Introduction to Linear Control Theory

Lecture 3 Math notes

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Outline

- Basic terminology and an example
- A short survey of control theory history
- Open-looped and closed loop systems
- Transfer function representation
- Examples
- N dimensional Laplace tranforms
- Stability analysis open-looped and closed loop systems
- Impulse response
- Closed loop transfer functions
- Frequency analysis closed-looped systems

Useful References

- Miller, R.K., *Introduction to Differential Equations*, Prentice-Hall,New Jersey, 1987

- Khoo, M.C.K., *Physiological Control Systems*, IEEE Press,New York, 2000

- Anderson, B.D.O., Moore, J.B.*Optimal Control: Linear Quadratic Methods*, Prentice-Hall ,New Jersey, 1989

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1 Introduction

Definition 1. A **System** is a complex organization of interacting factors, conditions or components, usually functioning to create some result. The components of the system often can exist in a wide range of **states**. These states influence each other producing a dynamic evolution of the system over time.

Definition 2. Often the mutual influence of states forces the system to a condition called the **steady state** in which all components states are constant.

Definition 3. If one or more of the states of a system at a steady state are abruptly changed or **perturbed** by outside influences the system may or may not return to its steady state.

- If the system does return to steady state the steady state is called **asymptotically stable**.
- If the system does not return to steady state but remains in some bounded region around the steady state the steady state is called **stable**.
- If the system wanders further and further from the steady state the steady state is termed **unstable**.

Definition 4. Mathematical Control theory is the study of the design of controls which can force a system to achieve certain goals in certain ways such as tracking a prescribed path (**servomechanism**) or regulating a system around a given state (**regulator mechanism**).

Example 1. In the absence of voluntary control of breathing or neurologically induced changes in breathing, the respiratory control system varies the ventilation rate in response to the levels of CO_2 and O_2 in the body.

The control mechanism which responds to the changing needs of the body to acquire oxygen and expel CO_2 acts to maintain the levels of these gases within very narrow limits (and to a less understood degree match ventilation and blood flow). The control system consists of three components:

- sensors which gather information;
- effectors which are nerve/muscle groups which control ventilation; and
- the control processor located in the brain which organizes information and sends commands to the effectors.

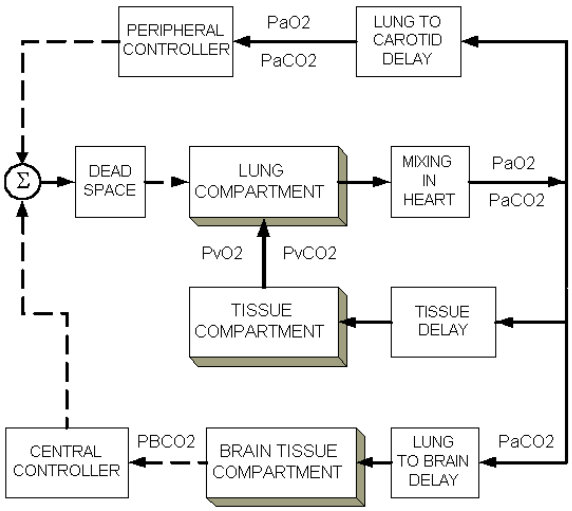

Figure 1: Block diagram of respiratory control system

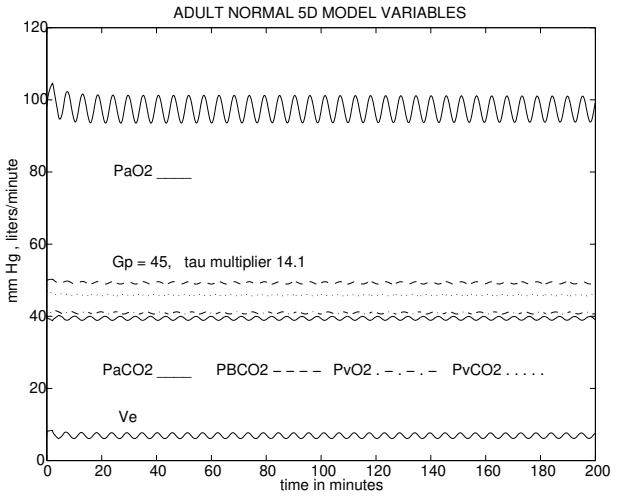

Figure 2: Respiratory control system oscillatory behavior

A **block diagram** of this system is given in Figure 1. In this block diagram the important organizational features of the system are diagramed. The signals from the control sensors are combined at the summation symbol and **fed back** to the control which is then altered in response to this information. The action of the control modifies the state of the system.

In this model, there are three compartments. One compartment models the partial pressures of carbon dioxide ($\mathrm{Pa}_{\mathrm{CO}_2}$) and partial pressure of oxygen and ($\mathrm{Pa}_{\mathrm{CO}_2}$) in the lungs. This is where the exchange of these gases with the outside environment occurs.

A second compartment models the general metabolic actions of the the body tissues where excess CO_2 ($P_{T_{CO_2}}$) builds up and O_2 ($P_{T_{O_2}}$) is used up. A third compartment monitors the level of CO_2 in the brain. The control sensors monitor CO_2 and O_2 levels in the brain and carotid arteries and the control processor in the brain translates this information into changes in ventilation to maintain steady state levels of these blood gases.

An example of stable but oscillatory behavior is given in Figure 2.

A Short Survey of Control Theory

- In 1840, the British Astronomer, G.B. Airy, developed a feedback mechanism for keeping the telescope at Greenwich England correctly pointed while compensating for the earth's rotation. Airy observed that strong oscillations could be produced in the system if the feedback was incorrectly set. He was the first to apply differential equations to analyze feedback control.

- In 1868, J.C. Maxwell carried out mathematical studies of the stability of Watt's flyball engine by linearizing the differential equations for the machine. He analyzed the characteristic equation of the system and established the link between the roots of the characteristic equation and the behavior of the system including the criterion that all roots have negative real parts if and only if the system is (asymptotically) stable. For complicated systems, E.J. Routh in 1877 developed a numerical method for testing when a characteristic equation has stable roots.

- In 1892, A.M. Lyapunov studied the stability of nonlinear differential equations using special functions which, by analogy to physical motion, captured a measure of the energy of the system. His work is central to the development of nonlinear analysis and modern control theory.

- The first use of the proportional-integral-derivative (PID) controller was developed by N. Minorsky in 1922 in his controller for the steering of ships.

- During the 1920's and 1930's Bell Labs and other research organizations involved in the design of communication systems and electrical engineering refined and extended frequency domain approaches developed much earlier earlier Laplace (1749-1827), Fourier (1768-1830) and others. By the 1950's automatic control theory using frequency-domain techniques had become established as the primary tool for the study and design of control systems.

- During the 1950's, L.S. Pontryagin developed the maximum principle, which solved optimal control problems using the calculus of variations developed earlier by L. Euler (1707-1783).

- The era of modern control theory had its primary impetus from three papers written in 1960 by R. Kalman and coworkers. These papers extended the work of Lyapunov in the time-domain control of nonlinear systems, introduced the linear quadratic regulator (LQR) in optimal control design, as well as applying optimal filtering, including the discrete Kalman filter.

- To provide for control mechanisms which are adaptive to disturbances, unmodelled dynamics, and measurement noise, many of the classical frequency-domain techniques were incorporated into modern design during the 1970's and 1980's by such researchers as I. Horowitz, I. Postlethwaite, and others. The modern approach is referred to as robust control theory. This theory combines powerful features of classical and modern techniques.

- As technology developed from the early computers of 1950, through the development of microprocessors and solid state design in the 1960's, to the introduction of the first PC's in 1983, control theory evolved to take advantage of these applications and addressed the issues of digital and discrete control and analysis.

Broadly classified, control theory can be classified into the following areas:

- Linear Systems vs Nonlinear Systems

- Time Varying Coefficients vs Time Invariant Coefficients

- Time Domain (State-space) analysis vs Frequency Domain Analysis

- Steady State Analysis vs Dynamical Analysis

- Continuous Control vs Discrete Control

- Input/Output design vs Optimal Control

2 Traditional approaches to control design

2.1 Open Loop, Closed Loop Systems and Block Diagrams

Traditional approaches to **Control Design** evolved from engineering fields, primarily electrical engineering. Indeed, early linear system theory borrowed heavily from electrical terminology and schematic diagrams as did early applications in physiology. Adaptations of models for computer simulations also used linear electrical circuit concepts and symbols such as resistance, capacitance, voltage, and current as well as analog devices to sum, differentiate, and integrate. There are clear and useful analogies between electrical systems and other dynamic systems as can be seen in the following table **??**. The primary method of analyzing **linear time-invariant** systems used Laplace transforms.

Table 1: Electrical-Physiology comparisons

Electrical	Physiological
Voltage	Fluid Pressure
Resistance	Vascular Resistance
Capacitance	Compliance
Inductance	Fluid Mass

The traditional techniques involved **steady state analysis** and dynamic input/output analysis utilizing special input signals such as **impulse**, **step input** and **sinusoidal**.

Figures (3) and (4) show **Block Diagrams** which describe the relationships between input and output in the two main forms of Control system. Figure (3) describes **open loop control**. In this format, the system involves some physical structure (or **plant**) which is reflected in the **output** which can be either the state of the system or some measurement (or function) of the state of the system. The state of the system is altered by the application of an input which can be modified by a controller. The signal from the controller alters the state of the system and thus the output.

According to Merriam Webster dictionary the term "control" denotes the action of "exercising restraining or directing influence". In the second sense, the input does indeed "control" the system by forcing the behavior of the plant to change. The controller cannot adapt to perturbations or noise in the system, however, and depends for its design a full knowledge of the behavior of the system to achieve desired goals.

Closed loop systems as depicted in Figure (4) monitor the output of the system and can **feed back** this knowledge to the control system to alter the control to respond to perturbations. Usually the aim is to maintain a steady state or track another signal.

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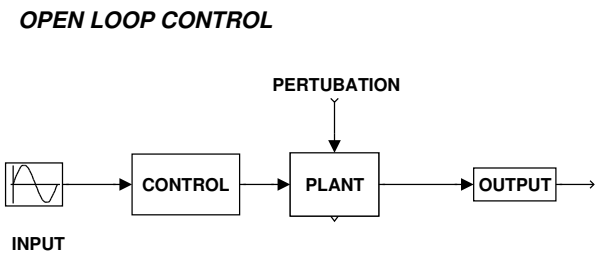

Figure 3: Open loop control system

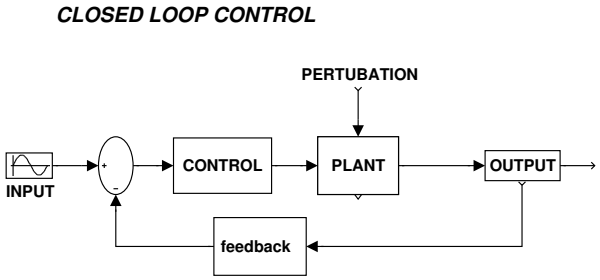

Figure 4: Closed loop control system

2.2 Laplace transform method

We want to give a short introduction to the methods and terminology of the technique for analyzing linear systems involving Laplace transforms.

This method is most successful when applied to linear systems with constant coefficients (linear-time-invariant or LTI), though it is sometimes useful for time dependent coefficients. The method can be used to analyze systems with single input and single output (**SISO**) as in Figure 3 but can be used to analyze multiple inputs multiple outputs (**MIMO**) by successive applications of the SISO method.

The system itself can be defined either by a set of linear differential equations in the **time domain** which are then transformed by the Laplace transform or by blocks directly defined by **transfer functions** representing electrical components in the "s" domain. Algebraic manipulations are employed to find the transformed unknown solution (with initial conditions incorporated).

The approach to analyzing a system employs the study of the system steady state response to various inputs including the **impulse input** and **step input** as well as the studying system response to trigonometric inputs (**frequency response**).

One important advantage of this approach to analyzing linear systems is that the linearity and the algebraic feature of the Laplace transform allow for the modularizing of the total system into subsystems and for the combining of subsystems into larger super systems in a convenient way.

The following examples illustrate the Laplace transform method for solving linear systems. For a more thorough treatment see [7] and [1].

Example 2. We let $x(t)$ denote the state of the system which can be thought of also as the **output** of the system, while $u(t)$ denotes the **input** either as a direct input to the system or a control signal. $u(t)$ is a function of t and affects the equations of the state. In this case $x(t)$ and $u(t)$ are functions but the discussion carries over for vectors as well with division replaced by matrix inversion. Let

$$a_2\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_0x(t) = u(t),$$
$$x(0) = 0, \quad x'(0) = 0.$$

If we take Laplace transform on each side of this equation, using the rules described in the first lecture we find that:

$$a_2s^2X(s) + a_1sX(s) + a_0X(s) = U(s),$$

or

$$X(s) = \frac{1}{a_2s^2 + a_1s + a_0}U(s).$$

The quantity $X(s)$ defines the Laplace transform of the state of the system or output. The quantity $U(s)$ is the Laplace transform of the input. The quantity

$$\frac{1}{a_2s^2 + a_1s + a_0}$$

relates the input to the output and is referred to as the **transfer function**, usually denoted by $H(s)$. Thus:

$$X(s) = H(s)U(s). \quad (1)$$

By taking the inverse Laplace transform of the expression $H(s)U(s)$ we find x(t). Systems can be analyzed either in the "t" domain or in the transformed 's" domain and then retransformed into the "t" domain. Furthermore, letting $h(t)$ be the inverse Laplace transform of $H(s)$, and recalling the convolution formula

$$L\{[f_1 * f_2](t)\} = L\{\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\} = L\{f_1(t)\}L\{f_2(t)\}$$

we have

$$x(t) = L^{-1}\{H(s)U(s)\} \quad (2)$$

$$= \int_0^t h(t-\tau)u(\tau)\,d\tau. \quad (3)$$

We see that once $h(t)$ (or the transfer function $H(s)$) is known the system response to any input $u(t)$ can be calculated. Note also from (1) and Figure (5) that if we could choose an input whose Laplace transform $U(s) = 1$ then the output $X(s)$ of the system for this input $U(s)$ would actually represent $H(s)$ and thus the time domain response $x(t)$ would actually represent $h(t)$. In other words, we could observe the system defining function $h(t)$ by applying a special test signal which would reveal it.

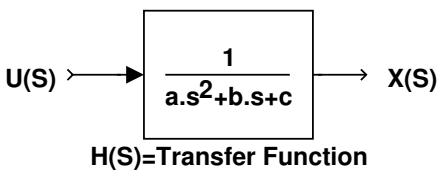

Figure 5: Open loop transfer function

2.3 Solutions of a second order linear time invariant system

An important result states that **every solution** of a linear system can be written as a sum of the general solution to the homogeneous problem (where $u(t) = 0$) plus a particular solution to the non-homogeneous system (where $u(t) \neq 0$).

Theorem 1. *The solution of the system defined by*

$$a_2\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_0x(t) = u(t), \tag{4}$$

$$x(0) = c_0, \quad x'(0) = c_1, \quad u(t) \neq 0, \tag{5}$$

can be written as the sum of the general solution $x_h(t)$ of the homogeneous system :

$$a_2\frac{d^2x_h(t)}{dt^2} + a_1\frac{dx_h(t)}{dt} + a_0x_h(t) = u(t), \tag{6}$$

$$x_h(0) = c_0, \quad x_h'(0) = c_1, \quad u(t) = 0, \tag{7}$$

and the solution $x_p(t)$, of the non homogeneous problem:

$$a_2\frac{d^2x_p(t)}{dt^2} + a_1\frac{dx_p(t)}{dt} + a_0x_p(t) = u(t), \tag{8}$$

$$x_p(0) = 0, \quad x_p'(0) = 0, \quad u(t) \neq 0. \tag{9}$$

The solution $x_h(t)$ is referred to as the **transient response** as it depends only on the uncontrolled behavior. $x_p(t)$ is referred to as the **forced response** because the control $u(t)$ changes the behavior. When the control persists over time the response is referred to as the **steady state** response.

Example 3. Consider the following second order2-D system:

$$\frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} = u(t),$$

$$x(0) = c_1, \quad x'(0) = c_2, \quad u(t) = \text{input}.$$

- If $u(t) = 0$, $x(0) = 0$,and $x'(0) = -1$ we have a homogeneous linear equation with no input control. Solving using the characteristic equation and the initial conditions gives

$$x(t) = -\frac{1}{2} + \frac{1}{2}e^{-2t}.$$

- In this case let $u(t) = 1$, $x(0) = 0$,and $x'(0) = 0$. Here we input a constant control signal $u(t) = 1$. Assuming $u(t) = 0$ for $t \leq 0$, $u(t)$ acts as a **step function** sometimes called the Heavyside function which jumps up to the constant value $u(t) = 1$ at $t = 0$. Note that in the earlier lectures this differential equation was called a non homogeneous equation. If we take Laplace transform on each side of this equation, using the rules described in the first lecture we find that:

$$s^2X(s) + 2sX(s) = U(s),$$

or

$$X(s) = \frac{1}{s^2 + 2s} U(s),$$

so that after applying partial fractions the transfer function $H(s)$ is

$$H(s) = \frac{1}{2s} - \frac{1}{2(s+2)}.$$

Using the tables from the first lecture we see that $h(t)$ (the inverse of the transfer function $H(s)$) is

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

and for $u(t) = 1$,

$$x(t) = \int_0^t (\frac{1}{2} - \frac{1}{2}e^{-2w})1dw = \frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2t}.$$

In this case as $t \to \infty$, $x(t) \to \infty$ so that the system is unstable.

- We let $u(t) = r(t) - Kx(t)$, $x(0) = 0, \quad x'(0) = 0$ and $0 < K < 1$. In this case, the control $r(t)$ is influenced by the state $x(t)$ and this influence is to reduce the control $r(t)$ when $x(t)$ increases thereby acting to feed back the growth of $x(t)$. The fed back information is a multiple of x and is called **proportional control**. Other feed back controls are possible which involve information about $\prime x$ (derivative control) or the integral of x (integral control).

  Applying Laplace transforms gives

$$H(s) = \frac{1}{s^2 + 2s + K}.$$

  Applying partial fractions and simplifying gives:

$$H(s) = \frac{1}{2\sqrt{1-K}} \left[\frac{1}{s + 1 - \sqrt{1-K}} - \frac{1}{s + 1 + \sqrt{1-K}} \right].$$

  The inverse Laplace transform gives $h(t)$ as

$$h(t) = \frac{1}{2\sqrt{1-K}} \left[e^{-(1-\sqrt{1-K})t} - e^{-(1+\sqrt{1-K})t} \right].$$

  Applying the convolution formula (2) when $r(t) = 1$ gives

$$x(t) = \frac{1}{K} + \frac{1}{2\sqrt{1-K}} \left[\frac{e^{-(1+\sqrt{1-K})t}}{1 + \sqrt{1-K}} - \frac{e^{-(1-\sqrt{1-K})t}}{1 - \sqrt{1-K}} \right]. \quad (10)$$

  In this case as $t \to \infty, x(t) \to \frac{1}{K}$ so that the system is asymptotically stable with steady state $x(t) = \frac{1}{K}$.

2.4 Laplace Transform for n-dimensional systems

The Laplace transform method works for n-dimensional systems as well. From the appendix, note that differentiation and integration are done componentwise and hence the basic process is the same as illustrated above in example (3). The main difference is in the formulation of the transfer matrix which involves finding the inverse of a matrix.

Recall that the general n dimensional linear system with constant coefficients takes the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Let $\mathbf{X}^*(s)$ be the Laplace transform of $\mathbf{x}(t)$ and $\mathbf{U}^*(s)$ be the Laplace transform of $\mathbf{u}(t)$. Then

$$s\mathbf{X}^*(s) - \mathbf{x}_0 \;=\; \mathbf{A}\mathbf{X}^*(s) + \mathbf{U}^*(s)$$

implies

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}^*(s) \;=\; \mathbf{x}_0 + \mathbf{U}^*(s).$$

Thus

$$\mathbf{X}^*(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{U}^*(s). \tag{11}$$

What is $(s\mathbf{I} - \mathbf{A})^{-1}$? Recall from lecture 2 that
$\mathbf{X}(t) = e^{\mathbf{A}t}$ satisfies the matrix relation

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$$

and hence satisfies the matrix differential equation

$$\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t) \quad \mathbf{X}(0) = \mathbf{I}. \tag{12}$$

where $\mathbf{X}', \mathbf{A}, \mathbf{X}$, and $\mathbf{X}(0)$ are matrices. Letting $\mathbf{X}^{**}(s) = L\{e^{\mathbf{A}t}\}$, take the Laplace transform of the system (12) component by component. This gives after simpification with matrix rules

$$(s\mathbf{X}^{**}(s) - \mathbf{I}) = \mathbf{A}\mathbf{X}^{**}(s),$$

or

$$(s\mathbf{I} - \mathbf{A})^{-1} = \mathbf{X}^{**}(s) = L\{e^{\mathbf{A}t}\}.$$

When we take the inverse Laplace transform of (11) we get the same solution as with variation of constants described briefly in lecture 2 using $\mathbf{X}^{**}(s) = L\{e^{\mathbf{A}t}\}$. Note that in terms of components

$$\mathbf{X}^*(s) = \begin{pmatrix} X_1(s) \\ \ldots \\ X_n(s) \end{pmatrix} = L\{\mathbf{x}(t)\} = \begin{pmatrix} L\{x_1(t)\} \\ \ldots \\ L\{x_n(t)\} \end{pmatrix}.$$

Example 4. Solve the initial value problem

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Taking Laplace transform of both sides of the differential equation gives

$$s\mathbf{X}(s) - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{X}(s) + \frac{1}{s-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Considering the system component-wise gives the following system to solve:

$$\begin{aligned} (s-1)X_1(s) - 4X_2(s) &= 2 + \frac{1}{s-1}, \\ X_1(s) + (s-1)X_2(s) &= 1 + \frac{1}{s-1}. \end{aligned}$$

The solution of these equations is

$$X_1(s) = \frac{2}{s-3} + \frac{1}{s^2-1}, \quad X_2(s) = \frac{1}{s-3} + \frac{s}{(s-1)(s+1)(s-3)}.$$

Now,

$$\frac{2}{s-3} = L\{2e^{3t}\}, \text{ and } \frac{1}{s^2-1} = L\{\frac{e^t - e^{-t}}{2}\}.$$

Hence,

$$x_1(t) = 2e^{3t} + \frac{e^t - e^{-t}}{2}.$$

To invert $X_2(t)$ we use partial fractions and get

$$\frac{s}{(s-1)(s+1)(s-3)} = \frac{-1/4}{s-1} + \frac{-1/8}{s+1} + \frac{3/8}{s-3}$$

and thus

$$x_2(t) = -\frac{1}{8}e^{-t} - \frac{1}{4}e^t + \frac{11}{8}e^{3t}.$$

In the examples (2) to (4) we have an input and output related by the state equations. For a more thorough treatment of the Laplace method and control design see [1].

2.5 Impulse response

Once the transfer function $H(s)$ is known, $h(t)$ can be found and the output for any input can be derived by applying the convolution formula (2) and integrating the control $u(t-\tau)$ with $h(t)$. In this set up, note that **if** there were a function whose Laplace transform was X(s) = 1 then the application of this input would produce a response which actually described the underlying transfer function which could be used to analyze systems whose internal structure was unknown. See Figure (5).

Strangely, there is no function which has Laplace transform $X(s) = 1$. The following family of functions in Equation (13) are concentrated around $x = 0$ and have Laplace transform as close to 1 as we want and can be thought of as approximating the ideal of a function whose Laplace transform **is** one. Let

$$f_0(t) = \frac{u_0(t) - u_a(t)}{a} \tag{13}$$

where $u_b(t)$ indicates a step function which jumps from 0 to 1 at $x = b$. That is:

$$u_b(t) = \begin{cases} 0 & \text{if } t < b \\ 1 & \text{if } t \geq b. \end{cases}$$

See the notes of lecture 1 for details on evaluating the Laplace transform of $f_0(t)$.

The idealized function is referred to as the **Dirac delta function** and is denote by $\delta(x)$. Note that these approximating functions become more and more concentrated around $x = 0$ but always with area under the curve equal to one. Hence, the applied input "impulse" gets more and more concentrated but with equal total "force". Figure (6) illustrates some of these functions with the impulse beginning at $x = 0$. The application of very short impulses, then, will give an approximate picture of the inverse Laplace of the transfer function $H(s)$ of the system. Note also, that the step response $u(t) = 1$ is found by

$$\int_0^t h(w)dw.$$

For analytical purposes, we can develop a more abstract theory of **generalized functions** in which the idealized $\delta(x)$ can be consistently defined as a linear function on a special class of functions referred to as test functions. For a short sample of this line of thinking see Miller [7]. For a more thorough treatment see [6].

Analytical tools for analyzing the steady state behavior of control systems includes then:

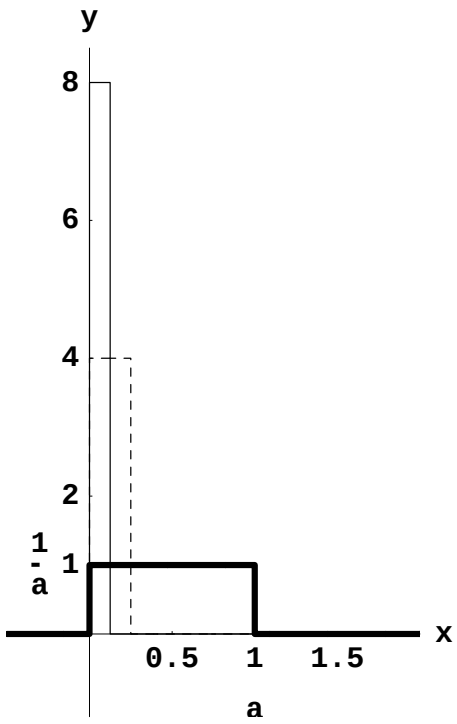

Figure 6: Impulse function approximations

- Finding the steady states of the system where the system is fixed and does not change over time. This is done by setting all derivatives equal to zero and solving for the states which produce this condition.
- Finding the transfer function by applying test inputs or controls which approximate the impulse δ dirac function
- Studying the system response to various controls such as the **step response**. In general there will be some alteration in the steady state of the system when controls are applied. This is referred to as steady state error.

2.6 Closed loop transfer functions for LTI system

The ideas involved in analyzing a closed loop system such as represented in Figure (7) can be illustrated using the following example.

Example 5. In this example, which is presented in Khoo 2000 ([5]), a simple linearized model is used to study the relation between the input of air pressure P_{ao} at the airway opening and the pressure P_A at the alveoli. Using

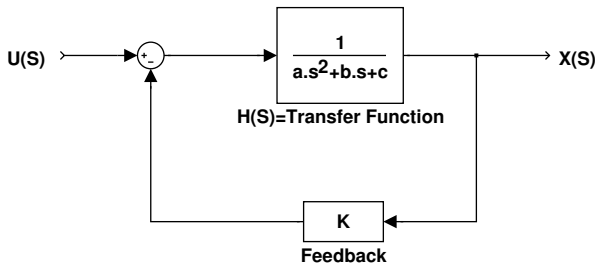

Figure 7: Closed loop transfer function

simple mechanical laws of pressure drop, air flow resistance, and lung compliance (similar to electrical diagrams), the following second order differential equation is derived:

$$P_{ao}(t) = LC\frac{d^2P_A(t)}{dt^2} + RC\frac{dP_A(t)}{dt} + P_A. \tag{14}$$

Here $P_A(0) = 0, \quad P_A{}'(0) = 0, \quad u(t) = P_{ao}, \quad LC$ and RC are constant. The transfer function of this system takes the form:

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + 1} = H(s) \tag{15}$$

where $H(s)$ is the transfer function of the open loop system as represented in Equation (1). Assume now that the air flow to the lungs needs to be regulated to avoid damage. We assume a mechanism that measures the alveolar pressure P_A and produces a negative feedback to the input control P_{ao}, that is the new control $u(t) = P_{ao}(t) - KP_A(t)$. The transfer function for this system (after rearranging terms) takes the form:

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + (1 + K)} = G(s) \tag{16}$$

where $G(s)$ is the transfer function of the closed loop system.

The above relationship allows for the study of transient (or impulse response), steady-state response to step inputs or sinusoidal inputs. Further, it can be used to study stability and the dependence of the response to changes in the parameters LC and RC. We see that

- the impulse response $h(t) = L^{-1}\{\frac{1}{LCs^2+RCs+(1+K)}\}$;
- the response to a step input or other input can be found by the convolution formula;

- the roots (called **poles**) of the expression $LCs^2 + RCs + (1 + K)$ are actually the same as the eigenvalues of the characteristic equation described earlier. These roots determine the stability qualities of the solution as described in lecture 2 and also reviewed in the appendix in this lecture.

A method for graphically representing the change in the poles as the parameters change is called the **root-locus method**. This representation draws the curves generated by the poles as plotted in the complex plane.

Example 6. In the lung mechanics model, which has a second degree polynomial in the denominator, there will be two roots for each choice of parameter values. The movement of these roots as K is varied from $K = 0$ to $K = \infty$ is represented in Figure(8). This example illustrates that "closing the loop" allows the designer to manipulate the characteristics of the the poles which determine the stability of the system. By varying K, it is possible to move the poles to the left half plane where they will have negative real part. It is also possible to adjust the magnitude of the negative real parts of poles to insure quicker stabilization. This is called **pole placement**. Recall, that the system will be unstable if at least one root has positive real part.

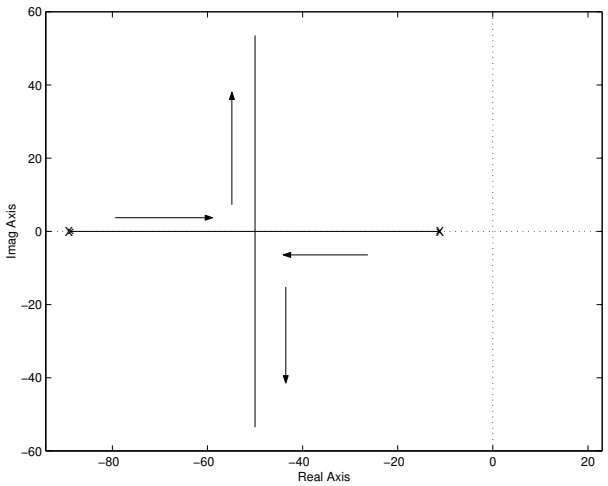

Figure 8: Closed loop root locus plot

2.7 Frequency domain analysis

We mention in passing that the expression in (16) can be used to study the response to a sinusoidal input as well. One need only replace $s = i\omega$ into (16)

and one can derive the amplitude and phase shift in the output for a sine wave of given frequency. The details can be found in ([5]). In Figure 9, we give the system response to various unit sine wave inputs (input frequencies of 1, 3, and 8 Hz (cycles/sec)) for the lung model with proportional feedback. In this figure, the change in amplitude and phase are represented as functions of the input frequency response in radians.

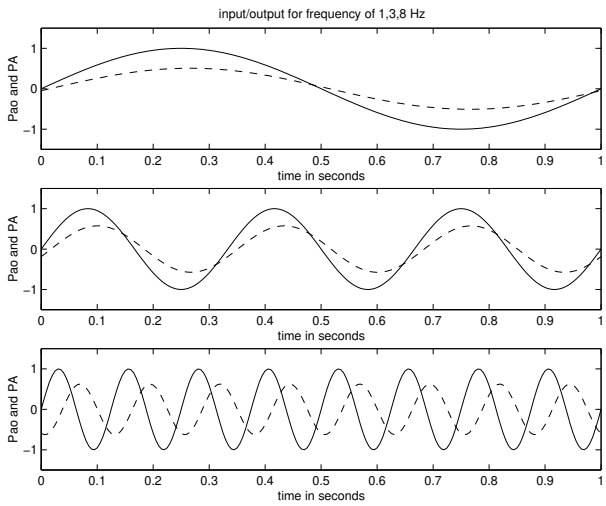

Figure 9: Closed loop frequency response

2.8 Stability analysis

For the general linear time invariant system with proportional feedback the equations for the transfer function follow directly as in Example 5. For a given input $u(t)$, output $x(t)$, feedback proportion K and open loop transfer function $H(s)$ we have by direct application of the Laplace transform:

$$\frac{X(s)}{U(s)} = \frac{H(s)}{1 + KH(s)} = G(s)$$

where $G(s)$ is the transfer function of the closed loop system. Compare this formulation with the one given equation (16). As in every case, we find the poles by solving

$$1 + KH(s) = 0.$$

That is we find the roots of the expression in the denominator of the transfer function. The techniques given in Example 5 can be applied to these expression to study the steady state and dynamic dependence of the general system on the parameters.

2.9 Summary

While the methods described above work very well for linear systems with constant coefficients and single inputs, the methods become more and more unwieldy when applied to systems with non-constant coefficients and multiple inputs and outputs. Modern state space analysis has advantages which includes the features that it can:

- handle MIMO,
- incorporate Optimal control design,
- provides the setting for designing numerical methods for solution.

2.10 Linear Systems Appendix

Definition 5 (Multiplication matrix with vector).

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{pmatrix}$$

Definition 6. A **transposed** matrix or vector is defined by exchange of columns and rows.

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} \quad \text{its transpose} \quad \mathbf{b}^T = \begin{pmatrix} b_{11} & b_{12} \end{pmatrix}.$$

Definition 7. The **identity matrix** has ones in the diagonal and zeros for the rest. For example the 3×3 identity matrix is

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 7. A vector or a matrix multiplied with the identity matrix stays identicaly.

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

Definition 8. The **inverse** of a $n \times n$ matrix satisfies the equation

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Note: The inverse of a matrix needs not to exist. It exists if and only if the **determinant** of the matrix is not equal to zero:

$$\det(\mathbf{A}) = |\mathbf{A}| \neq 0$$

Definition 9.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{then } \det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$$

Definition 10. A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ of n-dimensional vectors is called **linearly dependent** if there are real numbers $k_1, k_2, \ldots, k_m$, not all zero, such that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \ldots + k_m\mathbf{u}_m = 0.$$

If this equation is true, only if $k_1 = k_2 = \ldots = k_m = 0$ then the set S is called **linearly independent**.

Definition 11. The set of vectors generated by all possible linear combinations of a set of vectors is called a **vectorspace** spanned by these vectors.

Definition 12. A set of linearly independent vectors, which span a vectorspace V are called a **basis of V**.
Note: There exists more than one basis for most vectorspaces.

Theorem 2. *A set of n linearly independent vectors* $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ *defines a n-dimensional vectorspace V. If you add any other vector* $\mathbf{w}$ *to the set, then the set* $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \mathbf{w}\}$ *is linearly dependent, if you remove one they don't span the whole vectorspace V.*

Example 8.

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a set of linearly independent vectors. They are basis of the 2-dimensional vectorspace $V = \{(a, b)$ where a, b are real numbers$\}$.

Example 9. Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad \mathbf{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$$

then, because of

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} \Leftrightarrow \begin{cases} a_{11}x_1(t) + a_{12}x_2(t) &= b_1(t) \\ a_{21}x_1(t) + a_{22}x_2(t) &= b_2(t) \end{cases}$$

a **linear equation system** can be written as

$$\mathbf{A}\mathbf{x}(t) = \mathbf{b}(t)$$

Example 10. Thus, with the convention that

$$\mathbf{x}'(\mathbf{t}) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}' = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$$

we can write a **system of first-order linear differential equations**

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + g_1(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + g_2(t) \end{cases}$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t)$$

with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix}$.

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + g_1(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + g_2(t) \end{cases}$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

let $\mathbf{X}(s)$ be the Laplace transform of $\mathbf{x}(t)$ and $\mathbf{F}(s)$ the one of $\mathbf{f}(t)$. Then

$$\begin{aligned} s\mathbf{X}(s) - \mathbf{x}_0 &= \mathbf{A}\mathbf{X}(s) + \mathbf{F}(s) \\ (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{x}_0 + \mathbf{F}(s) \end{aligned}$$

The convolution formula is

$$L\{[f_1 * f_2](t)\} = L\{\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\} = L\{f_1(t)\}L\{f_2(t)\}.$$

note that $L\{[f_1 * f_2](t)\} = L\{[f_2 * f_1](t)\}$.

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s).$$

Note: $(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 = L\{e^{\mathbf{A}t}\mathbf{x}_0\}$.

The solution of the non-homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{u}(t)$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}t}\mathbf{u}(t)dt$$

Definition 13. Given is a system of linear differential equations

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

- If one eigenvalue λ of $\mathbf{A}$ has $\mathrm{Re}\lambda > 0$ then the matrix is called **unstable matrix**.
- If the matrix has only eigenvalues with $\mathrm{Re}\lambda < 0$ then it is called **stable** matrix. item A matrix $\mathbf{A}$ such that all eigenvalues λ of $\mathbf{A}$ satisfy $\mathrm{Re}\lambda \leq 0$ and at least one eigenvalue of $\mathbf{A}$ h as zero real part is called a **critical matrix**.

Theorem 3. *Let* $\mathbf{A}$ *be an unstable matrix, then the origin is an unstable equilibrium for the system*

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

If $\mathbf{A}$ *is stable, then the origin is an asymptotically stable equilibrium and it is the only equilibrium.*

Example 11. In case of 2 variables we can transform, by mean of change of the basis, the pro blem to one of these matrices - and assuming $t_0 = 0$ these solutions

1. $\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ thus $\mathbf{x}(t) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} \mathbf{x}_0$

   (a) $\lambda < 0 < \mu$: A saddle at the origin

   (b) $\lambda \leq \mu < 0$: A stable node at the origin

   (c) $0 < \lambda \leq \mu$: An unstable node at the origin

2. $\mathbf{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ thus $\mathbf{x}(t) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mathbf{x}_0$

   (a) $\lambda < 0$: A stable node at the origin

   (b) $\lambda > 0$: An unstable node at the origin

3. $\mathbf{A} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ thus $\mathbf{x}(t) = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} \mathbf{x}_0$

   (a) $a = 0$: A center at the origin

   (b) $a < 0$: A stable focus at the origin

   (c) $a > 0$: An unstable focus at the origin

Note: A change of the basis can be written as (with $\mathbf{P}$ is invertable)

$$\overline{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}.$$

We can show that $e^{\mathbf{P}\mathbf{A}\mathbf{P}^{-1}t} = \mathbf{P}e^{\mathbf{A}t}\mathbf{P}^{-1}$.

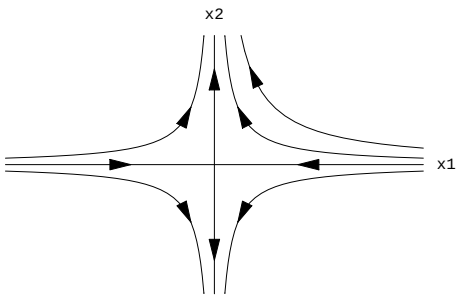

Figure 10: Ex. 8 - 1(a) saddle at the origin

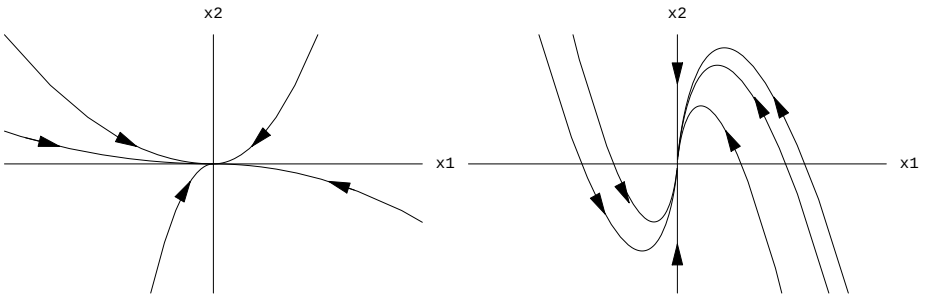

Figure 11: Ex. 8 - 1(b), 2(a) node at the origin

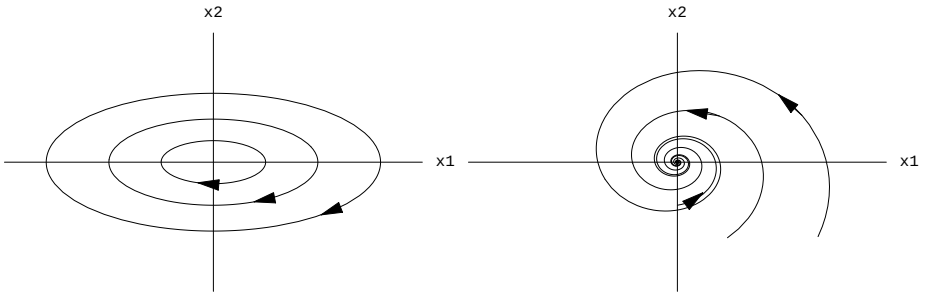

Figure 12: left: Ex. 8 - 3(a) center, right: Ex. 8 - 3(b) focus at the origin

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