Unsplittable minimal zero-sum subsequences over C_n

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Contents

Background

1961 EGZ Theorem:

Given a sequence S in C_n of length $2n-1$, we can extract a zero subsequence of length n in C_n.

Two examples:

In C_5, let $S = 0^3 1^2 2 3^2 4$, then $0^3 14, 01234$ are two zero subsequences of length 5.
Let $S = 0^4 1^4$, then there is no zero subsequences of length 5.

The Davenport constant

$D(G)$ of a group G is defined as the smallest integer $l \in \mathbb{N}$ such that every sequence S in G with length $|S| \geq l$ contains a zero-sum subsequence.

Notations

Sumset

Let G be an abelian group and $A, B \subset G$ finite non-empty subsets. We denote by

$$A + B = \{a + b, a \in A, b \in B\}$$

the **sumset** of A and B.

Free Monoid, Sequence

Let $\mathcal{F}(G)$ (multiplicatively written) be the free abelian monoid with basis G. An element $S \in \mathcal{F}(G)$ is called a **sequence** (in G) and will be written in the form

$$S = \prod_{g \in G} g^{v_g(S)} = \prod_{i=1}^{l} g_i \in \mathcal{F}(G).$$

Subsequence

A sequence $T \in \mathcal{F}(G)$ is called a **subsequence** of S, if there exists some $W \in \mathcal{F}(G)$ such that $WT = S$. If this holds, then $W = ST^{-1}$.

Sum

$$\sigma(S) = \sum_{v=1}^{l} g_v = \sum_{g \in G} v_g(S) g \in G$$

denotes the **sum** of S

Length

$$|S| = \sum_{g \in G} v_g(S) = l$$

the **length** of S

Subsums

$$\sum(S) = \{\sum_{i \in I} g_i | \emptyset \neq I \subseteq [1, l]\} \subseteq G$$

the set of all possible subsums of S.

Zero-sumfree

We say that the sequence S is
zero-sumfree , if $0 \notin \sum(S)$;
a zero-sum sequence, if $\sigma(S) = 0$;
a minimal zero-sum sequences, if it is zero-sum sequence and each proper subsequence is zero-sumfree.

Splitable, unsplitable

Let S be a minimal zero-sum (resp. zero-sumfree) sequence of elements in an abelian group G, we say that $a \in S$ is **splitable** if there exist two elements $x, y \in G$ such that $x + y = a$ and $Sa^{-1}xy$ is minimal zero-sum (resp. zero-sumfree) sequence as well, otherwise we say that $a \in S$ is **unsplitable**.
we say that a sequence S is **splitable** if there exists an element $a \in S$ such that a is splitable ; S is **unsplitable** if every $a \in S$ is unsplitable.

g-norm

Let G be an abelian group. Let $g \in G$ be a nonzero element with $\mathrm{ord}(g) = n > 1$. For a sequence $S = (n_1 g) \cdots (n_l g)$, where $l \in \mathbb{N}_0$ and $n_1, \cdots, n_l \in [1, n]$, we define

$$||S||_g = \frac{n_1 + \cdots + n_l}{n}$$

to be the g-**norm** of S. If $S = \emptyset$, then set $||S||_g = 0$.

Index

The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. The notion of the index of a sequence was introduced by Chapman, Freeze and Smith in 1999. It was first addressed by Kleitman-Lemke (in the conjecture in 1989), used as a key tool by Geroldinger in 1987, and then investigated by Gao in 2000 in a systematical way.

Definition

Let S be a nonzero sequence for which $< \mathrm{supp}(S) >\subset G$ is a nontrivial finite cyclic group. Then we call

$$\mathrm{index}(S) = \min\{||S||_g | g \in G \text{ with } < \mathrm{supp}(S) >=< g >\} \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the *index* of S.

$\mathbf{I}(C_n)$

Let $\mathbf{I}(C_n)$ be the minimal integer t such that every minimal zero-sum sequence S of at least t elements in C_n satisfies $index(S) = 1$.

$f(S)$

Let $f(S)$ denote $|\sum(S)|$.

$f(G_0, k)$

Let G be a finite abelian group, $G_0(\neq \emptyset) \subseteq G$ a subset and $k \in \mathbb{N}$. Define

$$f(G_0, k) = min\{f(S)\},$$

where $S \in \mathcal{F}(G_0$is a squarefree, zero-sum free sequence with $|S| = k$, and set $f(G_0, k) = \infty$, if there are no sequences in G_0 of the above form.

$\mathcal{B}(G)$

We denote by $\mathfrak{B}(G) = \{S \in \mathfrak{F}(G) : \sigma(S) = 0\}$ the set of all zero-sum sequences, by $\mathfrak{A}(G)$ the set of all minimal zero-sum sequences.

Remark:

Obviously, a zero-sum sequence can be decomposed into a product of some minimal zero-sum sequences (usually the decompositions are not unique). Every minimal zero-sum sequences can be derived from some unsplittable minimal zero-sum sequences.
Therefore, we will study unsplittable minimal zero-sum sequences over finite groups.

Some Problems

Lemke, Kleitman, 1989

Problem LK: Every sequence of n elements in C_n contains a non-empty subsequence T such that $Index(T) = 1$?

Gao, 2000 Integer

Definition The maximum index of minimal zero-sum sequences over C_n is defined as follow:

$$MI(C_n) = \max_S \{ind(S)\},$$

where S runs over all minimal zero sequences of elements in C_n. Gao proposed an upper bound for $MI(C_n)$ as follows.
Conjecture 4.2 $MI(C_n) \leq c \ln n$ for some absolute constant c.

Some Problems

Gao, 2000 Integer

Problem

Determine the value of $\mathbf{I}(C_n)$?
Gao obtained the bounds as: $\lfloor \mathbf{\frac{n+1}{2}} \rfloor + \mathbf{1} \leq \mathbf{I(C_n)} \leq \mathbf{n} - \lfloor \mathbf{\frac{n+1}{3}} \rfloor + \mathbf{1}$ for all $n \geq 8$, and $\mathbf{I}(C_n) = 1$ for $n = 1, 2, 3, 4, 5, 7$, $\mathbf{I}(C_6) = 5$.

Four Zero-sum Conjecture, 2000 Forklore

Conjecture Let n be a positive integer with $gcd(n, 6) = 1$. Suppose S is a minimal zero-sum sequence over C_n with $|S| = 4$, then $index(S) = 1$.

Some Results

Zhuang and Yuan, 2008

$\mathbf{I}(C_n) \leq \lfloor \frac{n}{2} \rfloor + 2$ for $n \geq 8$. For every integer k in $[1, MI(C_n)]$, there exists a minimal zero sequence S with $index(S) = k$.

Savechev-Chen (Discrete Math), Yuan(JCTA), 2007

$\mathbf{I}(C_n) = \lfloor \frac{n}{2} \rfloor + 2$ for $n \geq 8$; $\mathbf{I}(C_n) = 1$ for $n = 1, 2, 3, 4, 5, 7$ and $\mathbf{I}(C_6) = 5$.

Main idea: Determine S and $a \in C_n$ with

$$|\sum(Sa) \setminus \sum(S)| = 0, 1.$$

Some Results

Savchev-Chen

Let G be cyclic of order $n \geq 3$ and $S \in \mathcal{F}(G)$ a sequence of length $|S| \geq \frac{3n-1}{2}$. Then the following statements are equivalent:
(a) S has no zero-sum sequence of length n and $h(S) = v_0(S)$;
(b) $S = S_1 S_2$, where $S_1, S_2 \in \mathcal{F}(G)$ with $||S_1||_g < 1$ and $||g - S_2||_g < 1$ for some $g \in G$ with $ord(g) = n$.

Some Results

Gao and Gerldinger 2008

Definition Let H be an atomic monoid and $k \in \mathbb{N}$.
1. Let V_k denote the set of all $m \in \mathbb{N}$ for which there exist $u_1, \ldots, u_k; v_1, \ldots, v_m \in \mathfrak{A}(H)$ with $u_1 \cdots u_k = v_1 \cdots v_m$.
2. If $H = H^\times$, we set $\rho_k = \lambda_k = k$, and if $H \neq H^\times$, then we define

$$\rho_k = \sup V_k(H), \quad \lambda_k = \min V_k(H).$$

Theorem

Let H be a Krull monoid with cyclic class group G of order $|G| \geq 3$. Then for every $k \in \mathbb{N}$ we have

$$\rho_{2k}(H) \leq k|G| \quad \textit{and} \quad \rho_{2k+1}(H) \leq k|G| + 1$$

Moreover, if every class contains a prime, then equality holds.

Some Results

Gao, Li, Peng, Plyley, Wang2010

Let G be a cyclic group of order $n \geq 2$, where $n = 4k + 2$ for some $k \geq 5$, and let $g \in G$ with $ord(g) = n$. Then the sequence

$$S = g^{n/2-3}\left(\frac{n}{2}g\right)\left((\frac{n}{2}+1)g\right)^{n/2-1}\left((\frac{n}{2}+2)g\right)^{\lfloor\frac{n}{4}\rfloor-2}$$

has no subsequence T with $ind(T) = 1$.

Some Results

Zeng and Yuan 2011 EUJC

If S is a zero-sum sequence, we denote by $\mathfrak{L}(S)$ the maximum of all l such that $S = S_1 \cdot \ldots \cdot S_l$ with $S_i \in \mathfrak{A}(G)$ for all $i \in [1, l]$. In particular, we have $\mathfrak{L}(S) = 1$ for any minimal zero-sum sequence S.

Theorem

. Let G be a cyclic group of order n and $S \in \mathfrak{F}(G)$ a zero-sum sequence with $\mathfrak{L}(S) = k \geq 2$ and $|S| \geq k\frac{n}{2} + 2$. Then there exists some $g \in G$ with $ord(g) = n$ such that $||S||_g = k$.

Some Results

Yuan and Li 2015 Inter J. Number Theory

Theorem

Let n be a positive integer with $\lfloor \sqrt[3]{n} \rfloor \geq 4$, let d be a positive integer with $\lfloor \frac{\sqrt[3]{n}}{2} \rfloor \geq d \geq 2$, and set $n = dm - s$, where $m \geq 8d^2, 0 \leq s < d$. Then

$$S = g^{m-d-1}((n-m+2)g)^d((m-1)g)^{d-1}$$

is an unsplittable minimal zero-sum sequence over $G = C_n = \langle g \rangle$ and $ind(S) = d$. In particular, we may take $d = \lfloor \sqrt[3]{n}/2 \rfloor$, so $MI(C_n) \geq ind(S) = \lfloor \sqrt[3]{n}/2 \rfloor$.

$$|S| = m + d - 2.$$

Zeng, Li and Yuan 2015, Acta Arith.

Theorem

Let $n, d \geq 3$ be positive integers with $d|n$ and $n > d^3$. Let $\frac{n}{d} = d^2 t + r, 0 \leq r < d^2$. Then the sequence

$$S = \left(\frac{n}{d}g\right)^{d-1} g^{dt+r} \prod_{i=1}^{d-1} \left((1 + \frac{in}{d})g\right)^{dt}$$

is an unsplittable minimal zero-sum sequence over C_n . Moreover,

$$ind(S) = \frac{n}{2d} - \frac{dt + r}{2} + 1,$$

$$|S| = \frac{n}{d} + d - 1.$$

Zeng, Li and Yuan 2015, Acta Arith.

Theorem

Let $G = C_n = \langle g \rangle$ be a cyclic group of order n such that $2 \le d | n$ and $n > d^2(d^3 - d^2 + d + 1)$. Then the sequence

$$S = \left(\frac{n}{d}g\right)^{d-1} \left((\frac{n}{d} + d)g\right)^{\lfloor \frac{n}{d^2} \rfloor - d} \prod_{i=0}^{d-1} \left((1 + \frac{in}{d})g\right)^{l},$$

where $l = \frac{n}{d} - d(d-1) - 1$, has no subsequence T with $\mathrm{ind}(T) = 1$ and $|S| > n$.

Some Results

Zeng, Li and Yuan 2015 submitted, Acta Arith.

Let $n > 1$ be an odd integer and G an abelian group of order n. Let S be an unsplittable minimal zero-sum sequence of length $|S| \geq \lfloor \frac{n}{3} \rfloor + 3$ over G. Then G is cyclic and either $S = g^n$ or

$$S = g^{\frac{n-r}{2}-1-tr} \cdot \left(\frac{n+r}{2}g\right)^{2(t+1)} \cdot \left((\frac{n-r}{2}+1)g\right),$$

where g is a generator of G, $r, t \in \mathbb{N}_0$ with r odd and $3 \leq r \leq \frac{n-r}{2} - 1 - tr$. Moreover, $ind(S) = 2$ in the latter case.

Remark:

Xia and Yuan 2010 (Discrete Math) determined all unsplittable minimal zero-sum sequence of length $|S| = \lfloor \frac{n}{2} \rfloor + 1$. Peng and Sun 2014 determined all unsplittable minimal zero-sum sequence of length $|S| = \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - 1$ for prime n

New idea for the proof

Definition

Let G be a finite abelian group and $S = a^r b^t T$ be a sequence over G. If $ua = vb$ with $1 \le u \le r$ and $0 < 2v \le t$, then we can replace b^v by a^u and thus obtain a new sequence $S' = a^{r+u} b^{t-v} T$. This operation is called **Replacement operation** on S.

1. The Replacement operation is an equivalent relation.
2. $supp(S) = supp(S')$
3. S is unsplittable if and only if S' is unsplittable.

Determine S and $a \in C_n$ with

$$|\sum(Sa) \setminus \sum(S)| = 2.$$

Some Results on Four Zero-sum Conjecture

Li, Plyley,Yuan, Zeng 2010(JNT) for prime power n; Li, Peng 2013; Xia, Li 2013 have had some works on the conjecture

Shen, Xia, Li, 2014 Colloq Math

Let n be a positive integer with $gcd(n, 6) = 1$. Suppose $S = (n_1 g)(n_2 g)(n_3 g)(n_4 g), < g >= C_n$ is a minimal zero-sum sequence over C_n with $\gcd(n, n_1 n_2 n_3 n_4) > 1$, then $index(S) = 1$.

Zeng, 2015

Let n be a positive integer with $gcd(n, 30) = 1$. Suppose S is a minimal zero-sum sequence over C_n with $|S| = 4$, then $index(S) = 1$. **Zhiwei Sun 2015.12.20** told to me that the method of Zeng cannot solve the case with $5|n$.

Methods

- sunset sums; Let S be a zero-sum free sequence of elements in an abelian group, and let $S_1, S_2, \cdots, S_k$ be disjoint subsequences of S(i.e. $S_i \cap S_j = \emptyset$ if $i \neq j$). Then $|\sum(S)| \geq |\sum_{i=1}^{k}(S_i)|$.
- techniques from combinatorics

Methods

Lemma

(Gao 2008)*Let G be a cyclic group of order $n \geq 3$. If S is a zero-sum free sequence over G of length*

$$|S| \geq \frac{6n+28}{19},$$

then S contains an element $g \in G$ with multiplicity

$$v_g(S) \geq \frac{6|S|-n+1}{17}.$$

Methods

Lemma

(Xia, Yuan 2010) *Let S be a minimal zero-sum sequence in an abelian group of order n and $S_1, S_2, \ldots, S_k$ be non-empty subsequences such that $S = S_1 S_2 \cdots S_k$. Then*
$|\sum(S_1)| + \cdots + |\sum(S_{k-1})| + |\sum(S_k) \backslash \{\sigma(S_k)\}| < n.$

Lemma

(Yuan :2007) *Let S be an unsplittable minimal zero-sum sequence. If $a, ta \in supp(S)$ with $t \in [2, n-1]$, then $t \geq v_a(S) + 2$.*

Methods

Lemma

(Xia, Yuan [Lemma 2.14],2010) *Let S be a minimal zero-sum sequence in a finite abelian group G. Then an element a in S is unsplittable if and only if $\sum(Sa^{-1}) = G\backslash\{0\}$. Thus S is unsplittable if and only if for every element $a \in supp(S)$, we have $\sum(Sa^{-1}) = G\backslash\{0\}$.*

Lemma

(Xia, Yuan [Lemma 2.15] 2010) *Let S be a minimal zero-sum sequence consisting of two distinct elements. Then S is splittable.*

Methods

Lemma

(1) *We have* $f(k) \geq 2k$ *for* $k \geq 4$, *and* $f(k) \geq \frac{1}{9}k^2$.

(2) *If* p *is prime, then* $f_p(k) \geq \binom{k+1}{2} - \delta$, *where*

$$\delta = \begin{cases} 0, & k \equiv 0 \pmod 2; \\ 1, & k \equiv 1 \pmod 2. \end{cases}$$

Methods

Lemma

(Bhowmik, Halupczok, Schlage-Puchta Math Comp. [Page 2254])

(1) $f_n(3) \geq 6$ *when* $n \geq 7$.

(2) $f_n(4) \geq 10$ *when* $n \geq 11$ *and* $\gcd(n, 6) = 1$.

(3) $f_n(5) \geq 15$ *when* $n \geq 16$ *and* $\gcd(n, 30) = 1$.

(4) $f_n(k) \geq 3k$ *when* $k \geq 5$ *and* $\gcd(n, 30) = 1$.

Some Proofs

Let $S = (a_1 g)^{l_1}(a_2 g)^{l_2}\cdots(a_r g)^{l_r}$ be an unsplittable minimal zero-sum sequence of length $|S| \geq \lfloor \frac{n}{3} \rfloor + 3$ over C_n such that $h(S) = l_1 \geq l_2 \geq \cdots \geq l_r \geq 1$, $a_1 g, a_2 g, \ldots, a_r g$ are distinct nonzero elements of C_n, and $index(S) \geq 2$. $|supp(S)| \geq 3$. Let $h = \mathrm{h}(S)$, $a \in supp(S)$ with $\mathrm{v}_a(S) = h$ and $T = Sa^{-h}$.

Step 1: $h > \frac{n}{17}$ provided $n \geq 8$.

Some Proofs

Step 2: $\mathrm{h}(T) \le 31$.

Proof of Claim 2: Suppose to the contrary that there is $b \in supp(T)$ with $\mathsf{v}_b(T) \ge 32$. By Lemma, there is $k \in [1, t_n]$ and $s \in [-h, h]$ such that $kb = sa$, where $t_n = \left\lceil \frac{n-h}{h+1} \right\rceil \le \left\lceil \frac{n-n/17}{n/17+1} \right\rceil \le 16$ by Claim 1. Since S is minimal zero-sum, s cannot be in $[-h, 0]$. Hence $s \in [1, h]$. If $s > k$, then we can do the replacement operation: replacing b^k by a^s, and obtain a longer sequence $S' = Sa^s b^{-k}$, a contradiction with the definition of S. If $s = k$, then we can do the replacement operation: replacing b^k by a^s, and obtain a new sequence $S' = Sa^s b^{-k}$, which has the same length with S but larger height, that is $\mathrm{h}(S') > h$, also a contradiction. Finally we consider the case $s < k$. Note that $h \ge \mathsf{v}_b(T) \ge 32 \ge 2k > 2s$. We can do the replacement operation: replacing a^s by b^k, and obtain a longer sequence $S' = Sa^{-s}b^k$, a contradiction. This completes the proof of Claim 2.

Some Proofs

Step 3: $h > \frac{n}{4}$ provided $n \geq 2232$.

Proof of Claim 3: Suppose to the contrary that $h \leq \frac{n}{4}$.
Suppose first there is a length 2 subsequence U of T such that $|\sum(a^hU)| \geq 3h+2$. Since $|TU^{-1}| \geq \lfloor n/3 \rfloor + 3 - n/4 - 2 > n/12 \geq 6\max\{31,7\}$ provided $n \geq 2232$, by Lemma $TU^{-1} = T_1 \cdots T_t$, where each T_i is a square free and zero-sum free sequence of length 6 or 7. By Parts (2) and (3) of Theorem, $|\sum(T_i)| \geq 3|T_i|$ for $i \in [1,t]$. Hence by Lemma $\sum(S) \geq |\sum(a^hU)| + \sum_{i=1}^{t} |\sum(T_i)| \geq 3h+2+3(|T|-2) = 3|S|-4 \geq 3(\lfloor n/3 \rfloor + 3) - 4 > n$, a contradiction.
Next suppose that $|\sum(a^hU)| < 3h+2$ for any length 2 subsequence U of T. Let $b \in supp(T)$ with $\mathsf{v}_b(T) \geq 2$. Clearly $b \notin [-h, h+1]a$ by Lemma. Since $|supp(S)| \geq 3$, a^hb^2 is zero-sum free and thus $2b \notin [-h, 0]a$. The inequality $|\sum(a^hb^2)| < 3h+2$ implies that $2b \in [1,h]a$. The same proof as the one of Claim 2, we have $\mathsf{v}_b(S) \leq 3$. Hence $\mathrm{h}(T) \leq 3$.

Some Proofs

Let $g \in supp(T)$. By Lemma, $Tg^{-1} = T_1 T_2 T_3$, where each T_i is a square and zero-sum free sequence of length $|T_i| \geq \lfloor (|S| - h - 1)/3 \rfloor \geq (n + 4 - 3h)/9$. Since S is unsplittable, $|\sum(a^h g)| = 2h + 1$. Hence by Lemma and Part (1) of Theorem, we have

$$\begin{aligned} |\sum(S)| &\geq |\sum(a^h g)| + |\sum(T_1)| + |\sum(T_2)| + |\sum(T_3)| \\ &> 2h + 1 + 3 * \frac{1}{9}\left(\frac{n+4-3h}{9}\right)^2 \\ &= 1 + \frac{1}{243}((n+4-3h)^2 + 486h) \\ &= 1 + \frac{1}{243}(9h^2 - (6n - 462)h + (n+4)^2) \end{aligned}$$

The function $f : h \mapsto 9h^2 - (6n - 462)h + (n+4)^2$ is decreasing in the interval $[0, n/4]$ provided $n \geq 308$. Hence if provided $n \geq 1912$,

Some Proofs

$$\begin{aligned} |\sum(S)| &> 1+\frac{1}{243}\left(\frac{9n^2}{16}-\frac{(6n-462)n}{4}+(n+4)^2\right) \\ &= 1+\frac{1}{243}\left(\frac{n^2}{16}+\frac{247n}{2}+16\right) \\ &> \frac{1}{243}\left(\frac{n^2}{16}+\frac{247n}{2}\right) \\ &\geq n, \end{aligned}$$

a contradiction. This completes the proof of Step 3.
Step 4: $\mathrm{h}(T) \leq 5$.
Proof of Step 4: It is exactly the same as the one of Step 2.

Some Proofs

Claim 5: Let $U = g_1 g_2 \cdots g_r$ be a subsequence of T such that $g_{i+1} - g_i \in [0, h]a$ for all $i \in [1, r-1]$. Then $g_i = g_r$ for all $i \in [3, r]$. Moreover if $r \geq 3$ and $g_2 \neq g_3$, then $g_1 + g_2 + g_3 = a$. As a corollary, $r \leq 7$.

Proof of Claim 5: First we prove $g_i = g_r$ for all $i \in [3, r]$.

Let $t \in [1, r-1]$ be the maximal integer with $g_t \neq g_r$. If $t \leq 2$, we are done. Let $t \geq 3$. Since S is a unsplittable minimal zero-sum sequence, $S' = (Sg_r^{-1}) \cdot (g_r - a) \cdot a$ can be partitioned into two disjoint minimal zero-sum subsequences, of which one contains all a in S' while the other contains $g_r - a$ but no a. Let V be the subsequence containing $g_r - a$ but no a.

We now prove the statement: $g_1 g_2 \cdots g_t (g_r - a) | V$. If not, write $g_1 g_2 \cdots g_t = b_1^{l_1} \cdots b_s^{l_s}$, where $b_1 = g_1$, $b_s = g_t$ and $b_{i+1} - b_i \in [0, h]a$ for $i \in [1, s-1]$, and let $k \in [1, s]$ be the maximal integer such that $b_k^{l_k} \nmid V$. If $k = s$, then $V(g_r - a)^{-1} b_k a^{\ell}$ is a proper zero-sum subsequence of S, where $\ell \in [0, h-1]$ be such that $\ell a = g_r - a - b_k$. This is impossible because S is minimal zero-sum sequence.

Some Proofs

If $k < s$, then $b_{k+1}^{l_{k+1}} | V$ and thus $Vb_{k+1}^{-1} b_k (g_r - a)^{-1} g_r a^{\ell}$, where $\ell \in [0, h-1]$ be such that $\ell a = b_{k+1} - b_k - a$, is a proper zero-sum subsequence of S, a contradiction. This completes the proof of this statement.

Since $|V(g_r - a)^{-1}| \geq t \geq 3$, $V(g_r - a)^{-1}$ contains a subsequence V_0 with sum $\sigma(V_0) \in [-h, h]a$ by Lemma. Let $v_0 \in [-h, h]$ be such that $\sigma(V_0) = v_0 a$. Since S is minimal zero-sum, $v_0 \notin [-h, 0]$. Hence $V(g_r - a)^{-1} g_r V_0^{-1} a^{v_0 - 1}$ is a proper zero-sum subsequence of S, a contradiction. This completes the proof of the first part of this claim.

An Example

Example Let $S = g^{m-4}((n-m+2)g)^3((m-1)g)^2$ where $n = 3m - s$, $1 \leq s \leq 2$ and $m = \lfloor \frac{n}{3} \rfloor + 1$. Then S is an unsplittable minimal zero-sum sequence of length $|S| = \lfloor \frac{n}{3} \rfloor + 2$ and $index(S) = 3$. For example, we may take $n = 65537$ a prime, so $|S| = \lfloor \frac{n}{3} \rfloor + 2 = 21847$ and $index(S) = 3$.

Unsolved Problems

Problem 1: Determine $sp(C_n)$, where $sp(G)$ be the largest integer t such that every MZS of elements in G with $|S| \leq t$ is splitable. Determine the minimal length of all unsplitable MZS over C_n?
Remark: We conjecture that $sp(C_n) \geq c\sqrt{n}$, where c is an absolute constant. We have

$$sp(C_n) \geq d + \frac{n}{d} - 1, \quad d|n.$$

Problem 2: Let $I_n = \{t, |S| = t < n \text{ and } S \text{ is unsplittable over} C_n\}$. Is I_n an interval for $n > 12$? i.e., is $I_n = [sp(C_n), \mathbf{I}(C_n)]$?

Unsolved Problems

Problem 3:

Determine all unsplittable MZS S over C_n with $|supp(S)| = 3$.

Problem 4:

Compute $MI(C_n)$. We conjecture that

$$MI(C_n) \leq \frac{(p-1)n}{2p^2} + 1$$

for composite integer n with least prime divisor p. We conjecture that

$$MI(C_p) \leq c\sqrt{p}$$

when p is an odd prime. But we have not had any precise results of $MI(C_n)$ even for even n.

Unsolved Problems

Let G be an abelian group of rank greater than 1. Let $\mathbf{I}(G)$ be the minimal integer t such that every unsplittable minimal zero-sum sequence S of at least t elements in G satisfies $|S| = D(G)$.

Problem 5:

Determine $\mathbf{I}(G)$ when $rank(G) = 2$ and determine the structure of unsplittable minimal zero-sum sequence S with $|S| = \mathbf{I}(G)$?

Remark: When $G = C_n$, then $\mathbf{I}(G) = \mathbf{I}(C_n)$.

Unsolved Problems

Gao,

Conjecture 5.1. Let G be a cyclic group of prime order and S be a sequence over G of length $|S| = |G|$. Then S has a subsequence T with $ind(T) = 1$ and length $|T| \in [1, h(S)]$.

Let G be a cyclic group of order $n > 2$. We denote by

• $t(n)$ the smallest integer $l \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq l$ has a subsequence T with $ind(T) = 1$.

Open Problem. Determine $t(n)$ for all $n \geq 2$.

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Problem 6:

Four zero-sum conjecture? Determine all MZS S over C_n with $|S| = 5$ and $index(S) = 1$.

THANKS!