The Freiman $3k - 4$ Theorem

David Grynkiewicz

University of Memphis

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Sumsets

Definition

Let G be an abelian group and let $A, B \subseteq G$ be finite, nonempty subsets. Then their sumset is

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General Theme: $|A + B|$ "small" implies A, B and $A + B$ have "structure".

The Freiman $3k - 4$ Theorem

Theorem (Freiman 1959)

Let $A \subseteq \mathbb{Z}$ be a k-element subset with

$$|A + A| = |A| + |A| - 1 + r \leq 3|A| - 4 = 3k - 4.$$

Then there is an arithmetic progression $P_A \subseteq \mathbb{Z}$ with

$$A \subseteq P_A \quad \text{and} \quad |P_A \setminus A| \leq r.$$

The $3k - 4$ Theorem for Distinct Summands

Theorem (Lev and Smeliansky 1995; Freiman 1962)

Let A, $B \subseteq \mathbb{Z}$ *be finite and nonempty with* $\mathrm{diam}(A) \geq \mathrm{diam}(B)$, $\gcd(A - A) = 1$, *and*

$$|A + B| = |A| + |B| - 1 + r \leq |A| + 2|B| - 4.$$

Then there are arithmetic progressions P_A *and* P_B *having common difference* 1 *with*

$$A \subseteq P_A, \quad B \subseteq P_B, \quad |P_A \setminus A| \leq r, \quad \text{and} \quad |P_B \setminus B| \leq r.$$

Here $\mathrm{diam}(A) = \max A - \min A$.

The $3k - 4$ Theorem for Distinct Summands

Theorem (Stanchescu 1996)

Let A, $B \subseteq \mathbb{Z}$ be finite and nonempty with

$$|A+B| = |A| + |B| - 1 + r \leq |A| + |B| + \min\{|A|, |B|\} - 4.$$

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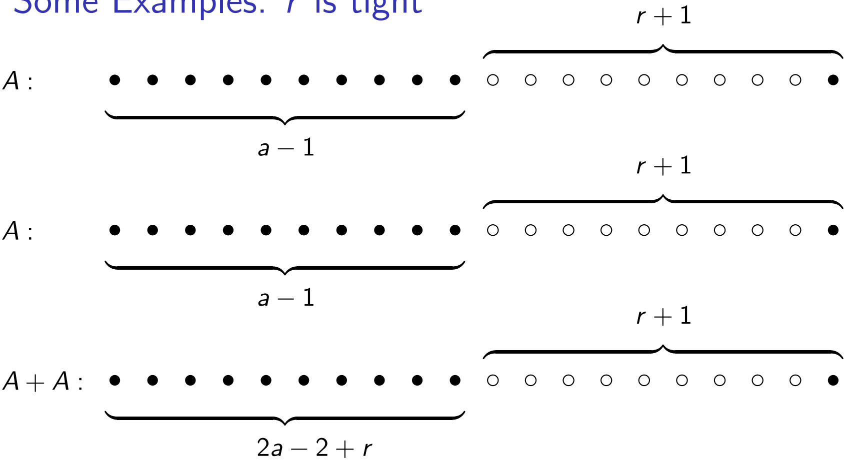
Some Examples: r is tight



$r \geq 0$ and $a \geq r + 2$

$$|A + A| = |A| + |A| - 1 + r \leq 3|A| - 3$$

$$|P_A \setminus A| = r$$

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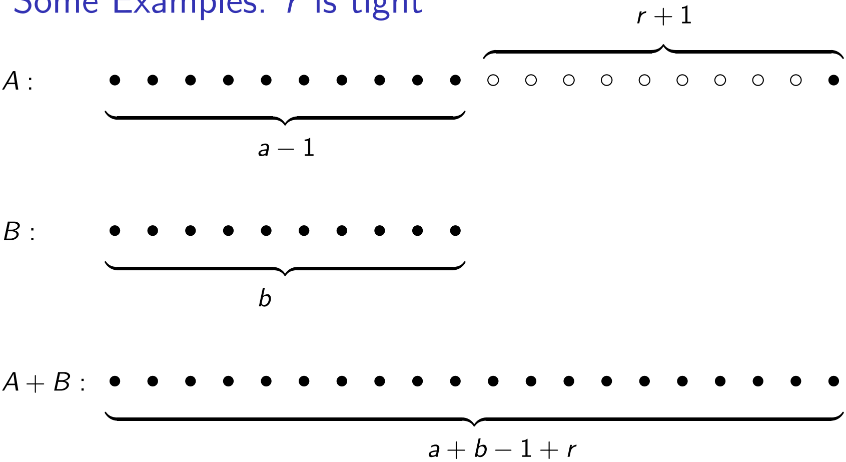

$r \geq 0$ and $b \geq r+1$

$$|A+B| = |A| + |B| - 1 + r \leq |A| + 2|B| - 2$$

$$|P_A \setminus A| = r \qquad \text{and} \qquad |P_B \setminus B| = 0.$$

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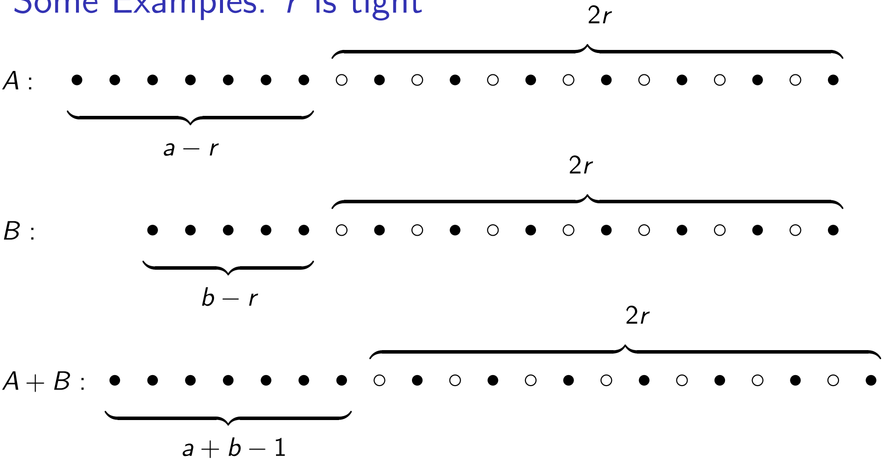

$r \geq 0$, $a \geq r+2$ and $b \geq r+2$.

$$|A+B| = |A| + |B| - 1 + r \leq |A| + |B| - 3 + \min\{|A|, |B|\}$$

$$|P_A \setminus A| = |P_B \setminus B| = r$$

Some Examples: $3k - 4$ is (nearly) tight

If $A = P_1 \cup P_2$ is the union of two arithmetic progressions (of common difference) spaced far enough apart, then

$$|A + A| = (2|P_1| - 1) + (|P_1| + |P_2| - 1) + (2|P_2| - 1) = 3|A| - 3.$$

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Likewise, if $B = P_B$ is also an arithmetic progression of the same difference, then

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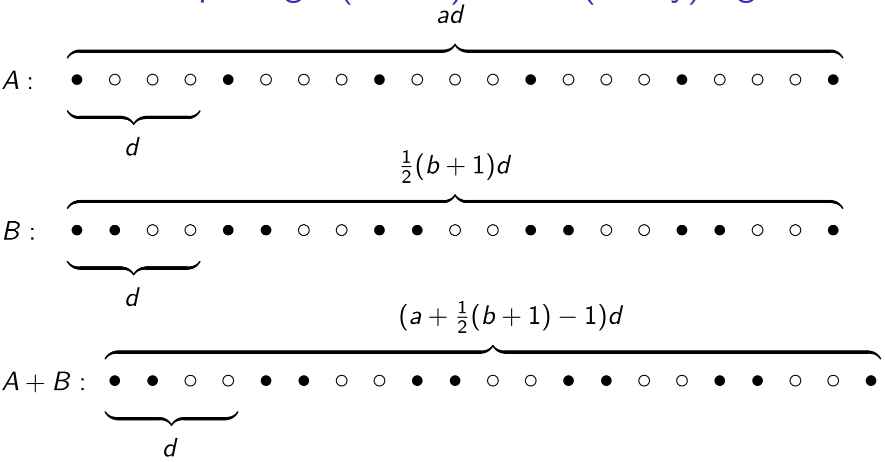
$$|A + B| = (|P_1| + |P_B| - 1) + (|P_2| + |P_B| - 1) = |A| + 2|B| - 2$$

In both cases, A can have arbitrarily many holes, so $|P_A \setminus A|$ is unbounded.

Some Examples: $\gcd(A - A) = 1$ is (nearly) tight



$$\gcd(A - A) = d \geq 3, \quad |A + B| = 2|A| + |B| - 2 \leq |A| + 2|B| - 3,$$

$$|P_A \setminus A| = (d - 1)(|A| - 1) = (d - 1)r \geq 2r.$$

Minor Touch-Ups

Theorem

Let $A, B \subseteq \mathbb{Z}$ *be finite and nonempty with* $\mathrm{diam}(A) \geq \mathrm{diam}(B)$, $\gcd(A-A) \leq 2$, *and*

$$|A+B| = |A| + |B| - 1 + r \leq |A| + 2|B| - 3 - \delta(A,B),$$

where

$$\delta(A,B) = \begin{cases} 1, & \text{if } x + A \subseteq B \text{ for some } x \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

Then there are arithmetic progressions P_A *and* P_B *having common difference* $d = \gcd(A+B-A-B)$ *with*

$$A \subseteq P_A, \quad B \subseteq P_B, \quad |P_A \setminus A| \leq r, \quad \text{and} \quad |P_B \setminus B| \leq r.$$

Minor Touch-Ups

Theorem
Let A, $B \subseteq \mathbb{Z}$ be finite and nonempty with

$$|A+B| = |A|+|B|-1+r \leq |A|+|B|-3+\min\{|A|-\delta(A,B), |B|-\delta(B,A)\}.$$

Then there are arithmetic progressions P_A and P_B having common difference with

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Trios

Definition

A **trio** in an abelian group G is a triple (A, B, C), where $A, B, C \subseteq G$ are *finite* or *cofinite*, such that $A + B + C \neq G$.

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Example

If A, $B \subseteq G$ are finite and $C = -\overline{A + B} := -G \setminus (A + B)$, then

$$0 \notin A + B + C = A + B - \overline{A + B},$$

as $a + b - c = 0$ with $a \in A$, $b \in B$ and $c \notin A + B$ is not possible. So

$$(A, B, -\overline{A + B})$$

is a G-trio.

Key Trio Facts

- The trio (A, B, C) is **nontrivial** if A, B and C are all nonempty.

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$$\delta(A, B, C) = |A| + |B| - |G \setminus C|,$$

  where $|A|, |B| \leq |C|$.
- If G is finite, then $\delta(A, B, C) = |A| + |B| + |C| - |G|$.

A Trio Formulation of the $3k-4$ Theorem

Theorem

Let (A,B,C) be a nontrivial $\mathbb{Z}$-trio. If

$$\delta(A,B,C) > -r \qquad \text{and} \qquad |A|,\ |B|,\ |C| \geq r+3,$$

then there exist subsets P_A, P_B and P_C, each either an arithmetic progression or complement of an arithmetic progression of common difference, such that

$$\begin{array}{lll} A \subseteq P_A, & B \subseteq P_B, & C \subseteq P_C \\ |P_A \setminus A| \leq r, & |P_B \setminus B| \leq r, & |P_C \setminus C| \leq r. \end{array}$$

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Note: If $A,\ B \subseteq \mathbb{Z}$ are finite and nonempty with

$$|A + B| = |A| + |B| - 1 + r \leq |A| + |B| - 4 + \min\{|A|,\ |B|\},$$

then (A, B, C) is a $\mathbb{Z}$-trio, where $C = -\overline{A + B}$, having

$$\delta(A, B, C) = |A| + |B| - |A + B| = -r + 1 \ \text{ and } \ |A|,\ |B|,\ |C| \geq r + 3.$$

What does $C \subseteq P_C$ with $|P_C \setminus C| \leq r$ mean?

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- Note $-\overline{A+B} = C \subseteq P_C$ implies $\overline{A+B} \subseteq -P_C$ implies $-\overline{P_C} \subseteq A+B$.
- Thus $-\overline{P_C} \subseteq A+B$ will be an arithmetic progression of length at least $|C| - r = |A| + |B| - 1 + r - r = |A| + |B| - 1$.

Long Arithmetic Progressions under the $3k - 4$ Theorem hypothesis

Theorem (Bardaji and G 2010; Freiman 2009, $A = B$)

Let $A, B \subseteq \mathbb{Z}$ be finite and nonempty with $\langle A + B - A - B \rangle = \mathbb{Z}$ and let $|A + B| = |A| + |B| - 1 + r$. If either

(i) $|A + B| \le |A| + |B| - 3 + \min\{|B| - \delta(A, B), |A| - \delta(B, A)\}$, *or*

(ii) $\mathrm{diam}\, B \le \mathrm{diam}\, A$, $\gcd(A - A) \le 2$ *and* $|A + B| \le |A| + 2|B| - 3 - \delta(A, B)$,

then $A + B$ contains an arithmetic progression with difference 1 and length at least $|A| + |B| - 1$.

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then $A+B$ contains an arithmetic progression with difference 1 and length at least $|A|+|B|-1$.

This result, combined with the $3k-4$ Theorem, can be used to deduce the Trio Formulation mentioned before, using saturation arguments.

A $3k-4$ Theorem for $\mathbb{Z}/p\mathbb{Z}$?

Conjecture ($3k-4$ Conjecture for $\mathbb{Z}/p\mathbb{Z}$.)
Let (A,B,C) be a nontrivial $\mathbb{Z}/p\mathbb{Z}$-trio, where p is a prime. If

$$\delta(A,B,C) > -r \quad \text{and} \quad |A|,\ |B|,\ |C| \geq r+3,$$

then there exist arithmetic progressions P_A, P_B and P_C of common difference such that

$$\begin{array}{lll} A \subseteq P_A, & B \subseteq P_B, & C \subseteq P_C \\ |P_A \setminus A| \leq r, & |P_B \setminus B| \leq r, & |P_C \setminus C| \leq r. \end{array}$$

A $3k-4$ Theorem for $\mathbb{Z}/p\mathbb{Z}$?

Equivalently:

Conjecture ($3k-4$ Conjecture for $\mathbb{Z}/p\mathbb{Z}$.)

Let $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ be nonempty subsets with p prime and $|A| \geq |B|$. If

$$|A+B| = |A|+|B|-1+r \leq p-r-3 \quad \text{and} \quad r \leq |B|-3,$$

then there exist arithmetic progressions P_A, P_B and P_C of common difference such that

$$\begin{array}{lll} A \subseteq P_A, & B \subseteq P_B, & P_C \subseteq A+B \\ |P_A \setminus A| \leq r, & |P_B \setminus B| \leq r, & |C| \geq |A|+|B|-1. \end{array}$$

Partial Progress in $\mathbb{Z}/p\mathbb{Z}$: Rectification Methods

- If $A + B \subseteq \mathbb{Z}/p\mathbb{Z}$ has $|A \cup B|$ "very small," then

$$A + B \cong A' + B'$$

with $A' + B' \subseteq \mathbb{Z}$, reducing consideration in $\mathbb{Z}/p\mathbb{Z}$ directly to the case of $\mathbb{Z}$.

Freiman Homomorphisms

Let G and G' be abelian groups.

- If $A + B \subseteq G$ is a sumset normalized by translation so that $0 \in A \cap B$, then a map $\psi : A + B \to G'$ is called a (normalized) **Freiman homomorphism** if

$$\psi(a+b) = \psi(a) + \psi(b) \quad \text{for all } a \in A \text{ and } b \in B.$$

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- If $\psi : A + B \to G'$ is injective, then

$$A + B \cong \psi(A) + \psi(B).$$

Universal Ambient Groups and Dimension

- Given a sumset $A + B$, there may be many groups G into which $A + B$ may be embedded, but there is always a "canonical" choice, called the **Universal Ambient Group** (UAG): $\mathcal{U}(A + B)$.

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- $\dim^+(A + B) = \mathrm{rk}(\mathcal{U}(A + B))$ is torsion free rank of $\mathcal{U}(A + B)$.
- If a sumset $A + B$ has an embedding into a torsion-free group, then $\dim^+(A + B) = d$ is the maximal $d \geq 1$ such that $A + B$ has an isomorphic copy $A' + B' \subseteq \mathbb{Z}^d$ with $\langle A' + B' \rangle = \mathbb{Z}^d$.

Partial Progress in $\mathbb{Z}/p\mathbb{Z}$: Rectification Methods

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- If $|A \cup B| \leq \lceil \log_2 p \rceil$, where p is the smallest prime divisor of the torsion subgroup $\text{Tor}(G)$, then $A + B \cong A' + B' \cong \mathbb{Z}$ (Lev, 2008).

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- If $A + B \subseteq G$ has $|A \cup B|$ "very small," then

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- If $|A \cup B| \leq \lceil \log_2 p \rceil$, where p is the smallest prime divisor of the torsion subgroup $\mathrm{Tor}(G)$, then $A + B \cong A' + B' \cong \mathbb{Z}$ (Lev, 2008).
- If $A + A \subseteq \mathbb{Z}/p\mathbb{Z}$ with $|A + A| \leq k|A|$ and $|A| \leq (32k)^{-12k}p$, then $A + A \cong A' + A' \subseteq \mathbb{Z}$ (Green and Ruzsa 2006; Bilu, Lev and Ruzsa 1998, weaker bounds).

Partial Progress in $\mathbb{Z}/p\mathbb{Z}$: Rectification Methods

Thus the $3k-4$ conjecture holds for $A+B \subseteq \mathbb{Z}/p\mathbb{Z}$ provided:

- $|A \cup B| \leq \lceil \log_2 p \rceil$, or
- $A = B$ and $|A| \leq cp$ for a *very* small constant $c > 0$, or

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- $A = B$ and $|A| \leq cp$ for a *very* small constant $c > 0$, or
- $||A| - |B|| \leq N$ and $|A \cup B| \leq c_N p$ for an even smaller constant $c_N > 0$ that depends on N (via Plünnecke bounds).

Partial Progress in $\mathbb{Z}/p\mathbb{Z}$: Refined Rectification via Exponential Sums

Lemma (Lev 2004 and 2007; Freiman 1961, weaker version)

Let $\varphi \in (0, \pi]$ be a real number and let $z_1 \cdot \ldots \cdot z_N$ be a sequence of complex numbers z_i from the unit circle such that every open arc of length φ contains at most n terms from the sequence S. Then

$$|\sum_{i=1}^{N} z_i| \leq 2n - N + 2(N-n)\cos(\varphi/2).$$

Partial Progress in $\mathbb{Z}/p\mathbb{Z}$: Refined Rectification via Exponential Sums

Theorem (Freimain 1961; Nathanson 1995; Roth 2006; G 2013)

Let $A+B \subseteq \mathbb{Z}/p\mathbb{Z}$ with

$$|A+B| = |A|+|B|-1+r \qquad \text{and} \qquad |A| \geq |B|.$$

Under any of the following conditions, the $3k-4$ Theorem holds for $A+B$.

$$\begin{array}{llll} A=B, & r \leq 0.4|B|-2 & \text{and} & |A+A| \leq 0.2125p; \\ A=B, & r \leq 0.29|B|-2 & \text{and} & |A+A| \leq \frac{p-1}{2}; \\ |A| \leq \frac{4}{3}|B|, & r \leq 0.05|B|-2 & \text{and} & |A+B| \leq \frac{p}{225}; \\ |A| \leq 1.12|B|, & r \leq 0.12|B|-2 & \text{and} & |A+B| \leq \frac{p}{55}; \\ |A| = |B|, & r \leq 0.15|B|-2 & \text{and} & |A+B| \leq 0.036p. \end{array}$$

Partial Progress in $\mathbb{Z}/p\mathbb{Z}$: Rectification+Plünnecke+Trios+UAG+Isoperimetric Method

Theorem (G 2013)

Let $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ with

$$|A+B| = |A|+|B|-1+r \leq p-r-3 \quad \text{and} \quad |A| \geq |B|.$$

If

$$r \leq |B|-3 \quad \text{and} \quad r \leq cp-1.2, \quad \text{where } c = 3.1 \cdot 10^{-1549},$$

then the $3k-4$ Conjecture holds for $A+B$.

Beyond $|A+B| \leq |A|+|B|-4+\min\{|A|, |B|\}$

- Problem: Finding similar precise bounds for the covering progression when $|A+B| > |A|+|B|-4+\min\{|A|, |B|\}$ when $A+B \subseteq \mathbb{Z}$.

Dimension and Sumset Cardinality

Theorem (Ruzsa 1994)

Let A, $B \subseteq \mathbb{Z}$ *be finite and nonempty with* $\dim^+(A+B) \geq d$ *with* $|A| \geq |B|$. *If* $\dim^+(A+B) \geq d$, *then*

$$|A+B| \geq |A| + d|B| - \frac{1}{2}d(d+1).$$

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In particular ($d = 3$): if $A + B$ is at least 3 dimensional, then $|A+B| \geq |A| + 3|B| - 6$.

Dimension and Sumset Cardinality

Theorem (G and Serra 2010)

Let $s \geq 2$ be an integer. Let A, $B \subseteq \mathbb{R}^2$ be finite subsets with $|A| \geq |B| \geq 2s^2 - 3s + 2$. If

$$|A + B| < |A| + (3 - \frac{2}{s})|B| - 2s + 1,$$

then there is a line ℓ such that each of A and B can be covered by at most $s - 1$ parallel translates of ℓ.

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then there is a line ℓ such that each of A and B can be covered by at most $s - 1$ parallel translates of ℓ.

In particular ($s = 3$): If $A + B \subseteq \mathbb{Z}$ has $|A + B| < |A| + \frac{7}{3}|B| - 5$, then either $\dim^{+}(A + B) = 1$ or $A + B$ has an isomorphic copy $A' + B' \subseteq \mathbb{Z}^2$ with A' and B' covered by two parallel lines.

The 2-Dimensional Case

Theorem (G 2016; Stanchescu 1998, $A = B$)

Let A, $B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets each covered by 2 horizontal lines. Suppose $\langle A + B - A - B \rangle = \mathbb{Z}^2$, $|A| \geq |B|$ and

$$|A + B| = |A| + 2|B| - 2 + r - \delta(A, B) \leq |A| + \frac{19}{7}|B| - 5.$$

Then there exist subsets P_A, P_B, $P \subseteq \mathbb{Z}^2$, each the union of two arithmetic progressions with difference $(1, 0)$, such that, after translating A and B appropriately,

$$A \subseteq P_A, \quad |P_A \setminus A| \leq r, \quad B \subseteq P_B, \quad |P_B \setminus B| \leq r, \quad A \cup B \subseteq P, \quad \text{and}$$

$$|P \setminus A| + |P \setminus B| \leq 2r + 2 + \Big||P_A| - |P_B|\Big| - \Big||A| - |B|\Big| \leq 3r + 2,$$

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$$|P \setminus A| + |P \setminus B| \leq 2r + 2 + \Big||P_A| - |P_B|\Big| - \Big||A| - |B|\Big| \leq 3r + 2,$$

Moreover, $|P \setminus A| + |P \setminus B| \leq 2r + 2 - \Big||A| - |B|\Big|$ *unless either*

$$P_B \subseteq P_A = P \qquad \text{and} \qquad |P \setminus A| + |P \setminus B| = 2|P_A \setminus A| + |A| - |B|, \qquad \text{or}$$
$$P_A \subseteq P_B = P \qquad \text{and} \qquad |P \setminus A| + |P \setminus B| = 2|P_B \setminus B| + |B| - |A|.$$

The 1-Dimensional Case?

Conjecture

Let $A, B \subseteq \mathbb{Z}$ *be finite, nonempty subsets with* $|A| \geq |B|$ *and* $\dim^+(A+B) = 1$. *If*

$$|A+B| = |A| + 2|B| - 2 + r - \delta(A,B) \quad \text{and} \quad 0 \leq r \leq |B| - 5,$$

then there arithmetic progressions P_A *and* P_B *of common difference with*

$$A \subseteq P_A, \quad B \subseteq P_B, \quad \text{and} \quad |P_A \setminus A|, |P_B \setminus B| \leq |B| - \delta(A,B) + 2r.$$

Other Ways Beyond $|A+B| \le |A| + |B| + \min\{|A|, |B|\} - 4$

- Problem: Finding similar precise bounds for the covering progression when $|A+B| > |A| + |B| - 4 + \min\{|A|, |B|\}$ when $A + B \subseteq \mathbb{Z}$.
- Question: If we have $\operatorname{diam} B < 2 \operatorname{diam} A$ instead of $\operatorname{diam} B \le \operatorname{diam} A$, can we achieve $|P_A \setminus A| \le r$ with a weaker bound than $|A+B| \le |A| + |B| - 4 + \min\{|A|, |B|\}$?
- How about when $\operatorname{diam} B < k \operatorname{diam} A$?

Thanks!