.

Example 11. Let $f(t) = e^{at}$ for some real number a. Then for $s > a$

$$F(s) = \int_0^{\infty} e^{-st} e^{at} = \lim_{T \to \infty} \int_o^T e^{-(s-a)t} dt = \frac{1}{s-a}$$

Hence, $L\{e^{at}\} = (s-a)^{-1}$ exists for all $s > a$.

Example 12. Analogous you can get $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Definition 12. A **function** defined on $[0, \infty)$ is said to be **of exponential order** if there are real constants $M \geq 0$ and a such that $|f(t)| \leq Me^{at}$ for all $t \geq 0$.

Theorem. *If f is of exponential order and at least piecewise continuous over $[0, T]$, $T > 0$ then the Laplace transform $L\{f\} = F(s)$ exists for all $s > a$. Moreover $|F(s)| \leq M(s-a)^{-1}$ for all $s > a$.*

Example 13. The unit step function is defined by

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

We see clearly that $u_c(t)$ is of exponential order, thus we can use Laplace transform

$$
\begin{aligned}
L\{u_c\} &= \int_0^\infty e^{-st}u_c(t)\,dt = \int_0^c e^{-st}\cdot 0\,dt + \int_c^\infty e^{-st}\cdot 1\,dt \\
&= \lim_{T\to\infty}\left[\frac{e^{-st}}{-s}\right]_c^T = \frac{e^{-sc}}{s} \qquad \text{for any } s>0
\end{aligned}
$$

4.6.1 Important rules

Important calculation rules for the **Laplace operator** (with c_1, c_2 real, f_1, f_2 of exponential order):

1. **Linearity:**
$$L\{c_1f_1 + c_2f_2\} = c_1L\{f_1\} + c_2L\{f_2\}$$

2. **Convolution:**
$$L\{\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\} = L\{f_1(t)\} * L\{f_2(t)\}$$

3. **Integration:**
$$L\{\int_0^t f(\tau)\,d\tau\} = \frac{1}{s}L\{f\}$$

4. **Differentiation:** There exists a real b depending on f, that for $s > b$
$$
\begin{aligned}
L\{\frac{d^n}{dt^n}f(t)\} &= s^nF(s) - s^{n-1}f_0 - \ldots - sf_0^{(n-2)} - f_0^{(n-1)} \\
\text{with } f_0^{(v)} &= \lim_{t\to 0+}\frac{d^vf(t)}{dt^v}
\end{aligned}
$$

5. **Shifting:**
$$L\{f(t-b)\} = e^{-bs}L\{f(t)\}$$

6. **Similarity:**
$$L\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \text{ with } a>0$$

7. **Damping:**
$$L\{e^{-\alpha t}f(t)\} = F(s+\alpha)$$

8. **Multiplicity:**
$$L\{t^nf(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

9. **Division:**

$$L\{\frac{1}{t}f(t)\} = \int_s^\infty F(q)\,dq$$

Example 14. Let $f(t) = t^\alpha$ then (with $r = st$)

$$\begin{aligned} L\{t^\alpha\} &= \int_0^\infty e^{-st}t^\alpha\,dt = \int_0^\infty e^{-r}\left(\frac{r}{s}\right)^\alpha \frac{dr}{s} \\ &= \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-r}r^\alpha\,dr = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \end{aligned}$$

The function *Gamma* is the **gamma function** with $\Gamma(v+1) = v\Gamma(v)$ for $v > 0$ and with $\Gamma(1) = 1$. Thus $\Gamma(n+1) = n!$ if n is a positive integer.

4.6.2 Solution of initial value problems

Example 15. Let's consider the insulin problem again

$$\begin{gathered} \frac{d^2g(t)}{dt^2} + 2\alpha\frac{dg(t)}{dt} + \omega_0^2 g = 0 \\ g(0) = g_0, \quad g'(0) = -m_1 g_0 \end{gathered}$$

If we take Laplace transform $G(s)$ of $g(t)$ on each side of this equation, using the rules described above we find that:

$$(s^2G(s) - sg_0 + m_1g_0) + 2\alpha(sG(s) - g_0) + \omega_0^2 G(s) = 0,$$

so that

$$(s^2 + 2\alpha s + \omega_0^2)G(s) = (s - 2\alpha + m_1)g_0$$

and

$$G(s) = \frac{(s - 2\alpha + m_1)g_0}{s^2 + 2\alpha s + \omega_0^2}.$$

If we recognize the right side of this equation as the Laplace of a known function, then we can compute the solution $g(t)$.

We use the method of partial fractions
with

$$\omega^2 = \omega_0^2 - \alpha^2$$

and

$$\frac{s - c}{(s+a)(s+b)} = \frac{\frac{a+c}{a-b}}{s+a} - \frac{\frac{b+c}{a-b}}{s+b}$$

we have

$$\begin{aligned}
\frac{(s-2\alpha+m_1)g_0}{s^2+2\alpha s+\omega_0^2} &= \frac{(s-2\alpha+m_1)g_0}{(s+2\alpha s+\alpha^2)+(\omega_0^2-\alpha^2)} \\
&= g_0\frac{s-2\alpha+m_1}{(s+\alpha)^2+\omega^2} \\
&= g_0\frac{s-2\alpha+m_1}{(s+\alpha+i\omega)(s+\alpha-i\omega)} \\
&= \frac{g_0}{2i\omega}\left[\frac{3\alpha+i\omega-m_1}{s+\alpha+i\omega}-\frac{3\alpha-i\omega-m_1}{s+\alpha-i\omega}\right]
\end{aligned}$$

Applying the previous rules for Laplace transform in reverse we can find the **inverse Laplace transform** $L^{-1}\{G(s)\}$ to recover $g(t)$.
With the knowledge that

$$L\{ce^{-at}\} = \frac{c}{s+a}$$

we find the searched inverse transform

$$\begin{aligned}
g(t) &= \frac{g_0}{2i\omega}\left[(3\alpha+i\omega-m_1)e^{(-\alpha-i\omega)t}-(3\alpha-i\omega-m_1)e^{(-\alpha+i\omega)t}\right] \\
&= \frac{g_0}{2i\omega}e^{-\alpha t}\left[(3\alpha-m_1)\left(e^{-i\omega t}-e^{i\omega t}\right)+i\omega\left(e^{-i\omega t}+e^{i\omega t}\right)\right] \\
&= g_0e^{-\alpha t}\left[\frac{m_1-3\alpha}{\omega}\sin\omega t+\cos\omega t\right].
\end{aligned}$$

which is

$$g(t) = Ae^{-\alpha t}\cos(\omega t-\delta)$$

with

$$\delta = \arctan\left(\frac{m_1-3\alpha}{\omega}\right), \quad A = \frac{g_0}{\cos\delta}$$

the same solution as before.

Notice that the initial values are automatically used when one Laplace transforms the differential equation.

To justify the inversion of the Laplace transform we have the following result:

Theorem. *If $x_1(t)$ and $x_2(t)$ are two functions of exponential order, both continuous and their Laplace transforms equal on an interval $s_0 < s < \infty$, then $x_1(t) = x_2(t)$ for all $t \geq 0$. This is not true for only piecewise-continuous functions! If they are piecewise-continuous, then $x_1(t) = x_2(t)$ on $0 \leq t < \infty$* except *on a set $\{t_n\}$ of isolated points.*

4.6.3 Laplace-transform tabulars

$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{1}{s}$	1	$\frac{1}{s+\alpha}$	$e^{-\alpha t}$
$\frac{1}{s^2}$	t	$\frac{1}{(s+\alpha)(s+\beta)}$	$\frac{1}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t})$
$\frac{1}{s^2+\alpha^2}$	$\frac{1}{\alpha}\sin(\alpha t)$	$\frac{s}{s^2+\alpha^2}$	$\cos(\alpha t)$
$\frac{1}{(s+\beta)^2+\alpha^2}$	$\frac{1}{\alpha}e^{-\beta t}\sin(\alpha t)$	$\frac{s}{(s+\beta)^2+\alpha^2}$	$e^{-\beta t}\left(\cos(\alpha t) - \frac{\beta}{\alpha}\sin(\alpha t)\right)$
$\frac{1}{s^n}$	$\frac{1}{(n-1)!}t^{n-1}$	$\frac{1}{(s+\alpha)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-\alpha t}$

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\ln\frac{s-\alpha}{s-\beta}$	$\frac{1}{t}(e^{\beta t} - e^{\alpha t})$	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
$\sqrt{\frac{\sqrt{s^2+\alpha^2}+s}{s^2+\alpha^2}}$	$\sqrt{\frac{2}{\pi t}}\cos(\alpha t)$	$\sqrt{\frac{\sqrt{s^2-\alpha^2}+s}{s^2-\alpha^2}}$	$\sqrt{\frac{2}{\pi t}}\cos(\alpha t)$
$\arctan\frac{\alpha}{s}$	$\frac{\sin(\alpha t)}{t}$		

$[t]$ =the biggest natural number n with $n \leq t$.

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{1}{s(e^{\alpha s}-1)}$	$\left[\frac{t}{\alpha}\right]$	$\frac{1}{s(1-e^{-\alpha s})}$	$\left[\frac{t}{\alpha}\right] + 1$

$F(s)$	$f(t)$
$\frac{e^{-\alpha s}-e^{-\beta s}}{s}$	$\begin{cases} 0 & \text{für} \quad 0 < t < \alpha \\ 1 & \text{für} \quad \alpha < t < \beta \\ 0 & \text{für} \quad \beta < t \end{cases}$
$\frac{\left(e^{-\alpha s}-e^{-\beta s}\right)^2}{s^2}$	$\begin{cases} 0 & \text{für} \quad 0 < t < 2\alpha \\ t-2\alpha & \text{für} \quad 2\alpha < t < \alpha+\beta \\ 2\beta - t & \text{für} \quad \alpha+\beta < t < 2\beta \\ 0 & \text{für} \quad 2\beta < t \end{cases}$
$\frac{e^{-\alpha s}}{s+\beta}$	$\begin{cases} 0 & \text{für} \quad 0 < t < \alpha \\ e^{-\beta(t-\alpha)} & \text{für} \quad \alpha < t \end{cases}$
$\frac{1}{s(1+e^{-\alpha s})}$	$\begin{cases} 1 & \text{für} \quad 2n\alpha < t < (2n+1)\alpha \\ 0 & \text{für} \quad (2n+1)\alpha < t < (2n+2)\alpha \end{cases}$ $n = 0, 1, 2, \ldots$