Introduction
to
Linear Control Theory

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Useful References

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1 State Space representation of dynamical systems

1.1 Introduction

The **state** of a system at time t_0 is the smallest amount of information about the system at t_0, which together with the input $u(t)$ for $t \in (t_0, \infty)$, determines uniquely the behavior of the system for all $t \geq t_0$.

The set of equations describing the unique relation between input, output, and state is called a **dynamical system**. The set of equations consist of two types. Equations which model the state (**state equations**) and equations which model the measurement of the state (**output equations**):

$$\begin{cases} \mathbf{x}'(t) = f(\mathbf{x}(t), \mathbf{u}(t), t) & \text{(state equation)} \\ \mathbf{y}(t) = g(\mathbf{x}(t), \mathbf{u}(t), t) & \text{(output equation)} \end{cases} \tag{1}$$

where $\mathbf{x}(t) \in R^n, \mathbf{u}(t) \in R^m$, and $\mathbf{y}(t) \in R^p$. Note $f(\mathbf{x}(t), \mathbf{u}(t), t)$ consists of component functions f_i for $i, = 1, 2, ...n$ and $g(\mathbf{x}(t), \mathbf{u}(t), t)$ has components g_i for $i, = 1, 2, ...p$.

Definition 1. To be considered a dynamical system we must have that for each initial condition $\mathbf{x}(t_0)$ and any given $\mathbf{u}(t)$ the above system has a unique solution.

The following theorem is important.

Theorem 1. *To guarantee that the system described by (1) will satisfy the existence and uniqueness requirement of a well-posed problem, it is sufficient that f_i and $\frac{\partial f_i}{\partial x_j}$ are continuous functions of t for $i, j = 1, 2, ...n$.*

1.2 Linear systems

Recall that a linear system in n dimensions takes the form:

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n \\ \ldots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n \end{cases}$$

where the functions $a_{ij}(t)$ (the **coefficients** of the system) are known while $x_1(t), x_2(t), \ldots, x_n(t)$ are unknown. The initial conditions for the above system are of the form

$$x_1(t_0) = c_1,\ x_2(t_0) = c_2,\ \ldots,\ x_n(t_0) = c_n.$$

Such a system of state equations together with the input equations and the output equations can be written in matrix notation as:

$$\begin{cases} \mathbf{x}'(\mathbf{t}) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t) \end{cases} \tag{2}$$

where $\mathbf{A}(\cdot) \in R^n \times R^n, \mathbf{B}(\cdot) \in R^n \times R^m, \mathbf{C}(\cdot) \in R^p \times R^n, \mathbf{D}(\cdot) \in R^p \times R^m$. Also, $\mathbf{x}(t) \in R^n, \mathbf{u}(t) \in R^m$, and $\mathbf{y}(t) \in R^p$.

When the coefficients are constant (linear time invariant or LTI) the form is:

$$\begin{cases} \mathbf{x}'(\mathbf{t}) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t). \end{cases} \tag{3}$$

In these equations, $\mathbf{x}(t)$ represents the state of the system, $\mathbf{u}(t)$ is the control and the vector variable $\mathbf{y}(t)$ is called the observed output. This output may be an indirect measurement of the state and/or the control.

Note that from Theorem (1), for existence and uniqueness in linear systems, it is sufficient that each element $a_{ij}(t)$ and $b_{ij}(t)$ be continuous in t. This condition automatically holds for the time invariant case.

From lecture 2, every n-th order linear differential equation can be written as a system of n first-order linear differential equations.

1.3 Controllability

Consider the Linear time invariant system (LTI):

$$\mathbf{x}'(\mathbf{t}) \quad = \quad \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t). \tag{4}$$

The function $\mathbf{u}(t)$ is real and m-dimensional and assumed to be piecewise continuous. The response $\mathbf{x}(t)$ of the system is influenced now by the control function $\mathbf{u}(t)$ and we consider the extent to which $\mathbf{u}(t)$ affects the behavior of the system.

Definition 1.1. *System (4) is called* **(completely) controllable** *if for any initial state x_0, any terminal state x_1, there is a $0 < t_1 < \infty$ and there exists a control function $u(t)$ defined on $[0, t_1]$ which drives the system from state $x_0 = x(0)$ to state $x_1 = x(t_1)$. That is, the system can be driven from any state to any other state within finite time.*

Sometimes we say the pair $(\mathbf{A}, \mathbf{B})$ is controllable. It can be shown that controllability is equivalent to being able to transfer the system from zero to any state (and also visa versa) in finite time.

There is an easy test for determining if a linear time-invariant systems is completely controllability.

Theorem 1.1. System (4) is completely controllable if and only if the column vectors of the **controllability matrix**

$$C(\mathbf{A},\mathbf{B}) = (\mathbf{B}, \mathbf{AB}, \mathbf{A^2B}, ..., \mathbf{A^{n-1}B}) \tag{5}$$

span R^n. Here $C(\mathbf{A},\mathbf{B})$ is a $n \times nm$ matrix

Recall that to span R^n means that the column vectors can be used to write any vector in R^n as a linear combination of these column vectors.

If the system is not completely controllable, we can define a useful quality of the system using the following terminology.

Definition 1.2. *The subspace spanned by the columns of* $C(\mathbf{A},\mathbf{B})$ *is called the* **controllable subspace** *of the system (4). Any state in the controllable subspace can be transferred to any other state in the controllable subspace within finite time.*

Example 1. Is the following system completely controllable?

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} u$$

where u is a constant, $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, and $B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The controllability matrix is given by

$$C(\mathbf{A},\mathbf{B}) = (\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}) \tag{6}$$

$$= \begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix} \tag{7}$$

and this matrix has linearly independent columns and so the system is completely controllable.

1.4 Stabilizability

Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\, x(t) + \mathbf{B}\, u(t). \tag{8}$$

If we set the control $\mathbf{u}(t) = 0$, then the system is uncontrolled. For this case:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\, x(t). \tag{9}$$

As described in lecture 2, if the system is not asymptotically stable, then there exist eigenvalues of $\mathbf{A}$ with nonnegative real parts. Is it always possible to find a control function $\mathbf{u}(t)$ which stabilizes the system, i.e., of steering the response of (8) to the zero state from any initial state?

If the system (8) is completely controllable, it can in particular be steered to the zero state. But a weaker property is sufficient, since only the unstable eigenvalues of system (9) need to be controlled. This leads us to the definition of stabilizability.

Definition 1.3. *System (8) is called* **stabilizable** *if its unstable subspace (space of eigenvectors associated with the eigenvalues with non-negative real part) is contained in its controllable subspace.*

1.5 Observability

Closely linked to the idea of controllability is the concept of observability. Loosely speaking, controllability studies the possibility of steering the state from a given input to a given output while observability studies the possibility of determining the state of the system from the output which may be a complicated indirect measurement of the system. Put another way, does there exist a finite time t_1 such that knowledge of $\mathbf{u}$ and $\mathbf{y}$ over $[t_0, t_1]$ suffice to determine $\mathbf{x_0} = \mathbf{x}(t_0)$ and hence $\mathbf{x}(t)$? For example, if the system is not completely coupled, then just measuring one variable may not help determine the behavior of another variable. Consider the system:

$$\begin{aligned} \mathbf{x'(t)} &= \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}(t)\,\mathbf{x}(t). \end{aligned} \tag{10}$$

Definition 1.4. *System (10) is called* **(completely) observable** *at t_0 if there exists a finite time $t_1 > t_0$ such that for any initial state $\mathbf{x}(t_0)$ the knowledge of $\mathbf{u}(t)$ and $\mathbf{y}(t)$ for $t \in [t_0, t_1]$ suffices to determine the state $\mathbf{x}(t_0)$ uniquely. Sometimes we say the pair $(\mathbf{A}, \mathbf{C})$ is observable.*

Again, for the LTI system we have a similar and simple test of observability.

Theorem 1.2. The LTI system

$$\begin{aligned} \mathbf{x'(t)} &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t), \end{aligned} \tag{11}$$

is (completely) observable if and only if the row vectors of the **observability matrix**

$$V(\mathbf{A}, \mathbf{C}) = (\mathbf{C}, \mathbf{CA}, \mathbf{CA^2}, ..., \mathbf{CA^{n-1}})^T \tag{12}$$

span R^n (see lecture 2). Here $V(\mathbf{A}, \mathbf{C})$ is a $pn \times n$ matrix.

1.6 Solutions of the linear system

The system of equations:

$$\begin{aligned} \mathbf{x}'(\mathbf{t}) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), && (13) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t), && (14) \end{aligned}$$

where $\mathbf{x}(t_0) = \mathbf{x}_0$, defines a solution implicitly. In this case (in general, the only case) we can find an explicit representation for the solution. To begin, consider the uncontrolled system:

$$\mathbf{x}'(\mathbf{t}) = \mathbf{A}\,\mathbf{x}(t). \tag{15}$$

where $\mathbf{x}(t_0) = \mathbf{x}_0$, and $\mathbf{u}(t) = 0$. We know from Lecture 2 that the solution is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0.$$

Consider next the case where:

$$\mathbf{x}'(\mathbf{t}) = \mathbf{B}\,\mathbf{u}(t) \tag{16}$$

where $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{u}(t)$ is integrable.

A simple integration (for vectors, entry by entry) yields the solution

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{B}\mathbf{u}(\tau)d\tau.$$

Now consider the system (13) above and let $\mathbf{x}(t)$ be the solution of this system. Further let

$$\mathbf{z}(t) = e^{-\mathbf{A}(t-t_0)}\mathbf{x}(t). \qquad \mathbf{z}(t_0) = \mathbf{x}(t_0). \tag{17}$$

Differentiating $\mathbf{z}(t)$ with respect to t (using the product rule) gives:

$$\begin{aligned} \mathbf{z}'(\mathbf{t}) &= -\mathbf{A}e^{-\mathbf{A}(t-t_0)}\mathbf{x}(t) \quad + e^{-\mathbf{A}(t-t_0)}\mathbf{x}'(\mathbf{t}) \\ &= -\mathbf{A}e^{-\mathbf{A}(t-t_0)}\mathbf{x}(t) + e^{-\mathbf{A}(t-t_0)}(\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)) \\ &= e^{-\mathbf{A}(t-t_0)}\mathbf{B}\mathbf{u}(t). \end{aligned}$$

Hence

$$\mathbf{z}(t) = \mathbf{z_0} + \int_{t_0}^{t} e^{-\mathbf{A}(\tau-t_0)}\mathbf{B}\mathbf{u}(\tau)d\tau.$$

Relating x and z by Equation (17) above we have:

$$\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}(t-t_0)}\mathbf{z}(t) \\
&= e^{\mathbf{A}(t-t_0)}\mathbf{z}(t_0) + e^{\mathbf{A}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{A}(\tau-t_0)}\mathbf{Bu}(\tau)d\tau \\
&= e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau,
\end{aligned}$$

and the observation output variable $\mathbf{y}(t)$ is given by

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau + \mathbf{Du}(t)$$

1.7 System modes

Suppose that $\mathbf{A}$ has the n distinct eigenvalues $\lambda_1, ..., \lambda_n$ and corresponding eigenvectors $\mathbf{e_1}, ..., \mathbf{e_n}$. Then these vectors form a basis for R^n and are linearly independent. Thus any initial vector $\mathbf{x}(0) = \sum_{i=1}^{n} \mu_i \, \mathbf{e_i}$, Let us furthermore define the $n \times n$ matrix $\mathbf{T}$ consisting of the eigenvectors of $\mathbf{A}$ as columns.

$$\mathbf{T} = (e_1, ..., e_n) \tag{18}$$

Then

Theorem 1.3. For the above defined vectors and matrices we have:

i) $\mathbf{T}$ diagonalizes $\mathbf{A}$, that is,

$$\mathbf{T}^{-1}\mathbf{AT} = diag(\lambda_1, ..., \lambda_n). \tag{19}$$

ii) The transition matrix has the form

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} = \mathbf{T}\mathrm{diag}\left(e^{\lambda_1 t} ... e^{\lambda_n t}\right)\mathbf{T}^{-1}. \tag{20}$$

iii) The solution of (15) can be written in the form

$$\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) \\
&= \sum_{i=1}^{n} \mu_i e^{\mathbf{A}t}\mathbf{e_i} \\
&= \sum_{i=1}^{n} \mu_i \, e^{\lambda_i t} \, \mathbf{e_i},
\end{aligned}$$

where the scalars μ_i depend on the initial condition $\mathbf{x}(0)$ via

$$(\mu_1, ..., \mu_n)^T = T^{-1}\mathbf{x}(0)$$

.

The unforced response decomposes into a linear combination of **fundamental modes** and as we have seen, the stability depends on the sign of the real part of these eigenvalues which are found via the characteristic equation.

1.8 State feedback law

Again consider the system:

$$\mathbf{x}'(\mathbf{t}) \;=\; \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{21}$$

where $\mathbf{A} \in R^n \times R^n, \mathbf{B} \in R^n \times R^m, \mathbf{C} \in R^p \times R^n, \mathbf{B} \in R^p \times R^m$. Let us again consider a control which feeds back information from the state of the system. Letting $\mathbf{K}$ be a matrix of appropriate dimensions, we consider a control of the form

$$\mathbf{u}(t) = \mathbf{K}\,\mathbf{x}(t).$$

The new system takes the form:

$$\begin{aligned} \mathbf{x}'(\mathbf{t}) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{BK}\,\mathbf{x}(t) \\ &= (\mathbf{A} + \mathbf{BK})\mathbf{x}(t). \end{aligned} \tag{22}$$

We first address controllability and observability of the new system with the following theorem.

Theorem 1.4. For the above defined systems (21) and (22) we have:

i) $(\mathbf{A}, \mathbf{B})$ is controllable if and only if $(\mathbf{A} + \mathbf{BK}, \mathbf{B})$ is controllable.

ii) If $(\mathbf{A}, \mathbf{B})$ is observable it is not necessarily true that $(\mathbf{A} + \mathbf{BK}, \mathbf{B})$ is observable.

We have seen that for linear systems to be asymptotically stable we want the eigenvalues (poles, system modes) to have negative real part. Furthermore, the speed of stabilization depends on the magnitude of these negative real parts. Thus we would like to be able to design into the feed back control eigenvalues (poles system modes) with characteristics of our choosing. Mathematically, we want to choose $\mathbf{K}$ so that the characteristic polynomial $\det(\lambda\mathbf{I} - (\mathbf{A} + \mathbf{BK}))$ has the roots which we wish. The following theorem gives us the result we need.

Theorem 1.5. If the time invariant linear system (21) is controllable then by the state feedback $\mathbf{u}(t) = \mathbf{K}\,\mathbf{x}(t)$, where $\mathbf{K}$ is an $m \times n$ constant, real matrix, the eigenvalues of the matrix $(\mathbf{A} + \mathbf{BK})$ can be arbitrarily placed anywhere in the complex plane.

Here is an example which shows how to choose $\mathbf{K}$. for more complicated systems there are better methods to apply.

Example 2. Consider the system

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

Here $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It is left as an exercise to show that $(\mathbf{A}, \mathbf{B})$ is controllable. To find the values for $\mathbf{K} = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$ we must solve the following system of equations defined by:

$$\det(\lambda \mathbf{I} - (\mathbf{A} + \mathbf{BK})) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

where k_1 and k_2 are real. Expanding the left and right sides we get:

$$\lambda^2 - \lambda(2 + 1 + k_2) + 2(1 + k_2) - k_1 \quad = \quad \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2. \tag{23}$$

Simplifying and equating the coefficients in λ we get the following two equations to solve simultaneously for k_1 and k_2:

$$\begin{aligned} 3 + k_2 &= \lambda_1 + \lambda_2, \\ 2 + 2k_2 - k_1 &= \lambda_1\lambda_2, \end{aligned}$$

from which we get

$$\begin{aligned} k_1 &= -\lambda_1\lambda_2 + 2(1 + k_2), \\ k_2 &= \lambda_1 + \lambda_2 - 3. \end{aligned}$$

Thus we can pick out the desired λ_1 and λ_2. For example, we could choose $\lambda_1 = -1$ and $\lambda_2 = -1$ Then $k_1 = -9$ and $k_2 = -5$ and $\mathbf{K} = \begin{pmatrix} -9 & -5 \end{pmatrix}$. The state feedback controlled system with desired poles would be defined by

$$\mathbf{A} + \mathbf{BK} = \begin{pmatrix} 2 & 1 \\ -9 & -4 \end{pmatrix}.$$

2 Optimization

Johann Bernoulli first mentioned the Principle of Optimality in connection with the Brachistochrome Problem in 1696. This problem helped to establish the search for manifestations of optimality as a fundamental property of action in natural systems.

Under many conditions, it is important to consider the ability of a control mechanism to stabilize a system at its equilibrium point. That is to say, we want the control to return the system to the equilibrium point when the initial conditions have displaced the system from this equilibrium state. For linear systems we can rearrange the problem to reflect deviations in the system from the equilibrium state and require that these deviations be reduced to zero by the control. When considering performance, we also are concerned that the control carry out its task in some desired optimal way such as using minimal time or energy to achieve its goal.

We again consider the linear time-invariant system (LTI):

$$\begin{aligned} \mathbf{x}'(\mathbf{t}) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) && (24) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t). && (25) \end{aligned}$$

Note that $\mathbf{y}(t)$ may very well just be the unaltered state $\mathbf{x}(t)$. Further, $\mathbf{x}(t)$ will often represent deviations from an equilibrium state.

A common approach to defining optimality involves the following considerations:

i) We may want to move the controlled variable $\mathbf{y}$ from an arbitrary initial value to zero with as little effort (total deviation in $\mathbf{y}$) as possible. One criterion to express this is the quadratic cost functional

$$\int_0^\infty \mathbf{y}^T(t)\mathbf{R}_1\mathbf{y}(t)\,dt, \qquad (26)$$

where $\mathbf{R}_1$ is a positive definite (i.e. has positive eigenvalues) symmetric matrix. The expression $\mathbf{y}^T(t)\mathbf{R}_1\mathbf{y}(t)$ is a measure of the extent to which $\mathbf{y}$ at time t deviates from zero. The **w**eighting matrix $\mathbf{R}_1$ determines how much weight is attached to each of the components of $\mathbf{y}$. Usually $\mathbf{R}_1$ is diagonal, and hence $\mathbf{y}^T(t)\mathbf{R}_1\mathbf{y}(t)$ is the weighted sum of the deviations from zero of each of the components of $\mathbf{y}$:

$$\|\mathbf{y}\| = \sqrt{\sum_{i=1}^{n} r_i y_i^2}. \qquad (27)$$

The integral (26) is a measure for the cumulative deviation of $\mathbf{y}(t)$ from zero during the interval $[0, \infty]$.

ii) Now, minimizing the criterion (26) can result in an indefinitely large control driving $\mathbf{y}$ rapidly to zero. Excessively powerful controls can be physiologically unrealistic however. To moderate the amplitude of the control $\mathbf{u}$ which up to now has not been restricted, we need to include a second term in the quadratic cost criterion that prevents the control from growing indefinitely large. We thus consider the criterion:

$$\int_0^\infty \left(\mathbf{y}^T(t)\mathbf{R}_1\mathbf{y}(t) + \mathbf{u}^T(t)\mathbf{R}_2\mathbf{u}(t)\right)\, dt. \tag{28}$$

We again consider here the linear time invariant (LTI) system. The **linear quadratic regulator problem** is defined as follows:

Definition 2. Given the system

$$\begin{aligned} \mathbf{x}'(\mathbf{t}) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t), \end{aligned} \tag{29}$$

and the cost functional

$$\int_0^\infty \left(\mathbf{y}^T(t)\mathbf{R}_1\mathbf{y}(t) + \mathbf{u}^T(t)\mathbf{R}_2\mathbf{u}(t)\right)\, dt, \tag{30}$$

the linear quadratic regulator problem seeks to find the control $\mathbf{u}(t)$ which transfers the controlled variable $\mathbf{y}(t)$ to zero with minimal cost as defined by the cost functional.

The following result holds

Theorem 2.1. Consider the system

$$\begin{aligned} \mathbf{x}'(\mathbf{t}) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t), \end{aligned} \tag{31}$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ and cost criterion:

$$\int_0^\infty \left(\mathbf{y}^T(t)\mathbf{R}_1\mathbf{y}(t) + \mathbf{u}^T(t)\mathbf{R}_2\mathbf{u}(t)\right)\, dt. \tag{32}$$

The matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are constant and the weighting matrices $\mathbf{R}_1$, and $\mathbf{R}_2$ are constant positive definite symmetric matrices. Furthermore assume the system is stabilizable and detectable. The optimal input $\mathbf{u}(t)$ as defined by the cost criterion and which transfers the system from $\mathbf{x}(0) = \mathbf{x}_0$ to zero can be generated with a linear control law of the form

$$\mathbf{u}(t) = -\mathbf{F}\mathbf{x}(t) \tag{33}$$

where

$$\mathbf{F} = \mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P}. \tag{34}$$

The matrix $\mathbf{P}$ is the unique symmetric, non-negative definite solution of the associated algebraic $\mathbf{R}$iccati equation:

$$0 = \mathbf{C}^T\mathbf{R}_1\mathbf{C} - \mathbf{P}\mathbf{B}\mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P}. \tag{35}$$

The minimal value of the cost functional achieved by the optimal control is given by $\mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0)$.

2.1 Linearization

Having considered linear systems, we present an important theorem which allows for a **local** analysis of nonlinear systems by **linearizing** the system around equilibrium points.

Let $\mathbf{x}$ be a n-dimensional vector and let

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{36}$$

This measures the length or norm of the vector $\mathbf{x}$.

A result given in Russell [9], furnishes the following result.

Theorem 2.2. Consider the nonlinear control problem (NLCP)

$$\begin{aligned} \mathbf{x}' &= \mathcal{F}(\mathbf{x}, \mathbf{u}), \\ \mathbf{y} &= \mathbf{C}\mathbf{x}, \end{aligned}$$

and let $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ be an equilibrium of the system such that $\mathcal{F}$ is at least two times continuously differentiable at $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$. Consider the criterion

$$J(\mathbf{u}, \mathbf{x}) = \int_0^\infty ((\mathbf{x} - \bar{\mathbf{x}})^T\mathbf{R}_1(\mathbf{x} - \bar{\mathbf{x}}) + (\mathbf{u} - \bar{\mathbf{u}})^T(t)\mathbf{R}_2(\mathbf{u} - \bar{\mathbf{u}}))\,dt. \tag{37}$$

Consider furthermore the linearized system around $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$.,

$$\begin{aligned} \mathbf{x}'(\mathbf{t}) &= \mathbf{A}(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{B}(\mathbf{u} - \bar{\mathbf{u}}), \\ \mathbf{y} &= \mathbf{C}\,(\mathbf{x} - \bar{\mathbf{x}}), \end{aligned}$$

where $\mathbf{A}$ and $\mathbf{B}$ denotes the Jacobian of $\mathcal{F}(\mathbf{x}, \mathbf{u})$ with respect to $\mathbf{x}$ and $\mathbf{u}$ at $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$, respectively. Let the linearized system be stabilizable and observable. Then there exists a neighbourhood N of $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ such that the following holds.

(i) There exists a unique feedback law $\mathbf{u} = \mathcal{K}(\mathbf{x})$ which solves the NLCP for $\mathbf{x} \in N$.

(ii) The stationary solution $\mathbf{x} = \bar{\mathbf{x}}$ is asymptotically stable for the closed-loop system $\mathbf{x}' = \mathcal{F}(\mathbf{x}, \mathcal{K}(\mathbf{x}))$, $\mathbf{x}(0) \in N$.

(iii) The feedback $\mathcal{K}$ has the form $\mathcal{K} = \mathbf{K}\,(\mathbf{x} - \bar{\mathbf{x}}) + \bar{\mathbf{u}} + o(\|\mathbf{x} - \bar{\mathbf{x}}\|)$, where $\mathbf{K}$ is the optimal feedback matrix for the linearized problem. Here $o(\cdot)$ denotes the Landau symbol $(f(x) = o(g(x)) :\Leftrightarrow \|f(x)\|/\|g(x)\| \longrightarrow 0$ as $x \longrightarrow \infty)$.

(iv) The feedback law $u_\ell = \mathbf{K}\,(\mathbf{x} - \bar{\mathbf{x}}) + \bar{\mathbf{u}}$ stabilizes also the nonlinear system for $\mathbf{x}(0) \in N$.

3 Example

3.1 Cardiovascular physiology

A model of the cardiovascular-respiratory system is outlined below. We now sketch some features of this system, starting with the cardiovascular part.

- Blood flow to various regions of the body depends on a large number of factors including cardiac output, various blood pressures, cross-section of arteries and veins, and partial pressures of CO_2 and O_2 in the blood. Blood flow is regulated by both central mechanisms of control which can affect cardiac output, general vascular resistance, and arterial pressure and by local mechanisms found in the each tissue region which can alter the local vascular resistance. Furthermore, blood gas levels affect both vascular resistance and cardiac output in several ways. The interaction of the central and local controllers of blood flow, cardiac output, and blood pressure allow for a very adaptive but very complex system.

- The sympathetic system is a part of the nervous system that acts to increase heart rate, constrict blood vessels, and raise blood pressure. The sympathetic nervous system and the parasympathetic nervous system together constitute the autonomic nervous system, which controls involuntary functions.

- Local control of resistance consists of changes in the cross-section of the arterioles leading into the organs. If the metabolic rate increases or less oxygen is available to a tissue area, local vascular resistance will be reduced, allowing for increased blood flow which can then remove excess CO_2 and replenish O_2. There is also a central mechanism controlling systemic resistance which responds to changes in system $P_{a_{CO_2}}$, $P_{a_{O_2}}$, pH, and blood pressure.

- The heart's pumping activity or cardiac output Q depends on the interaction of a number of factors including heart rate, blood pressure, myocardial contractility (the force of contraction of heart muscle), the degree of ventricular filling (Starling's law), and the blood gases $\mathrm{Pa}_{\mathrm{CO}_2}$ and $\mathrm{Pa}_{\mathrm{O}_2}$. Sympathetic activity increases cardiac output by stimulating heart rate and contractility.

- The baroreceptor sensors respond quickly to changes in blood pressure by detecting stretching at the peripheral sites. When blood pressure falls the vasomotor center increases the sympathetic tone and decreases the parasympathetic tone which raises blood pressure because sympathetic activity increases both the heart pumping frequency and stroke volume.

- The purpose of the respiratory system is to exchange unwanted gas byproducts of metabolism, such as CO_2, for O_2, which is necessary for metabolism. The site of this exchange is the boundary between the alveoli and the cardiovascular capillary network intertwined with the alveoli. The efficiency of exchange is accomplished by varying the ventilation rate $\dot{V}_A$ which is the rate at which fresh air is introduced into the alveoli. The reverse exchange occurs in the tissues where metabolic activity utilizes O_2 and produces as a byproduct CO_2. In the absence of voluntary control of breathing or neurologically induced changes in breathing, the human respiratory control system varies the ventilation rate in response to the levels of CO_2 and O_2 in the body (via partial pressures $P_{a_{CO_2}}$ and $P_{a_{O_2}}$). This is referred to as the chemical control system and involves two sensory sites which monitor the blood gas levels producing a negative feedback control loop as described in Example (1) of lecture 3.

- The cardiovascular and respiratory systems are linked in a number of ways and influence each other's performance. The cardiovascular control system affects the performance of the respiratory control system by determining the transport delay between the lungs where $\mathrm{Pa}_{\mathrm{CO}_2}$ and $\mathrm{Pa}_{\mathrm{O}_2}$ levels are adjusted by ventilation and the sites where these levels are measured. The rate of blood flow also influences the efficiency of gas exchange in the lungs and various tissue compartments. Vasomotor activity is related to respiratory center activity in general and almost any factor that increases the degree of vasomotor activity will also at least somewhat increase respiration.

- The state of the respiratory system influences cardiovascular function in several ways. The chemosensors of the cardiovascular system (perhaps the same or closely connected to the respiratory chemosensors in the carotid and aortic bodies) respond to $\mathrm{Pa}_{\mathrm{CO}_2}$ and $\mathrm{Pa}_{\mathrm{O}_2}$. As mentioned previously, activation of these sensors stimulates the vasomotor center. This stimulation in general acts to modulate heart rate, contractility, and vasoconstriction in rather complicated ways and it is experimentally difficult to isolate specific interactions. It appears that chemoreceptor stimulation results in bradychardia, systemic vasoconstriction, reduced cardiac output, and a negative left ventricular inotropic response.

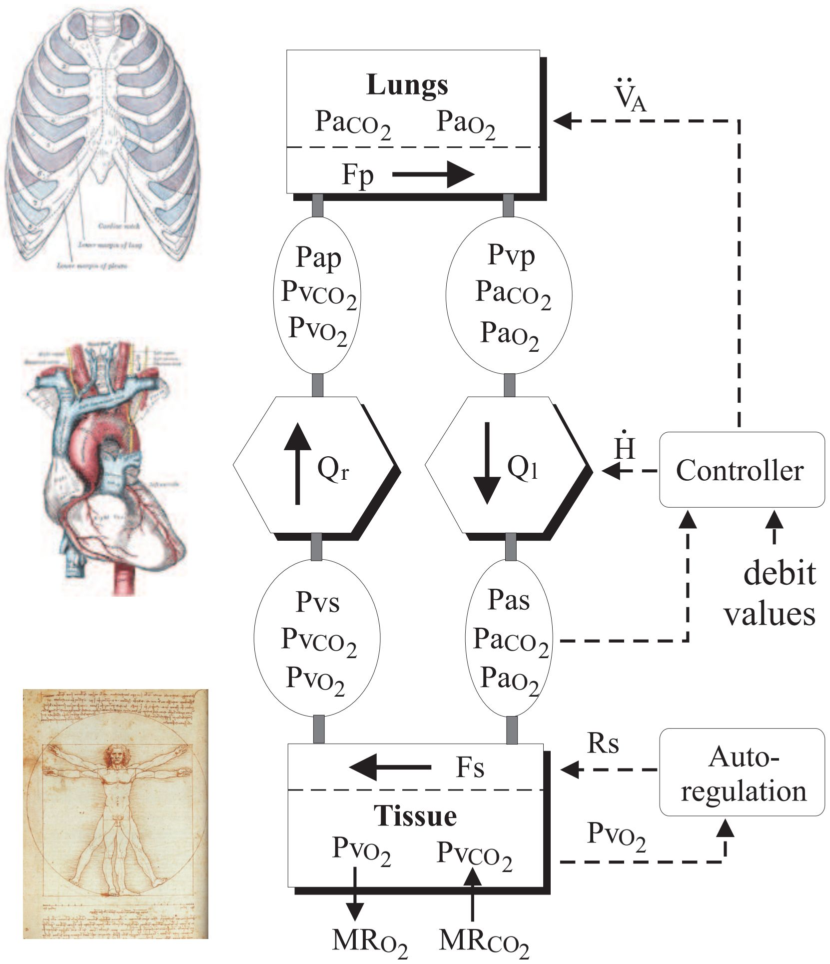

Figure 1: cardiovascular-respiratory block diagram

A block diagram of the cardiovascular-respiratory control system is given in Figure (1).

3.2 A cardiovascular-respiratory control model

The following set of 13 differential equations describes the model of the cardiovascular-respiratory control system corresponding to the block diagram in Figure (1).

$$
\begin{aligned}
V_{A_{CO_2}} \dot{P}_{a_{CO_2}}(t) &= 863 F_p(t)(C_{v_{CO_2}}(t) - C_{a_{CO_2}}(t)) \\
&\quad + \dot{V}_A(t)(P_{I_{CO_2}} - P_{a_{CO_2}}(t)), && (38) \\
V_{A_{O_2}} \dot{P}_{a_{O_2}}(t) &= 863 F_p(t)(C_{v_{O_2}}(t) - C_{a_{O_2}}(t)) \\
&\quad + \dot{V}_A(t)(P_{I_{O_2}} - P_{a_{O_2}}(t)), && (39) \\
V_{T_{CO_2}} \dot{C}_{v_{CO_2}}(t) &= MR_{CO_2} + F_s(t)(C_{a_{CO_2}}(t) - C_{v_{CO_2}}(t)), && (40) \\
V_{T_{O_2}} \dot{C}_{v_{O_2}}(t) &= -MR_{O_2} + F_s(t)(C_{a_{O_2}}(t) - C_{v_{O_2}}(t)), && (41) \\
c_{as} \dot{P}_{as}(t) &= Q_l(t) - F_s(t), && (42) \\
c_{vs} \dot{P}_{vs}(t) &= F_s(t) - Q_r(t), && (43) \\
c_{vp} \dot{P}_{vp}(t) &= F_p(t) - Q_l(t), && (44) \\
\dot{S}_l(t) &= \sigma_l(t), && (45) \\
\dot{S}_r(t) &= \sigma_r(t), && (46) \\
\dot{\sigma}_l(t) &= -\gamma_l \sigma_l(t) - \alpha_l S_l(t) + \beta_l H(t), && (47) \\
\dot{\sigma}_r(t) &= -\gamma_r \sigma_r(t) - \alpha_r S_r(t) + \beta_r H(t), && (48) \\
\dot{H}(t) &= u_1(t), && (49) \\
\ddot{V}_A(t) &= u_2(t). && (50)
\end{aligned}
$$

This combined model was developed and studied in Timischl [11]. A complete list of symbols is given in Tables 2 and 3 at the end of these notes.

The respiratory component is defined by equations (38) to (41) and is based on equations given in Khoo [3]. The respiratory system is modeled using two compartments, a lung compartment and a general tissue compartment (see Figure 1). The lung compartment equations (38) and (39) represent mass balance equations for CO_2 and O_2, respectively. Equations (40) and (41) are the state equations, respectively, for CO_2 and O_2 in the tissue compartment.

The cardiovascular model equations are based on the work of Grodins [1] and Kappel et al. [2] and is described by equations (42) to (48). The model consists of two circuits (systemic and pulmonary) which are arranged in series, and two pumps (left and right ventricle) (see Figure ??).

The complex of arteries and veins, arterioles, and capillary networks of each circuit is lumped into three components: a single elastic artery, a single elastic vein, and a single resistance vessel. We assume a unidirectional non-pulsatile blood flow through the heart. Hence, blood flow and blood pressure have to be interpreted as mean values over the length of a pulse.

Equations (42) and (43) are the blood mass balance equations in the systemic artery and vein, respectively. Equation (44) refers to the mass balance equation in the pulmonary vein. Utilizing the assumption of a fixed blood volume V_0, the equation for the pulmonary artery can be eliminated by a fixed algebraic relation.

Cardiac output Q is defined as the mean blood flow over the length of a pulse,

$$Q(t) = H(t)V_{str}(t), \tag{51}$$

where H is the heart rate and V_{str} is the stroke volume. The relationship between stroke volume and blood pressure is given in Kappel et al. [2]. It is based on the Frank Starling mechanism which is essentially described by

$$V_{str}(t) = S(t)\frac{cP_v(t)}{P_a(t)}. \tag{52}$$

Here S denotes the contractility, P_v is the venous filling pressure, P_a is the arterial blood pressure opposing the ejection of blood, and c is a constant. Equations (47) through (48) describe the observation that the contractility S_l respectively S_r increases if heart rate increases (Bowditch effect). This is modeled via a second order differential equation. For details see Kappel et al. [2].

The respiratory mass balance equations include expressions for the blood flows F_s and F_p. The respiratory system in turn influences the cardiovascular sys tem via the local metabolic autoregulation. This is modeled by the assumption th at systemic resistance R_s depends on venous oxygen concentration $C_{v_{O_2}}$,

$$R_s(t) = A_{pesk}C_{v_{O_2}}(t), \tag{53}$$

where A_{pesk} is a parameter.

Finally, equations (49) and (50) define the variation of heart rate $\dot{H}$ and the variation of ventilation rate $\ddot{V}_A$ as control variables. The functions $u_1(t)$ and $u_2(t)$ will be determined by an optimality criterion.

Motivation for an optimal control approach are as follows:

- Many physiologists consider optimization as a basic mechanism in the evolution of biological systems (see, e.g., Swan [10]). For example, it is reasonable to postulate that during the millions of years of evolution biological systems would tend to eliminate mechanisms that use too much energy and consequently too many basic nutrients to fuel the system. Therefore, minimum energy is a reasonable design requirement for a control system. Other optimal design considerations include minimizing stress on organs or systems and maintaining functioning within optimal bounds.

- It is the task of the baroreceptor loop to stabilize the arterial systemic pressure $P_{as}(t)$ around an equilibrium operating point P_{as}^f. The respiratory control acts to keep the extracellular CO_2 partial pressure as close as possible to an equilibrium value $P_{a_{CO_2}}^f$ and to a lesser extent O_2 to the equilibrium $P_{a_{O_2}}^f$. We assume that the variables P_{as}, $P_{a_{CO_2}}$, and $P_{a_{O_2}}$ are stabilized to their stable equilibrium values as quickly as possible when perturbed but with the constraint that the heart rate and the alveolar ventilation must not change too fast. This constraint corresponds to the requirement that the control effort should be as small as possible. In this way, the baroreceptor loop and the respiratory control system can be considered not only as a stabilizing but also as an optimizing feedback.

- The two quantities which we use to control the system are the heart rate H and the ventilation rate $\dot{V}_A$. H and $\dot{V}_A$ are adjusted via the baroreceptor loop and the respiratory control loop such that the mean arterial blood pressure P_{as} and the partial pressures of carbon dioxide $P_{a_{CO_2}}$ and oxygen $P_{a_{O_2}}$ in arterial blood are stabilized to their operating points when deviations occur. These operating points are the steady state values for a given condition defined by the choice of the parameters of the system. We model the control action as an optimal control.

- The transition is optimal in the sense that P_{as}, $P_{a_{CO_2}}$, and $P_{a_{O_2}}$ are stabilized such that deviations from their final steady state values are as small as possible. Furthermore, heart rate and ventilation rate are prevented from changing too fast.

Mathematically, this can be formulated in the following way. We determine control functions u_1 and u_2 such that the cost functional

$$\int_0^\infty \Big(q_{as}(P_{as}(t) - P_{as}^f)^2 + q_c(P_{a_{CO_2}}(t) - P_{a_{CO_2}}^f)^2 \tag{54}$$

$$+q_o(P_{a_{O_2}}(t) - P^f_{a_{O_2}})^2 + q_1 u_1(t)^2 + q_2 u_2(t)^2\Big)\ dt$$

is minimized under the restriction

$$\dot{x}(t) = g(x(t)) + B\, u(t), \qquad x(0) = x^i. \tag{55}$$

The positive scalar coefficients q_{as}, q_c, q_o, q_1, and q_2 determine how much weight is attached to each term in the integrand.

The superscript "i" refers to the initial state or perturbation and superscr ipt "f" refers to the final equilibrium or steady state of the system.

Equation (55) refers to the system equations (38) to (50) which is a non-linear system of thirteen differential equations. Here $x(t) \in R^{13}$ is given by

$$x(t) = (P_{a_{CO_2}}, P_{a_{O_2}}, P_{v_{CO_2}}, P_{v_{O_2}}, P_{as}, P_{vs}, P_{vp}, S_l, S_r, \sigma_l, \sigma_r, H, \dot{V}_A)^T. \tag{56}$$

$u(t) \in R^2$ denotes the control vector

$$u(t) = (u_1(t), u_2(t))^T = (\dot{H}, \ddot{V}_A)^T. \tag{57}$$

We linearize the system around x^f with initial condition $x(0) = x^i$.

Explicitly, we compute a feedback control

$$u(t) = -F_m\, x(t) \tag{58}$$

where F_m is the feedback gain matrix. This control is optimal for the linearized system around the final steady state x^f and suboptimal in the sense of Russell [9].

The calculated control transfers the linear system from the initial perturbation or state to the final steady state of the system in the optimal way as defined above. Table 1 refers to the parameters which appear in the cost functional. In the next section, we will apply this concept to the transition from resting awakeness to sleep. We will view the "awake state" as an initial perturbation of the steady state "stage 4 sleep". For a survey of applications of optimal control theory in biomedicine see, e.g. , Swan [10] or Noordergraaf and Melbin [8]. For gene ral control theory see Russell [9].

3.3 Simulation

In modeling the transition from the "resting awake" steady state to "stage 4 (NREM) sleep" the following steps are carried out:

- Compute the steady states "resting awake", x^a, and "stage 4 sleep", x^s.
  Here the steady state "sleep" is defined by the following parameter changes:
  - lower O_2 demand (MR_{O_2}) and lower CO_2 (MR_{CO_2}) production,
  - lower heart rate H,
  - higher $P_{a_{CO_2}}$ concentration in arterial blood.
- The control functions u_1 and u_2 which transfer system (55) from the initial steady state "awake", x^a, to the final steady state "sleep", x^s, are found as follows. We consider the linearized system around x^s with initial condition $x(0) = x^a$, and the cost functional

$$\int_0^\infty \Big(q_{as}(P_{as}(t) - P^s_{as})^2 + q_c(P_{a_{CO_2}}(t) - P^s_{a_{CO_2}})^2 \qquad (59)$$
$$+ q_o(P_{a_{O_2}}(t) - P^s_{a_{O_2}})^2 + q_1 u_1(t)^2 + q_2 u_2(t)^2 \Big)\, dt.$$

  Superscript "a" and superscript "s" will refer to the steady state values "awake" and "sleep", respectively. We then compute the control functions u_1 and u_2 such that the cost functional is minimized subject to the linearized system. This is accomplished by solving an algebraic matrix-Riccati equation. In particular, u_1 and u_2 are given as feedback control functions.
- This control is used to stabilize the nonlinear system (55). This control will be suboptimal in the sense of Russell [9].

The linearization proceeds as follows:

$$\dot{x}(t) = f(x(t); W^s) + B\,u(t), \qquad x_0 = x(0). \qquad (60)$$

where W^s represents the associated weights for sleep steady state. In general, we use the same weights for "awake" and "sleep" states. As controlled variables we have

$$y(t) = C\,x(t) = (P_{a_{CO_2}}(t), P_{a_{O_2}}(t) P_{as}(t))^T, \qquad (61)$$

where C is a 3×13 matrix given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \ldots & 0 \\ 0 & .3 & 0 & 0 & 0 & 0 \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \ldots & 0 \end{pmatrix}, \qquad (62)$$

and where

$$x(t) = (P_{a_{CO_2}}, P_{a_{O_2}}, P_{v_{CO_2}}, P_{v_{O_2}}, P_{as}, P_{vs}, P_{vp}, S_l, S_r, \sigma_l, \sigma_r, H, \dot{V}_A)^T. \quad (63)$$

$u(t) \in R^2$ denotes the control vector

$$u(t) = (u_1(t), u_2(t))^T = (\dot{H}, \ddot{V}_A)^T. \quad (64)$$

Explicitly, we compute a feedback control

$$u(t) = -F_m\, x(t) \quad (65)$$

where F_m is the feedback gain matrix.

We want to stabilize system (38) to (50) around the stage 4 sleep equilibrium x^s. Therefore our first step is a shift in the origin of the state space by introdu cing new variables ξ and η

$$\begin{aligned}\xi(t) &= x(t) - x^s, \\ \eta(t) &= y(t) - y^s.\end{aligned}$$

Next, we approximate $\dot{\xi}$ linearly. To this aim we replace x in (60) by ξ and make a Taylor expansion around x^s for fixed time t. This yields

$$\begin{aligned}\dot{x}(t) &= \dot{\xi}(t) = f(x^s + \xi(t); W^s) + B\, u \\ &= A_1\, \xi(t) + B\, u + o(\xi).\end{aligned}$$

Here $o(\cdot)$ denotes the Landau symbol

$$(h(x) = o(k(x)) :\Leftrightarrow \|h(x)\ |/\|k(x)\| \longrightarrow 0 \,\mathrm{as}\, x \longrightarrow \infty).$$

Note that the original state equations were already linear with respect to the control u.

The matrix A_1 is a 13×13 matrix of the Jacobian of f with respect to the components of $x(t)$ evaluated at $x = x^s$:

$$A_1 = \frac{\partial f}{\partial x(t)}(x^s; W^s).$$

Analogously,

$$\begin{aligned}\eta(t) &= (P_{a_{CO_2}}(t) - P^s_{a_{CO_2}}, P_{a_{O_2}}(t) - P^s_{a_{O_2}}, P_{as}(t) - P^s_{as})^T \\ &= C\, \xi(t).\end{aligned}$$

By neglecting terms of order $o(\xi)$ we get linear approximations $\xi_\ell(t)$ and $\eta_\ell(t)$ for $\xi(t)$ and $\eta(t)$, respectively,

$$\begin{aligned}
\dot{\xi}_\ell(t) &= A_1\xi_\ell(t) + B\,u(t), \\
\eta_\ell(t) &= C\xi_\ell(t), \\
\xi_\ell(0) &= x^a - x^s.
\end{aligned}$$

This is a special case of the general linear control system

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t).
\end{aligned}$$

Here $x(t) \in R^{13}$, $u(t) \in R^2$, and $y(t) \in R^3$.

In the following Figures (2) and (3) we illustrate the dynamics predicted by the system for the state variables P_{as}, P_{vs}, and the controls $\dot{H}$, and $\ddot{V}_A$). These figures illustrate the transition from the initial condition "awake" to the steady state "stage four" sleep.

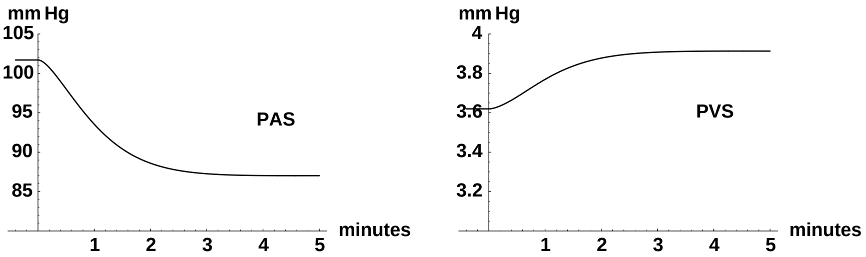

Figure 2: Systemic pressures dynamics

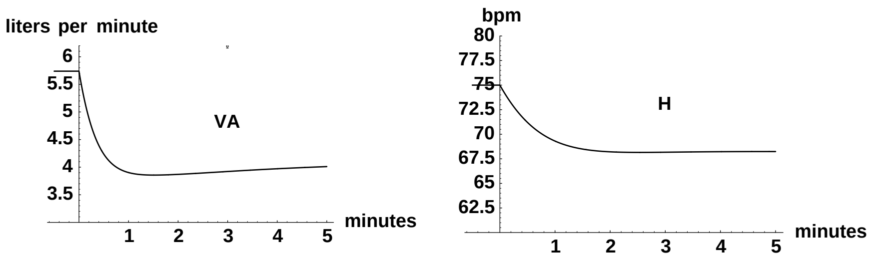

Figure 3: Control dynamics

Table 1: Control parameters

Symbol	Meaning	unit
q_{as}	weighting factor of P_{as} in the cost fun ctional	mmHg^{-2}
q_c	weighting factor of $P_{a_{CO_2}}$ in the cost functional	mmHg^{-2}
q_o	weighting factor of $P_{a_{O_2}}$ in the cost functional	mmHg^{-2}
q_1	weighting factor of u_1 in the cost functional	min^4
q_2	weighting factor of u_2 in the cost functional	$\mathrm{min}^4 \cdot \mathrm{l}_{\mathrm{BTPS}}^{-2}$
u_1	control function, $u_1 = \dot{H}$	min^{-2}
u_2	control function, $u_2 = \ddot{V}_A$	$\mathrm{l}_{\mathrm{BTPS}} \cdot \mathrm{min}^{-2}$

Table 2: Respiratory parameters

Symbol	Meaning	unit
$C_{a_{CO_2}}$	concentration of bound and dissolved CO_2 in arterial blood	$\mathrm{l}_{\mathrm{STPD}} \cdot \mathrm{l}^{-1}$
$C_{a_{O_2}}$	concentration of bound and dissolved O_2 in arterial blood	$\mathrm{l}_{\mathrm{STPD}} \cdot \mathrm{l}^{-1}$
$C_{v_{CO_2}}$	concentration of bound and dissolved CO_2 in mixed venous blood entering the lungs	$\mathrm{l}_{\mathrm{STP\ D}} \cdot \mathrm{l}^{-1}$
$C_{v_{O_2}}$	concentration of bound and dissolved O_2 in the mixed venous blood entering the lungs	$\mathrm{l}_{\mathrm{S\ TPD}} \cdot \mathrm{l}^{-1}$
MR_{CO_2}	metabolic CO_2 production rate	$\mathrm{l}_{\mathrm{STPD}} \cdot \mathrm{min}^{-1}$
MR_{O_2}	metabolic O_2 consumption rate	$\mathrm{l}_{\mathrm{STPD}} \cdot \mathrm{min}^{-1}$
$P_{a_{CO_2}}$	partial pressure of CO_2 in arteri al blood	mmHg
$P_{a_{O_2}}$	partial pressure of O_2 in arterial blood	mmHg
$P_{v_{CO_2}}$	partial pressure of CO_2 in mixed venous blood	mmHg
$P_{v_{O_2}}$	partial pressure of O_2 in mixed v enous blood	mmHg
u_2	control function, $u_2 = \ddot{V}_A$	$\mathrm{l}_{\mathrm{BTPS}} \cdot \mathrm{min}^{-2}$
$\dot{V}_A$	alveolar ventilation	$\mathrm{l}_{\mathrm{BTPS}} \cdot \mathrm{min}^{-1}$
$\ddot{V}_A$	time derivative of alveolar ventilation	$\mathrm{l}_{\mathrm{BTPS}} \cdot \mathrm{min}^{-2}$
$V_{A_{CO_2}}$	effective CO_2 storage volume of the lung compart-ment	$\mathrm{l}_{\mathrm{BTPS}}$
$V_{A_{O_2}}$	effective O_2 storage volume of the lung compartment	$\mathrm{l}_{\mathrm{BTPS}}$
$V_{T_{CO_2}}$	effective tissue storage volume for CO_2	l
$V_{T_{O_2}}$	effective tissue storage volume for O_2	l

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Table 3: Cardiovascular parameters

Symbol	Meaning	Unit
α_l	coefficient of S_l in the differential equation for σ_l	mi n^{-2}
α_r	coefficient of S_r in the differential equation for σ_r	mi n^{-2}
A_{pesk}	$R_s = A_{\text{pesk}} C_{v_{O_2}}$	$\mathrm{mmHg} \cdot \mathrm{min} \cdot \mathrm{l}^{-1}$
β_l	coefficient of H in the differential equation for σ_l	$\mathrm{mmHg} \cdot \mathrm{min}^{-1}$
β_r	coefficient of H in the differential equation for σ_r	$\mathrm{mmHg} \cdot \mathrm{min}^{-1}$
c_{as}	compliance of the arterial part of the systemic circuit	$\mathrm{l} \cdot \mathrm{mmHg}^{-1}$
c_{ap}	compliance of the arterial part of the pulmonary circuit	$\mathrm{l} \cdot \mathrm{mmHg}^{-1}$
c_{vs}	compliance of the venous part of the systemic circuit	$\mathrm{l} \cdot \mathrm{mmHg}^{-1}$
c_{vp}	compliance of the venous part of the pulmonary circuit	$\mathrm{l} \cdot \mathrm{mmHg}^{-1}$
F_p	blood flow perfusing the lung compartment	$\mathrm{l} \cdot \mathrm{min}^{-1}$
F_s	blood flow perfusing the tissue compartment	$\mathrm{l} \cdot \mathrm{min}^{-1}$
H	heart rate	min^{-1}
γ_l	coefficient of σ_l in the differential equation for σ_l	min^{-1}
γ_r	coefficient of σ_r in the differential equation for σ_r	min^{-1}
P_{as}	mean blood pressure in arterial region: systemic circuit	mmHg
P_{ap}	mean blood pressure in arterial region: pulmonary circuit	mmHg
P_{vs}	mean blood pressure in venous region: systemic circuit	mmHg
P_{vp}	mean blood pressure in venous region: pulmonary circuit	mmHg
Q_l	left cardiac output	$\mathrm{l} \cdot \mathrm{min}^{-1}$
Q_r	right cardiac output	$\mathrm{l} \cdot \mathrm{min}^{-1}$
R_p	resistance in the peripheral region of the pulmonary circuit	$\mathrm{mmHg} \cdot \mathrm{min} \cdot \mathrm{l}^{-1}$
R_s	peripheral resistance in the systemic circuit	$\mathrm{mmHg} \cdot \mathrm{min} \cdot \mathrm{l}^{-1}$
S_l	contractility of the left ventricle	mmHg
S_r	contractility of the right ventricle	mmHg
σ_l	derivative of S_l	$\mathrm{mmHg} \cdot \mathrm{min}^{-1}$
σ_r	derivative of S_r	$\mathrm{mmHg} \cdot \mathrm{min}^{-1}$
u_1	control function, $u_1 = \dot{H}$	min^{-2}
$V_{str,l}$	stroke volume of the left ventricle	l
$V_{str,r}$	stroke volume of the right ventricle	l
V_0	total blood volume	l

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