Dynamical Systems

Martin Fink

SFB F003 "Optimierung und Kontrolle"
Graz, Austria
April 3, 2001

Outline

- Systems of linear equations (Existence and uniqueness)
- Linear algebra
- Systems of linear equations in matrix form
- Solving homogeneous linear systems
- Solving non-homogeneous linear systems, Laplace transforms
- Stability of linear systems
- Stability of nonlinear systems

Referring to:
Miller, R.K., *Introduction to Differential Equations*, Prentice-Hall,New Jersey, 1987
Boyce, W.E., DiPrima, R.C., *Gewöhnliche Differentialgleichungen*, Spektrum Verlag, Heidelberg, 1995
Perko, L., *Differential Equations and Dynamical Systems*, Springer-Verlag, New York, 1991

1 Systems of linear equations

1.1 Introduction

Definition 1. By a system of **first-order linear differential equations** we mean a set of n simultaneous equations of the form

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + g_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + g_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + g_n(t) \end{cases}$$

The functions $a_{ij}(t)$ (the **coefficients** of the system) and $g_k(t)$ (the **forcing terms**) are known functions while $x_1(t), x_2(t), \ldots, x_n(t)$ are the unknowns. The system of differential equations is called **homogeneous** when $g_k(t) \equiv 0$ for all k, otherwise it is called **non-homogeneous**.
The initial conditions for the above system are of the form

$$x_1(t_0) = b_1,\ x_2(t_0) = b_2,\ \ldots,\ x_n(t_0) = b_n$$

Note: Every n-th order linear differential equation can be written as a system of n first-order linear differential equations.

Example 1. If we have

$$x''(t) = b_1(t)x'(t) + b_0(t)x(t) + g(t)$$

then we define

$$x_1(t) := x(t) \quad x_2(t) := x'(t)$$

and thus get the system

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = b_1(t)x_2(t) + b_0(t)x_1(t) + g(t) \end{cases}$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b_0(t) & b_1(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

Note: Only some systems of first-order equations can be transformed into an n-th order differential equation - not all!

1.2 Linear algebra

Definition 2. We will consider a **vector** to be defined as a row or column of elements, functions or variables.

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \end{pmatrix}$$

Definition 3. If a vector consists of a row or column of **n** elements, then we will say it has **dimension n**.

Definition 4. A $n \times n$ **matrix** is an array of elements, functions, variables. It can be written as

$$\mathbf{A} = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Example 2.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}$$

Definition 5 (Addition and multiplications of vectors or matrices). Adding vectors or matrices is done term by term, thus they have to have the same dimensions. In multiplication each term is multiplied by the scalar.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$$

Example 3.

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} a_1 + x_1(t) \\ a_2 + x_2(t) \\ a_3 + x_3(t) \end{pmatrix}$$

$$t^2 \begin{pmatrix} 3 & -7 & 2.5 \\ 1 & -2 & 0 \\ 5 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 3t^2 & -7t^2 & 2.5t^2 \\ t^2 & -2t^2 & 0 \\ 5t^2 & 5t^2 & t^2 \end{pmatrix}$$

Definition 6 (Multiplication matrix with vector).

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{pmatrix}$$

Definition 7. A **transposed** matrix or vector is defined by exchange of columns and rows.

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} \quad \text{its transpose} \quad \mathbf{b}^T = \begin{pmatrix} b_{11} & b_{12} \end{pmatrix}.$$

Definition 8. The **identity matrix** has ones in the diagonal and zeros for the rest.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 4. A vector or a matrix multiplied with the identity matrix stays identically.

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

Definition 9. The **inverse** of a $n \times n$ matrix satisfies the equation

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Note: The inverse of a matrix needs not to exist. It exists if and only if the **determinant** of the matrix is not equal to zero:

$$\det(\mathbf{A}) = |\mathbf{A}| \neq 0$$

Definition 10 (determinant).

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{then } \det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$$

Note: The determinant can't be calculated that easily in higher dimensions.

Definition 11. A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ of n-dimensional vectors is called **linearly dependent** if there are real numbers $k_1, k_2, \ldots, k_m$, not all zero, such that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \ldots + k_m\mathbf{u}_m = 0.$$

If this equation is true only if $k_1 = k_2 = \ldots = k_m = 0$ then the set S is called **linearly independent**.

Definition 12. The set of vectors generated by all possible linear combinations of a set of vectors is called a **vector space** spanned by these vectors.

Definition 13. A set of linearly independent vectors, which span a vector space V are called a **basis of V**.If a vector space can be generated by n linearly independent vectors, then we say the vector space has dimension n.
Note: There exists more than one basis for most vector spaces.

Example 5.

$$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \quad \text{or} \quad \left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$$

are sets of linearly independent vectors. They are bases of the 2-dimensional vector space

$$V = \{(a, b) \text{ where } a, b \text{ are real numbers}\}.$$

Any other 2-dimensional vector is a linear combination of a basis. For example

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

1.3 Systems of linear equations with constant coefficients in matrix form

Example 6. Thus, with the convention that

$$\mathbf{x}'(\mathbf{t}) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}' = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$$

we can write a **system of first-order linear differential equations**

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + g_1(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + g_2(t) \end{cases}$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t)$$

$$\text{with initial conditions } \mathbf{x}(t_0) = \mathbf{x}_0 = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix}.$$

1.4 Homogeneous linear systems with constant coefficients

Given a system of linear ODE's with constant coefficients, we can write them

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0. \qquad (H)$$

Consider $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions and c be a constant then, because of linearity, $\mathbf{x}(t) + \mathbf{y}(t)$ and $c\mathbf{x}(t)$ are solutions, too.

1.4.1 Solutions with the characteristic polynomial

Our method of solution is a natural extension of the methods developed last lecture. We take as a trial solution

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v} = e^{\lambda t}\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} e^{\lambda t}v_1 \\ \vdots \\ e^{\lambda t}v_n \end{pmatrix}$$

where λ and $\mathbf{v}$ are constant and must be determined. On substituting into the homogeneous equation it follows that

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{A}\mathbf{x}(t) \\ \lambda e^{\lambda t}\mathbf{v} &= \mathbf{A}(e^{\lambda t}\mathbf{v}) = e^{\lambda t}\mathbf{A}\mathbf{v} \end{aligned}$$

or because of $e^{\lambda t} \neq 0$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

When there exists the inverse of $(\mathbf{A} - \lambda\mathbf{I})$ then the vector $\mathbf{v} = \mathbf{0}$ is the only possible solution ($\Rightarrow \mathbf{x} = \mathbf{0}$). The inverse does not exist if and only if

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

$p(\lambda)$ is a polynomial of nth degree. We have to determine the values λ_i, called **eigenvalues** of $\mathbf{A}$, which are the roots of the polynomial, such that the above equation is true.
We get exactly n solutions

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

which don't have to be distinct, but may be.
Note: Using the eigenvalues, we get, with the methods below, exactly n solutions, which can be shown to be linearly independent. These solutions

span an n-dimensional vector space. By the linearity of the system all linear combinations of these solutions are also solutions. Thus we have an n-dimensional vector space of solutions.

Could there be other solutions, not involving exponentials? - We will see by the fundamental existence theorem, that this cannot happen.

Real distinct eigenvalues

In this case the corresponding **eigenvectors** $\mathbf{v_i}$, can be calculated by solving

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_i = \mathbf{0} \;\Leftrightarrow\; \lambda_i \mathbf{v}_i = \lambda_i \mathbf{A}.$$

The solutions are

$$\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i \quad i = 1 \dots n$$

Complex distinct eigenvalues

If $\lambda = \alpha + i\beta$ is a complex eigenvalue (solution of $p(\lambda)$), then the conjugate $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue. Suppose that $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue λ, so that

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0} \Rightarrow (\mathbf{A} - \overline{\lambda}\mathbf{I})\overline{\mathbf{v}} = \mathbf{0}.$$

Hence $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}$ is a solution and $\mathbf{x}_2(t) = \overline{\mathbf{x}_1(t)}$ is a solution, too. To obtain two equivalent real-valued solutions, we take the real and imaginary parts, thus

$$\mathbf{x}_3(t) = \mathrm{Re}[\mathbf{x}_1(t)] = \frac{1}{2}[\mathbf{x}_1(t) + \mathbf{x}_2(t)]$$

and

$$\mathbf{x}_4(t) = \mathrm{Im}[\mathbf{x}_1(t)] = \frac{1}{2i}[\mathbf{x}_1(t) - \mathbf{x}_2(t)]$$

The real-valued general solution is

$$\mathbf{x}(t) = c_3\mathbf{x}_3(t) + c_4\mathbf{x}_4(t)$$

Multiple eigenvalues If there exist multiple eigenvalues, then there is just one corresponding eigenvector for each of them. Thus we have to find a method to find more linearly independent eigenvectors. We say that $\mathbf{v}$ is a **generalized eigenvector of A of rank k** associated with the eigenvalue λ if

$$(\mathbf{A} - \lambda\mathbf{I})^k\mathbf{v} = \mathbf{0} \text{ and } (\mathbf{A} - \lambda\mathbf{I})^{k-1}\mathbf{v} \neq \mathbf{0}.$$

For such an eigenvector we define a **chain of generalized eigenvectors of length k** as follows.

$$\mathbf{v}_k = \mathbf{v},$$

$$\begin{aligned}
\mathbf{v}_{k-1} &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_k = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}, \\
\mathbf{v}_{k-2} &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{k-1} = (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}, \\
&\vdots \\
\mathbf{v}_1 &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})^{k-1}\mathbf{v}.
\end{aligned}$$

So, how do we get the $\mathbf{x}_i$?

$$\begin{aligned}
\mathbf{x}_1(t) &= \mathbf{v}_1 e^{\lambda t} \\
\mathbf{x}_2(t) &= (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} \\
&\vdots \\
\mathbf{x}_k(t) &= \left[\mathbf{v}_1\left(\frac{t^{k-1}}{(k-1)!}\right) + \mathbf{v}_2\left(\frac{t^{k-2}}{(k-2)!}\right) + \cdots + \mathbf{v}_{k-1}t + \mathbf{v}_k\right]e^{\lambda t}
\end{aligned}$$

Thus, we get k linearly independent solutions with the eigenvalue λ. They are also linearly independent to all other solutions with other eigenvalues.
Note: Using the eigenvalues, we get, with these methods, exactly n solutions, which can be shown to be linearly independent. These solutions span an n-dimensional vector space. By the linearity of the system all linear combinations of these solutions are also solutions. Thus we have an n-dimensional vector space of solutions.

Could there be other solutions, not involving exponentials? - We will see by the fundamental existence theorem, that this cannot happen.

Example 7. Find a real-valued set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \mathbf{x}.$$

The characteristic equation is

$$\begin{aligned}
\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} &= (1-\lambda)(1+\lambda)^2 - 2(\lambda+1) \\
&= (\lambda+1)(-1-\lambda^2) = 0.
\end{aligned}$$

Hence the eigenvalues are $\lambda_1 = i$, $\lambda_2 = -i$ and $\lambda_3 = -1$. For $\lambda_1 = i$ the eigenvector is

$$\begin{pmatrix} 1-i & 2 & 0 \\ -1 & -(1+i) & 0 \\ 1 & 0 & -(1+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (1-i)v_1 + 2v_2 \\ -v_1 - (1+i)v_2 \\ v_1 - (1+i)v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution is $v_1 = -(1+i)v_2$ and $v_3 = (1+i)^{-1}v_1 = -v_2$. We can choose $v_1 = -1$. Thus

$$\mathbf{v}_1 = \begin{pmatrix} 1+i \\ -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_2 = \overline{\mathbf{v}}_1 = \begin{pmatrix} 1-i \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = -1$ the eigenvector equation is

$$\begin{pmatrix} 2 & 2 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_1 + 2v_2 \\ -v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $v_1 = 0$, $v_2 = 0$, and v_3 is arbitrary. We take $v_3 = 1$ so that

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since a solution for the first eigenvalue would be

$$\begin{aligned}
\begin{pmatrix} 1+i \\ -1 \\ 1 \end{pmatrix} e^{it} &= \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] (\cos t = i \sin t) \\
&= \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sin t \right] + i \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos t \right] \\
&= \begin{pmatrix} \cos t - \sin t \\ -\cos t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \sin t + \cos t \\ -\sin t \\ \sin t \end{pmatrix}
\end{aligned}$$

and for the second eigenvalue the complex conjugate, we get the real-valued solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} \cos t - \sin t \\ -\cos t \\ \cos t \end{pmatrix}, \ \mathbf{x}_2(t) = \begin{pmatrix} \sin t + \cos t \\ -\sin t \\ \sin t \end{pmatrix}, \ \mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix}$$

1.4.2 Solutions with the exponential of the matrix

Definition 14. We define the **exponential of a matrix** as

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}.$$

We can show that for the exponential of matrices similar rules apply as for the usual exponential function, just with some restrictions:

$$e^{\mathbf{S}+\mathbf{T}} = e^{\mathbf{S}}e^{\mathbf{T}} \quad \text{if and only if } \mathbf{ST} = \mathbf{TS} \text{ i.e. they commute}$$

$$\left(e^{\mathbf{T}}\right)^{-1} = e^{-\mathbf{T}}$$

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$$

Theorem. *Given an initial value problem*

$$\begin{cases} \mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

we can prove that

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

is the unique solution of the initial value problem.

Proof. If $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$, then

$$\mathbf{x}'(t) = \frac{d}{dt}e^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{A}\mathbf{x}(t)$$

for all real values t. Also $\mathbf{x}(0) = \mathbf{I}\mathbf{x}_0 = \mathbf{x}_0$. Thus $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$ is a solution. To see that this is the only solution, let $\mathbf{x}(t)$ be any solution of the initial value problem and set

$$\mathbf{y}(t) = e^{-\mathbf{A}t}\mathbf{x}(t).$$

Then, using that $\mathbf{x}(t)$ is a solution

$$\begin{aligned} \mathbf{y}' &= -\mathbf{A}e^{-\mathbf{A}t}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{x}'(t) \\ &= -\mathbf{A}e^{-\mathbf{A}t}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) \\ &= \mathbf{0} \end{aligned}$$

for all real values t, since $e^{-\mathbf{A}t}$ and $\mathbf{A}$ commute. Thus $\mathbf{y}(t)$ is constant. Setting $t = 0$ shows that $\mathbf{y}(t) = \mathbf{x}_0$ and therefore any solution of the initial value problem is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{y}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

This completes the proof of the theorem.

1.4.3 Merging both solution methods

- Let $\mathbf{X}(t)$ be the matrix with columns formed by the n solutions found by the characteristic polynomial. The set of all solutions is a vector space of dimension
at least n, $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c}$.
- We have just shown, that *all* solutions are of the form $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$. $e^{\mathbf{A}t}$ is a matrix, whose n columns span *at most* an n-dimensional vector space.
- If the dimension of this vector space is less than n, then the vector space would not include all solutions generated by the characteristic polynomial. *Thus the dimension must be* n.

$\Rightarrow$ $e^{\mathbf{A}t}$ and $\mathbf{X}(t)$ span the same vector space. Thus the columns of $\mathbf{X}(t)$ and of $e^{\mathbf{A}t}$ are *bases* of the vector space *of all solutions.*

Definition 15. Such matrices, where the columns form a basis of the solution space are called **fundamental matrices**.

Lemma. *Let* $\mathbf{X}(t)$ *and* $\mathbf{Y}(t)$ *be two fundamental matrices.*
$\Rightarrow$ *Then there exists a constant matrix* $\mathbf{C}$ *such that* $\mathbf{Y}(t) = \mathbf{X}(t)\mathbf{C}$.

From the above follows

$$e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{C} \qquad \mathbf{C} = \text{const.}$$

Setting $t = 0$ yields $\mathbf{I} = \mathbf{X}(0)\mathbf{C}$, which implies that $\mathbf{C} = \mathbf{X}^{-1}(0)$.
Hence $e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$.

We know that the solution of the homogeneous system (H) is given by use of a fundamental matrix as

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c}.$$

With the initial conditions, the vector $\mathbf{c}$ is determined by

$$\mathbf{c} = \mathbf{X}^{-1}(t_0)\mathbf{x}_0.$$

Thus, for an initial value problem with the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ we get the solution

$$\begin{aligned}
\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 &= \mathbf{X}(t)\mathbf{X}^{-1}(0)\left(\mathbf{X}^{-1}(0)\right)^{-1}\mathbf{X}^{-1}(t_0)\mathbf{x}_0 \\
&= \mathbf{X}(t)\mathbf{X}^{-1}(0)[\mathbf{X}(t_0)\mathbf{X}^{-1}(0)]^{-1}\mathbf{x}_0 \\
&= e^{\mathbf{A}t}[e^{\mathbf{A}t_0}]^{-1}\mathbf{x}_0 \\
&= e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{x}_0 \\
\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 &= e^{\mathbf{A}(t-t_0)}\mathbf{x}_0.
\end{aligned}$$

1.5 Non-homogeneous linear systems

Given a continuous $n \times n$ matrix $\mathbf{A}$ and a continuous n-vector-valued function $\mathbf{g}(t)$ defined on an open interval J, consider the non-homogeneous (or forced) linear system

$$\begin{cases} \mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

We assume that a fundamental matrix $\mathbf{X}(t)$ can be found for the homogeneous system. Then we assume a trial solution of the form

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{y}(t)$$

where $\mathbf{y}(t)$ is an unknown vector function. Substituting this in the given problem, we find that

$$\mathbf{X}'(t)\mathbf{y}(t) + \mathbf{X}(t)\mathbf{y}'(t) = \mathbf{x}'(t) = \mathbf{A}[\mathbf{X}(t)\mathbf{y}(t)] + \mathbf{g}(t)$$

Since $\mathbf{X}' = \mathbf{A}\mathbf{X}$, 'cause it is a fundamental set, this becomes

$$\mathbf{A}\mathbf{X}(t)\mathbf{y}(t) + \mathbf{X}(t)\mathbf{y}'(t) = \mathbf{A}\mathbf{X}(t)\mathbf{y}(t) + \mathbf{g}(t)$$

Hence

$$\mathbf{y}'(t) = \mathbf{X}^{-1}(t)\mathbf{g}(t) \text{ for all } t \text{ in } J$$

or for any fixed τ in J we have

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \int_\tau^t \mathbf{X}(t)\mathbf{X}^{-1}(s)\mathbf{g}(s)\, ds$$

since $\mathbf{c}$ is an arbitrary constant, this is a general solution of the non-homogeneous system. For the initial value problem we get (in analogy to the last section)

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^t \mathbf{X}(t)\mathbf{X}^{-1}(s)\mathbf{g}(s)\, ds$$

or

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)\, ds$$

Similar as in one dimension this is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$$

with $\mathbf{x}_p(t)$ =particular solution of the non-homogeneous problem
$\mathbf{x}_c(t)$ =solution of the corresponding homogeneous system.

1.6 Laplace Transforms

Given

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

let $\mathbf{X}(s)$ be the Laplace transform of $\mathbf{x}(t)$ and $\mathbf{F}(s)$ the one of $\mathbf{f}(t)$. Then

$$\begin{aligned} s\mathbf{X}(s) - \mathbf{x}_0 &= \mathbf{A}\mathbf{X}(s) + \mathbf{F}(s) \\ (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{x}_0 + \mathbf{F}(s) \end{aligned}$$

Thus we get

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s).$$

Note: $(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 = L\{e^{\mathbf{A}t}\mathbf{x}_0\}$ - So we get the same result as with variation of constants. When we consider that $\mathbf{X}(s)$ is the Laplace transform of $\mathbf{x}(t)$, which means

$$\mathbf{X}(s) = \begin{pmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{pmatrix} = L\{\mathbf{x}(t)\} = \begin{pmatrix} L\{x_1(t)\} \\ \vdots \\ L\{x_n(t)\} \end{pmatrix}$$

When we have calculated the vector $\mathbf{X}(s)$, we use inverse Laplace transform for each of its elements and get the solution.

Example 8. Solve the initial value problem

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Taking Laplace transform of both sides of the differential equation gives

$$s\mathbf{X}(s) - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{X}(s) + \frac{1}{s-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or

$$(s-1)X_1(s) - 4X_2(s) = 2 + \frac{1}{s-1}$$

$$X_1(s) + (s-1)X_2(s) = 1 + \frac{1}{s-1}$$

The solution of these equations is

$$X_1(s) = \frac{2}{s-3} + \frac{1}{s^2-1}, \quad X_2(s) = \frac{1}{s-3} + \frac{s}{(s-1)(s+1)(s-3)}.$$

Now,

$$\frac{2}{s-3} = L\{2e^{3t}\}, \text{ and } \frac{1}{s^2-1} = L\{\frac{e^t - e^{-t}}{2}\}.$$

Hence,

$$x_1(t) = 2e^{3t} + \frac{e^t - e^{-t}}{2}.$$

to invert $X_2(t)$ we use partial fractions and get

$$\frac{s}{(s-1)(s+1)(s-3)} = \frac{-1/4}{s-1} + \frac{-1/8}{s+1} + \frac{3/8}{s-3}$$

Consequently,

$$x_2(t) = -\frac{1}{8}e^{-t} - \frac{1}{4}e^{t} + \frac{11}{8}e^{3t}$$

2 Qualitative Analysis - Stability

Until now we just tried to solve the equations. In this section we want to see what information we could gain concerning the solutions without actually solving it. We will look at equations of the form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(t, \mathbf{x}(t)) \quad \mathbf{x}(t_0) = \mathbf{a}$$

Definition 16. Equations with $\mathbf{f}$ independent of t, we call **autonomous equations**, we write

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{a}$$

We will only consider equations of this type.

2.1 Analysis of scalar equations

We assume that we have an autonomous equation for $\mathbf{x}$ and all properties such that there exists a unique solution $\mathbf{x}(t, t_0, \mathbf{a})$. Since $\mathbf{f}$ is independent of t the solutions $\mathbf{x}(t, t_0, \mathbf{a})$ satisfy the following property

$$\mathbf{x}(t, t_0, \mathbf{a}) = \mathbf{x}(t - t_0, 0, \mathbf{a})$$

Thus we can always assume that the initial point $t_0 = 0$.

Definition 17. An **equilibrium point** (or **critical point**) of the equation

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)) \quad \text{is the point } \mathbf{a} : \mathbf{f}(\mathbf{a}) = \mathbf{0}.$$

Note: $\mathbf{a}$ is an equilibrium point if and only if the constant function $\mathbf{x}(t) \equiv \mathbf{a}$ is a solution the differential equation.

The **sign analysis** yields, that the (one dimensional) function $x(t)$ must be increasing at a point $x(t) = b$, when $f(b) > 0$. Let $[a_1, a_2]$ be a proper interval such that $f(a_1) = f(a_2) = 0$, while $f(x(t)) > 0$ when $a_1 < x(t) < a_2$. Then it can be shown that

$$\lim_{t\to\infty} x(t) = a_2 \qquad \lim_{t\to-\infty} x(t) = a_1$$

2.2 Stability

If we have got an autonomous differential equation and the point a is an equilibrium point of it, then we call **a stable** if all solutions that start near $\mathbf{a}$ remain near $\mathbf{a}$ for all future time. We say $\mathbf{a}$ is **unstable** if this is not true. Thus the formal definitions are

Definition 18. Let $\mathbf{a}$ be a critical point (or equilibrium point) then

- $\mathbf{a}$ is called **stable** if for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $\mathbf{x}(t)$ is any solution with $||\mathbf{x}(0) - \mathbf{a}|| < \delta$, then $||\mathbf{x}(t) - \mathbf{a}|| < \varepsilon$ for all $t \geq 0$,
- $\mathbf{a}$ is called **unstable** if it is not stable,
- $\mathbf{a}$ is called **asymptotically stable** if it is stable and in addition there is an $R > 0$ such that when $\mathbf{x}(t)$ is a solution with $||\mathbf{x}(0) - \mathbf{a}|| < R$, then $\mathbf{x}(t) \to \mathbf{a}$ as $t \to \infty$ (i.e. $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{a}$).

The given definitions can easily be generalized for systems of differential equations.

Definition 19. When we have a 2nd-order differential equation it is often very interesting to consider its **trajectory** or **path** in the **phase plane**, which is the parametric solution curve $(x_1(t), x_2(t))$.

2.3 Linear Systems

Definition 20. Given is a system of linear differential equations

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

- If one eigenvalue λ of $\mathbf{A}$ has $\mathrm{Re}\lambda > 0$ then the matrix is called **unstable matrix**.
- If the matrix has only eigenvalues with $\mathrm{Re}\lambda < 0$ then it is called **stable** matrix.

- A matrix $\mathbf{A}$ such that all eigenvalues λ of $\mathbf{A}$ satisfy $\mathrm{Re}\lambda \leq 0$ and at least one eigenvalue of $\mathbf{A}$ has zero real part is called a **critical matrix**.

Theorem. *Let $\mathbf{A}$ be an unstable matrix, then the origin is an unstable equilibrium for the system*

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

If $\mathbf{A}$ is stable, then the origin is an asymptotically stable equilibrium and it is the only equilibrium.

Example 9. In case of 2 variables we can transform, by mean of change of the basis, the problem to one of these matrices - and assuming $t_0 = 0$ these solutions

1. $\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ thus $\mathbf{x}(t) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} \mathbf{x}_0$
   (a) $\lambda < 0 < \mu$: A saddle at the origin
   (b) $\lambda \leq \mu < 0$: A stable node at the origin
   (c) $0 < \lambda \leq \mu$: An unstable node at the origin
2. $\mathbf{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ thus $\mathbf{x}(t) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mathbf{x}_0$
   (a) $\lambda < 0$: A stable node at the origin
   (b) $\lambda > 0$: An unstable node at the origin
3. $\mathbf{A} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ thus $\mathbf{x}(t) = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} \mathbf{x}_0$
   (a) $a = 0$: A center at the origin
   (b) $a < 0$: A stable focus at the origin
   (c) $a > 0$: An unstable focus at the origin

Note: A change of the basis can be written as (with $\mathbf{P}$ is invertable)

$$\overline{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}.$$

We can show that $e^{\mathbf{P}\mathbf{A}\mathbf{P}^{-1}t} = \mathbf{P}e^{\mathbf{A}t}\mathbf{P}^{-1}$.

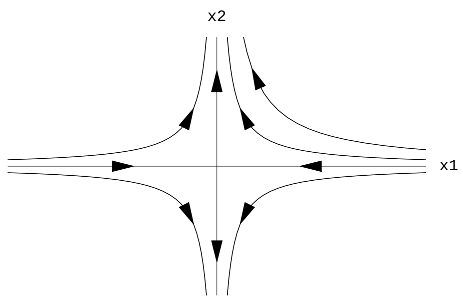
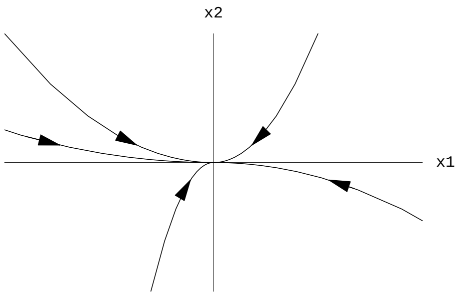


Figure 1: Ex. 8 - 1(a) saddle at the origin



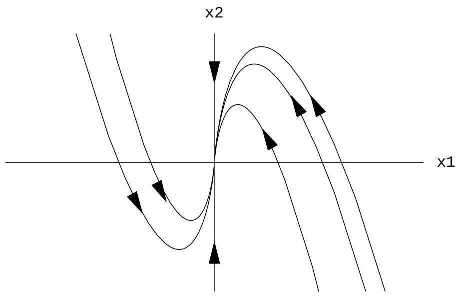
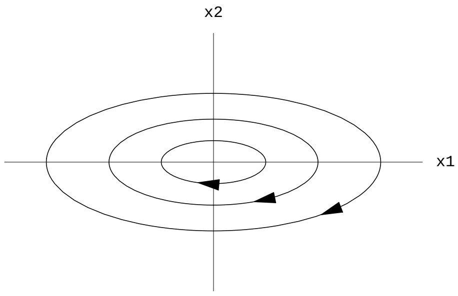


Figure 2: Ex. 8 - 1(b), 2(a) node at the origin



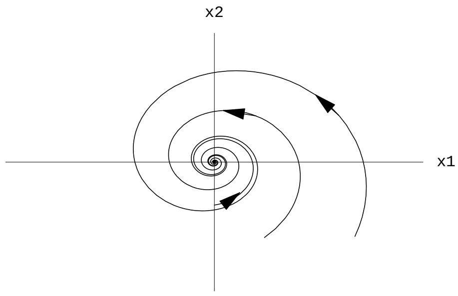


Figure 3: left: Ex. 8 - 3(a) center, right: Ex. 8 - 3(b) focus at the origin

2.4 Nonlinear systems

Given is a continuous function $\mathbf{f}(\mathbf{x}(t))$ and we want to know about the behavior of the solution of

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)) = \begin{pmatrix} f_1(\mathbf{x}(t)) \\ \vdots \\ f_n(\mathbf{x}(t)) \end{pmatrix}$$

in the neighborhood of an equilibrium point $\mathbf{x}(t_0) = \mathbf{x}_0$.

Definition 21. The **Jacobi matrix** of $\mathbf{f}(\mathbf{x}(t))$ is

$$D\mathbf{f}(\mathbf{x}(t)) = \begin{pmatrix} \frac{df_1(\mathbf{x}(t))}{dx_1(t)} & \frac{df_1(\mathbf{x}(t))}{dx_2(t)} & \cdots & \frac{df_1(\mathbf{x}(t))}{dx_n(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{df_n(\mathbf{x}(t))}{dx_1(t)} & \frac{df_n(\mathbf{x}(t))}{dx_2(t)} & \cdots & \frac{df_n(\mathbf{x}(t))}{dx_n(t)} \end{pmatrix}$$

Note: To do this, the partial derivatives have to exist - and they have to be continuous, if we want to be sure that the linearized system has unique solutions.

Definition 22. The system

$$\mathbf{x}'(t) = D\mathbf{f}(\mathbf{x}_0)\mathbf{x}(t)$$

is called the **linearization** of the system

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$$

Note: The linearization is the beginning of the matrix Taylor series at the point $\mathbf{x}_0$ which is similar to the scalar case, where $\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{u}(\mathbf{t})$ and

$$\mathbf{f}(\mathbf{x}_0 + \mathbf{u}(t)) = D\mathbf{f}(\mathbf{x_0})\mathbf{u}(t) + \frac{1}{2}D^2\mathbf{f}(\mathbf{x_0})(\mathbf{u}(t))^2 + \cdots.$$

Thus

$$\begin{aligned} \mathbf{x}'(t) = \mathbf{u}'(t) &= \mathbf{f}(\mathbf{x}_0 + \mathbf{u}(t)) \\ &\approx D\mathbf{f}(\mathbf{x_0})\mathbf{u}(t) \end{aligned}$$

From now on we assume, that the equilibrium point is at the origin, which can easily be reached by affine transformation of coordinates $\mathbf{x} \to \mathbf{x} - \mathbf{x}_0$.

Theorem. *Let I be an open subset of $\mathbf{R}^n$ containing the origin, let $\mathbf{f} \in C^1(E)$. Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Then in a small neighborhood the nonlinear system will behave like the linear.*

1. If $D\mathbf{f}(\mathbf{x}_0)$ is stable, then the origin is an local asymptotic stable equilibrium point for the nonlinear system,
2. if $D\mathbf{f}(\mathbf{x}_0)$ is unstable, then the origin is unstable,
3. otherwise $D\mathbf{f}(\mathbf{x}_0)$ is critical and linearization provides no stability information.

The following example is out of:
Khoo, M.C.K., *Physiological Control Systems*, IEEE Press, New York, 1999

Example 10 (The Bonhoeffer - van der Pol Model (BvP)). Although the Hodkin-Huxley equation set is able to reproduce many features of neuronal dynamics, it constitutes a somewhat unwieldy model, containing several state variables and a large number of empirical constants. Fitzhugh considered the BvP model as a simplified alternative, demonstrating the similarity of the phase-space charcteristics of the former to the reduced phase-space behavior of the Hodgkin-Huxley model. The differential equations representing the BvP model are very similar to those of the van der Pol, these are

$$\frac{dx}{dt} = c\left(y - \frac{x^3}{3} + x + z\right)$$

and

$$\frac{dy}{dt} = -\frac{1}{c}(x - a + by)$$

where a, b, and c are constants that satisfy the following constraints:

$$\begin{gathered} 1 - \tfrac{2b}{3} < a < 1 \\ 0 < b < 1 \\ b < c^2 \end{gathered}$$

The variable z in the equations represents the magnitude of the stimulus applied to the model. For simplification, we will set this term equal to zero. Let's look for the nullclines by setting dx/dt and dy/dt to zero, we get

$$\begin{gathered} y = \frac{x^3}{3} - x \qquad \text{with} \qquad \frac{dx}{dt} = 0 \\ \text{and} \\ y = \frac{(a-x)}{b} \qquad \text{with} \qquad \frac{dy}{dt} = 0. \end{gathered}$$

At the equilibrium point both derivatives have to be zero. Thus both x and y of the above equations must be the same. We get a polynomial of third degree for x, which can be solved. The real valued solution is

$$\begin{aligned} x_0 \;=\; & \frac{54^{1/3}(1-b)}{(-81ab^2 + \sqrt{2916(1-b)^3b^3 + 6561a^2b^4})^{1/3}} \\ & - \frac{(-81ab^2 + \sqrt{2916(1-b)^3b^3 + 6561a^2b^4})^{1/3}}{54^{1/3}b} \end{aligned}$$

and so

$$y_0 = \frac{a}{b} + \frac{-2b + 2b^2 + 2^{1/3}(-3ab^2 + \sqrt{b^3(4(1-b)^3 + 9a^2b)})^{2/3}}{2^{2/3}b^2(-3ab^2 + \sqrt{b^3(4(1-b)^3 + 9a^2b)})^{1/3}}$$

You see, it is useful to calculate the example for specific values a, b and c. We assume

$$a = 0.7 \quad b = 0.8 \quad c = 3$$

It follows, that the equilibrium point is $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1.1994 \\ -0.6243 \end{pmatrix}$.
We can write the equation system in vector form like

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)) = \begin{pmatrix} c(x_1(t) - \dfrac{x_1(t)^3}{3} + x_2(t)) \\ -\dfrac{1}{c}(x_1(t) + bx_2(t) - a) \end{pmatrix}$$

Thus, with

$$D\mathbf{f}(\mathbf{x}(t)) = \begin{pmatrix} c(1 - x_1(t)^2) & c \\ -\frac{1}{c} & -\frac{b}{c} \end{pmatrix} \Rightarrow D\mathbf{f}(\mathbf{x}_0) = \begin{pmatrix} 3(1 - 1.43858) & 3 \\ -\frac{1}{3} & -\frac{0.8}{3} \end{pmatrix}$$

we get the linearized equation

$$\mathbf{x}'(t) = \begin{pmatrix} -1.31574 & 3 \\ -\frac{1}{3} & -\frac{0.8}{3} \end{pmatrix} \mathbf{x}(t)$$

We can solve it with the polynomial equation

$$\lambda^2 + 1.5824\lambda + 1.3509 = 0$$

The eigenvalues are $\lambda_1 = -0.791 + 0.851i$ and $\lambda_2 = \overline{\lambda_1}$. This means that both real parts are negative, thus the linearized matrix is stable and the equilibrium point is an asymptotic stable point for the non linear system. Note that the linearized system looks like Figure 3, right side.

When we want to look at the nonlinearized system, we can draw the equilibrium point and both nullclines in the phase plane $(x_1(t), x_2(t))$. And we can draw on several points the derivatives. Thus we get a picture like

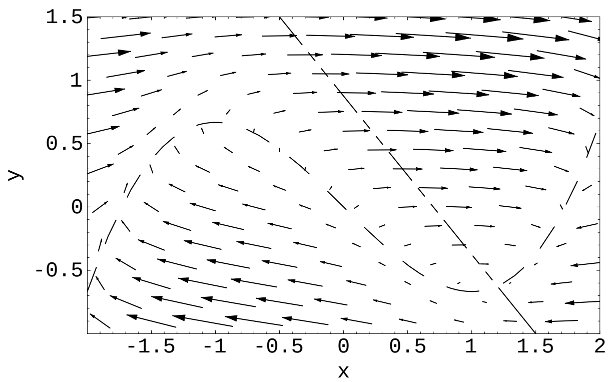


Figure 4: Equilibrium point, Nullclines and some derivatives of the Bonhoeffer - van der Pol Model, with $a = 0.7, b = 0.8c = 3$.