Introduction
to
Mathematical Modeling

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<u>Outline</u>

- Basic terminology for mathematical modeling
- What are differential equations and why use them?
- What is a solution of a differential equation ?
- Existence and uniqueness of solutions of differential equations
- Linear versus Non Linear systems

Reference:
Miller, R.K., *Introduction to Differential Equations*, Prentice-Hall,New Jersey, 1987

Motivation

In medicine and most other sciences we use mathematics to model systems. **Functions** describe how one quantity depends on other quantities in a causal way. **Ordinary derivatives** describe the rate at which one quantity will vary as a second quantity is varied. Many models involve derivatives because:

- Rates of change are often easier to observe and measure.
- Many physical situations involve the interaction of forces which influence and generate rates of change.

Example 1. If a function $x(t)$ has constant derivative

$$\frac{dx(t)}{dt} = x'(t) = k, \quad k = const$$

we get

$$x(t) = k \cdot t + d$$

Terminology

Definition 1. Ordinary differential equations contain only functions and derivatives of a single variable. For example:

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + \sin(x(t)) = 0$$

In general

$$x^{(n)}(t) = f(t, x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t))$$

with

$$x^{(j)}(t) := \frac{d^jx(t)}{dt^j}$$

Definition 2. Partial differential equations contain the derivatives of a function of two or more variables

$$\frac{\partial x(t,s)}{\partial t} + \frac{\partial x(t,s)}{\partial s} \sin(x(t,s)) + x(t,s) = 0$$

Definition 3. An **initial value problem** consists of a differential equation and **initial conditions.**

$$\begin{cases} x^{(')}(t) = f(t, x(t)) \\ x(t_0) = a_0, \end{cases}$$

Definition 4. A **solution** $Y(t)$ to a differential equation is a function (formula) which satifies the conditons of the system described by the differential equation.

To test if a function $Y(t)$ is a solution you plug the function into the differential equation and see if both sides of the equation are equal.

Example 2. $Y(t) = sin(t)$ is a solution of:

$$\begin{cases} x^{('')}(t) = -x(t) \\ x(0) = 0 \end{cases}$$

since $sin(t)'' = -sin(t)$ and $sin(0) = 0$.

A differential equation may have no solution or even more than one solution, so it is interesting under what constraints a solution exists and is unique.

Definition 5. A **well posed problem** has the following characteristics:

- There exists a solution of the differential equation
- The solution is unique
- There is continuous dependence on the initial conditions

What do these requirements mean in practical terms?

1. Clearly existence of a solution is required if the model describes a physical reality. Numerical approximations would be meaningless if there is no underlying solution to approximate.
2. The solution is unique means that the model has predictive ability. We expect that a physical model will behave deterministically (unless probability is involved in the process).
3. Continuous dependence on the initial conditions says that small changes pt discontinous alterations in the system. This is also important for iuseable numerical approximation

Definition 6. An operator is called a **linear operator**, if

$$L(x(t) + y(t)) = L(x(t)) + L(y(t))$$

$$L(cx(t)) = cL(x(t)), \quad c \text{ is real}$$

Note: If $x_1(t), \ldots, x_k(t)$ are solutions of $L(x(t)) = 0$ and if $c_1, \ldots, c_k$ are any real constants, then, since L is a linear operator,

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + \ldots + c_k x_k(t)$$

is also a solution of $L(x(t)) = 0$. Why is this condition important?

- Sometimes called the superposition property, it says that complicated solutions can be analyzed by breaking down into simpler solutions
- Also, in some sense outputs of a system are proportional to inputs.