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Monoids of weighted zero-sum sequences

This is a joint work with A. Geroldinger and F. Halter-Koch

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Outline

Weighted zero-sum sequences

Algebraic properties

Arithmetic properties

Sequences over G

Let $(G,+,0)$ be an abelian group and let $G_0 \subset G$ nonempty.
By a sequence over G_0, we mean

- a finite unordered sequence with terms from G_0 and repetition of terms allowed;
- a multiset with elements taking from G_0;
- an element of the free abelian group $(\mathcal{F}(G_0),\cdot)$ with basis G_0, where the operation '$\cdot$' is the concatenation of sequences.

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- an element of the free abelian group $(\mathcal{F}(G_0),\cdot)$ with basis G_0, where the operation '$\cdot$' is the concatenation of sequences.

Then a sequence over G_0 is also a sequence over G. Let

$$S = g_1 \cdot g_2 \cdot \ldots \cdot g_\ell = g_2 \cdot g_1 \cdot g_3 \cdot \ldots \cdot g_\ell = \prod_{g\in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G)$$

be a sequence over G. Then we say

- $\mathsf{v}_g(S) \in \mathbb{N}_0$ is the *multiplicity* of g in S;
- $|S| = \ell = \sum_{g\in G} \mathsf{v}_g(S)$ is the *length* of S.

Weighted zero-sum sequences

Let $S = g_1 \cdot g_2 \cdot \ldots \cdot g_\ell \in \mathcal{F}(G_0)$ and let $\Gamma \subset \mathrm{End}(G)$ be a nonempty subset of the endomorphism group of G.

We say

- S is a *zero-sum sequence* if $\sigma(S) = g_1 + g_2 + \ldots + g_\ell = 0$;
- S is a *Γ-weighted zero-sum sequence* if there exist $\gamma_1, \ldots, \gamma_\ell \in \Gamma$ such that $\gamma_1(g_1) + \ldots + \gamma_\ell(g_\ell) = 0$.

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Let $\mathcal{B}(G_0)$ denote the *monoid of zero-sum sequences over* G_0, and let $\mathcal{B}_\Gamma(G_0)$ denote the *monoid of Γ-weighted zero-sum sequences over* G_0. Then $\mathcal{B}(G_0) \subset \mathcal{B}_\Gamma(G_0) \subset \mathcal{F}(G_0)$.

In particular, if $\Gamma = \{\mathrm{id}_G\}$, then $\mathcal{B}_\Gamma(G_0) = \mathcal{B}(G_0)$.

Examples

- Let $\gamma \in \mathrm{End}(G)$ and $\Gamma = \{\gamma\}$. Then

$$\mathcal{B}_\Gamma(G) = \{S \in \mathcal{F}(G) \colon \sigma(S) \in \ker(\gamma)\}\,.$$

In particular, if $\gamma \in \mathrm{Aut}(G)$, then $\mathcal{B}_\Gamma(G) = \mathcal{B}(G)$; if $\gamma(g) = 0$ for all $g \in G$, then $\ker(\gamma) = G$ and hence $\mathcal{B}_\Gamma(G) = \mathcal{F}(G)$.

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- Let $\Gamma \supset \{\mathrm{id}_G, -\mathrm{id}_G\}$. Then for every $S \in \mathcal{F}(G)$, we have $S^2 \in \mathcal{B}_\Gamma(G)$. If $G = C_3 = \{0, g, -g\}$, then

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- Let $\Gamma = \mathrm{Aut}(G)$ and $G = C_3^r$. Then

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- Interesting case: $\Gamma \subset \mathrm{Aut}(G)$ with $\{\mathrm{id}_G, -\mathrm{id}_G\} \subset \Gamma$. We will study the algebraic and arithmetic properties of $\mathcal{B}_\Gamma(G)$.

Motivation

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- There are transfer homomorphisms from norm monoids (of Galois-invariant orders in algebraic number fields) and from monoids of positive integers (that can be represented by binary quadratic forms) to monoids of weighted zero-sum sequences.
- Arithmetic investigations of monoids of weighted zero-sum sequences were initiated by Schmid et al.

Outline

seminormal, root closed, and completely integrally closed

A *monoid* H means a multiplicative commutative cancellative semigroup with identity element $1_H = 1 \in H$. We denote by $\mathsf{q}(H)$ the quotient group of H and denote

- by $H' = \{x \in \mathsf{q}(H) \colon \text{there is some } N \in \mathbb{N} \text{ such that } x^n \in H \text{ for all } n \geq N\}$ the *seminormalization* of H;
- by $\widetilde{H} = \{x \in \mathsf{q}(H) \colon \text{there is some } n \in \mathbb{N} \text{ such that } x^n \in H\}$ the *root closure* of H;
- by $\widehat{H} = \{x \in \mathsf{q}(H) \colon \text{there is some } c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}\}$ the *complete integral closure* of H.

Then $H \subset H' \subset \widetilde{H} \subset \widehat{H} \subset \mathsf{q}(H)$.

H is said to be *seminormal, root closed, or completely integrally closed* if $H = H'$, $H = \widetilde{H}$, or $H = \widehat{H}$.

Krull, weakly Krull, and transfer Krull

A monoid H is said to be

- a *weakly Krull monoid* if $H = \bigcap_{\mathfrak{p} \in \mathfrak{X}(H)} H_\mathfrak{p}$ and $\{\mathfrak{p} \in \mathfrak{X}(H) \colon a \in \mathfrak{p}\}$ is finite for all $a \in H$.
- a *Krull monoid* if H is weakly Krull and $H_\mathfrak{p}$ is a discrete valuation monoid for all $\mathfrak{p} \in \mathfrak{X}(H)$,
  or equivalently, if there is a divisor homomorphism $\varphi \colon H \to D$, where D is a free abelian monoid;
- a *transfer Krull monoid* if there is a transfer homomorphism $\theta \colon H \to B$, where B is a Krull monoid.

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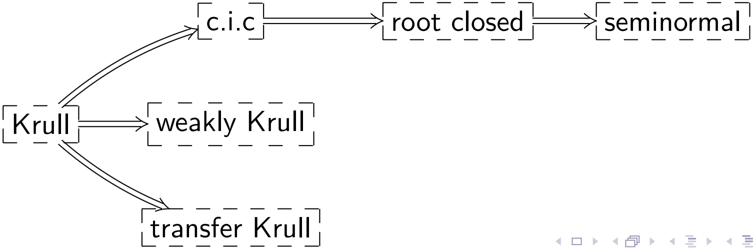

Theorem (Geroldinger+Halter-Koch+Z., 2022)

Let G be a finite abelian group and let $\Gamma \subset \mathrm{Aut}(G)$ be a subset with $\{\mathrm{id}_G, -\mathrm{id}_G\} \subset \Gamma$. Then the following statements are equivalent.

(a) $\mathcal{B}_\Gamma(G)$ *is root closed.*

(b) $\mathcal{B}_\Gamma(G)$ *is Krull.*

(c) $\mathcal{B}_\Gamma(G)$ *is transfer Krull.*

(d) $\mathcal{B}_\Gamma(G)$ *is weakly Krull.*

(e) G *is an elementary* 2*-group and* $\Gamma = \{\mathrm{id}_G, -\mathrm{id}_G\}$.

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Remark: Suppose $G = C_2^r$. Then $\mathrm{id}_G = -\mathrm{id}_G$.

- If $\Gamma = \{\mathrm{id}_G, -\mathrm{id}_G\}$, then $B_\Gamma(G) = \mathcal{B}(G)$ is Krull.
- If $\{\mathrm{id}_G, -\mathrm{id}_G\} \subsetneq \Gamma$, then $r \geq 2$ and there exist $\gamma \in \Gamma \setminus \{\mathrm{id}_G\}$, $e \in G$ such that $\gamma(e) \neq e$. Note that e^2 and $e^2(e + \gamma(e))$ are both Γ-weighted zero-sum, whence $B_\Gamma(G)$ is not root closed.

Theorem (Geroldinger+Halter-Koch+Z., 2022)

Let G be a finite abelian group, $|G| > 1$, and let $\Gamma \subset \mathrm{Aut}(G)$ be a subset such that $\{\mathrm{id}_G, -\mathrm{id}_G\} \subset \Gamma$.

1. *If $\mathcal{B}_\Gamma(G)$ is seminormal, then $\exp(G)$ is a power of 2.*

   Assume to the contrary that there exists $g \in G$ such that $\mathrm{ord}(g) = m \geq 3$ is odd. Then g^m and g^2 are both Γ-weighted zero-sum, whence $g \in (\mathcal{B}_\Gamma(G))' \setminus \mathcal{B}_\Gamma(G)$.

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1. *If $\mathcal{B}_\Gamma(G)$ is seminormal, then $\exp(G)$ is a power of 2.*
2. *$\mathcal{B}_\pm(G)$ is seminormal if and only if $\exp(G) \mid 4$. If this holds, then*
$$\widehat{\mathcal{B}_\pm(G)} = \{S \in \mathcal{F}(G) \colon \sigma(S) \in 2G\}.$$

(⇒) Assume to the contrary that there exists $g \in G$ such that $\mathrm{ord}(g) = 8$. Let $S = g(5g)$. Then S^2 and $S^3 = (g^3(5g))(5g)^2$ are both $\pm$-weighted zero-sum, whence $S \in (\mathcal{B}_\pm(G))' \setminus \mathcal{B}_\pm(G)$.

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$$\widehat{\mathcal{B}_\pm(G)} = \{S \in \mathcal{F}(G) \colon \sigma(S) \in 2G\}.$$
3. *Let $\Gamma = \mathrm{Aut}(G)$.*
   (i) *$\mathcal{B}_\Gamma(G)$ is seminormal if and only if $G \cong C_{2^{t_1}} \oplus \ldots \oplus C_{2^{t_r}}$, where $r, t_1, \ldots, t_r \in \mathbb{N}$ and $t_1 < \ldots < t_r$.*
   (ii) *Let $G \cong C_{2^{t_1}} \oplus \ldots \oplus C_{2^{t_r}}$, where $r, t_1, \ldots, t_r \in \mathbb{N}$ and $t_1 < \ldots < t_r$. For $S = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{F}(G)$, where $\ell \in \mathbb{N}$ and $g_1, \ldots, g_\ell \in G$, we set $\mathrm{n}(S) = |\{j \in [1, \ell] \colon \mathrm{ord}(g_j) = 2^{t_r}\}|$. Then*
$$\widehat{\mathcal{B}_\Gamma(G)} = \{S \in \mathcal{F}(G) \colon \mathrm{n}(S) \text{ is even } \}.$$

C-monoid

- Let H be a submonoid of a factorial monoid F.
- We define a congruence relation on F. We write $y \sim y'$, if for all $x \in F$, we have that

$$xy \in H \quad \text{if and only if} \quad xy' \in H\,.$$

We denote by $[y]_H^F = [y]$ its congruence class.

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  We denote by $[y]_H^F = [y]$ its congruence class.
- $\mathcal{C}^*(H,F) = \{[y] \colon y \in (F \setminus F^\times) \cup \{1\}\} \subset \mathcal{C}(H,F) = \{[y] \colon y \in F\}$ are commutative semigroups with identity element $[1]$.
- $\mathcal{C}(H,F)$ is the *class semigroup* and $\mathcal{C}^*(H,F)$ is the *reduced class semigroup* of H in F.
- A monoid H is said to be a C-*monoid* (defined in a factorial monoid F) if it is a submonoid of F such that $H \cap F^\times = H^\times$ and $\mathcal{C}^*(H,F)$ is finite.

Clifford semigroup

- Let $\mathcal{C}$ be an additively written, finite, commutative semigroup with zero element.
- Let $\mathsf{E}(\mathcal{C})$ be the set of idempotents of $\mathcal{C}$, endowed with the Rees order, defined by $e \le f$ if $e + f = e$.
- If $\mathsf{E}(\mathcal{C}) = \{0 = e_0, \ldots, e_n\}$, then $0 = e_0$ is the largest and $e_0 + \ldots + e_n$ is the smallest element of $\mathsf{E}(\mathcal{C})$ in the Rees order.

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- For every $e \in \mathsf{E}(\mathcal{C})$, we denote by $\mathcal{C}_e$ the set of all $x \in \mathcal{C}$ such that $x + e = x$ and $x + y = e$ for some $y \in \mathcal{C}$. Then $\mathcal{C}_e$ is a group with identity element e, called the *constituent group* of e. If $e, f \in \mathsf{E}(\mathcal{C})$ and $e \ne f$, then $\mathcal{C}_e \cap \mathcal{C}_f = \emptyset$.

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- $\mathcal{C}$ is a union of groups if and only if

$$\mathcal{C} = \biguplus_{e \in \mathsf{E}(\mathcal{C})} \mathcal{C}_e,$$

and, if this is the case, then $\mathcal{C}$ is called a *Clifford semigroup*.

Theorem (Boukheche, Merito, Ordaz, and Schmid, 2022)

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Definition

Let $S = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{F}(G)$. We define

$$\sigma_\pm(S) = \{\epsilon_1 g_1 + \ldots + \epsilon_\ell g_\ell \colon \epsilon_1, \ldots, \epsilon_\ell \in \{1, -1\}\}.$$

Theorem (Geroldinger+Halter-Koch+Z., 2022)

Let $G = C_2^r \oplus C_4^t$ *and set* $\mathcal{C} = \mathcal{C}(\mathcal{B}_\pm(G), \mathcal{F}(G))$.

1. *Let* $S \in \mathcal{F}(G)$. *Then* $[S] \in \mathsf{E}(\mathcal{C})$ *if and only if* $S \in \mathcal{B}_\pm(G)$ *if and only if* $\sigma_\pm(S)$ *is a subgroup of* $2G$.

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2. *Let* $G_1 \subset 2G$ *be a subgroup, let* $g_1, \ldots, g_s \in G$ *such that* $(2g_1, \ldots, 2g_s)$ *is a basis of* G_1, *and set* $S_0 = g_1^2 \cdot \ldots \cdot g_s^2 \in \mathcal{B}_{\pm}(G)$. *Then* $\sigma_{\pm}(S_0) = G_1$.

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3. *Let* $S_1, S_2 \in \mathcal{B}_\pm(G)$. *Then* $[S_1] = [S_2]$ *if and only if* $\mathcal{C}_{[S_1]} = \mathcal{C}_{[S_2]}$ *if and only if* $\sigma_\pm(S_1) = \sigma_\pm(S_2)$.

$$|\mathsf{E}(\mathcal{C})| = \#\text{ subgroups of } 2G \cong C_2^{r+t} = \#\text{ subspaces of } \mathbb{F}_2^{r+t}$$

$$= 1 + (2^{r+t} - 1) + \frac{(2^{r+t}-1)(2^{r+t}-2)}{(2^2-1)(2^2-2)} + \ldots$$

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4. $\mathcal{C}$ *is a Clifford semigroup and* $\{\mathcal{C}_{[S]} \colon S \in \mathcal{B}_{\pm}(G)\}$ *is the set of its constituent groups.*

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3. *Let* $S_1, S_2 \in \mathcal{B}_\pm(G)$. *Then* $[S_1] = [S_2]$ *if and only if* $\mathcal{C}_{[S_1]} = \mathcal{C}_{[S_2]}$ *if and only if* $\sigma_\pm(S_1) = \sigma_\pm(S_2)$.
4. $\mathcal{C}$ *is a Clifford semigroup and* $\{\mathcal{C}_{[S]} \colon S \in \mathcal{B}_\pm(G)\}$ *is the set of its constituent groups.*
5. *Let* $S_0 \in \mathcal{B}_\pm(G)$. *Then*
$$\mathcal{C}_{[S_0]} = [S_0] + \{[g] \colon g \in G \ \text{ such that } \ 2g \in \sigma_\pm(S_0)\} \cong C_2^{r+t}.$$

Theorem (Geroldinger+Halter-Koch+Z., 2022)

Let $G = C_2^r \oplus C_4^t$ *and set* $\mathcal{C} = \mathcal{C}(\mathcal{B}_\pm(G), \mathcal{F}(G))$.

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6. *Let* $S_1, S_2 \in \mathcal{F}(G)$. *Then*
$$[S_1] + [S_2] = [S_1 S_2] = [S_1^2 S_2^2] + [\sigma(S_1 S_2)].$$

Outline

Weighted zero-sum sequences

Algebraic properties

Arithmetic properties

System of sets of lengths

Let H be a monoid and denote by $\mathcal{A}(H)$ the set of its *atoms* (i. e., irreducible elements).

- If $a = u_1 \cdot \ldots \cdot u_k$ where $u_1, \ldots, u_k \in \mathcal{A}(H)$, then
- k is called the length of the factorization, and
- $\mathsf{L}_H(a) = \{k \colon a \text{ has a factorization of length } k\} \subset \mathbb{N}$ is the set of lengths of a.
- By convention, we let $\mathsf{L}(a) = \{0\}$ if $a \in H^\times$.
- The system of all sets of lengths

$$\mathcal{L}(H) = \{\mathsf{L}(a) \colon a \in H\}.$$

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A monoid H is called

- atomic if $\mathsf{L}(a) \neq \emptyset$ for all $a \in H$.
- a BF-monoid if H is atomic and L is finite for all $L \in \mathcal{L}(H)$.
- half-factorial if $|L| = 1$ for all $L \in \mathcal{L}(H)$.

Let $G = C_{n_1} \oplus C_{n_2} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$ and let $t \in [1, r]$ be minimal such that n_t is even.
We denote

$$\mathsf{D}(G) = \sup\{|U| \colon U \in \mathcal{A}(\mathcal{B}(G))\}$$
$$\text{and } \mathsf{D}_{\pm}(G) = \sup\{|U| \colon U \in \mathcal{A}(\mathcal{B}_{\pm}(G))\}\,.$$

Then $\mathsf{D}(G) \ge \mathsf{D}^*(G) := 1 + \sum_{i=1}^{r}(n_i - 1)$ and

$$\mathsf{D}(G) \ge \mathsf{D}_{\pm}(G) \ge 1 + \sum_{i=1}^{t-1}(n_i - 1) + 1 + \sum_{i=t}^{r} \frac{n_i}{2}\,.$$

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Theorem (Boukheche, Merito, Ordaz, and Schmid, 2022)

1. *Let G be an abelian group such that $|G|$ is odd.*
   - $\mathcal{A}(\mathcal{B}(G)) \subset \mathcal{A}(\mathcal{B}_\pm(G))$;
   - $\mathsf{D}(G) = \mathsf{D}_\pm(G)$;
   - $\mathsf{L}_{\mathcal{B}(G)}(B) \subset \mathsf{L}_{\mathcal{B}_\pm(G)}(B)$ *for every* $B \in \mathcal{B}(G)$.
2. *Let G be a cyclic group of even order n. Then* $\mathsf{D}_\pm(G) = 1 + \frac{n}{2}$.

Union of sets of lengths

For every $k \in \mathbb{N}$, we let

$$\mathcal{U}_k(H) = \bigcup_{k \in L \in \mathcal{L}(H)} L \,.$$

Furthermore, we set $\rho_k(H) = \sup \mathcal{U}_k(H)$ and $\lambda_k(H) = \min \mathcal{U}_k(H)$.

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Proposition

Let G be a finite abelian group.

1. *If $|G| \le 2$, then $\mathcal{U}_k(\mathcal{B}(G)) = \{k\}$ for all $k \in \mathbb{N}$.*
2. *If $|G| \ge 3$, then, for all $k \in \mathbb{N}_0$, we have $\mathcal{U}_{k+1}(\mathcal{B}(G)) = [\lambda_{k+1}(\mathcal{B}(G)), \rho_{k+1}(\mathcal{B}(G))]$ and if $k\mathsf{D}(G) + j \ge 1$, then*

$$\lambda_{k\mathsf{D}(G)+j}(\mathcal{B}(G)) = \begin{cases} 2k, & j = 0 \\ 2k+1, & j \in [1, \rho_{2k+1}(G) - k\mathsf{D}(G)] \\ 2k+2, & j \in [\rho_{2k+1}(G) - k\mathsf{D}(G) + 1, \mathsf{D}(G) - 1] \end{cases}$$

Theorem (Boukheche, Merito, Ordaz, and Schmid, 2022)

Let G be a finite abelian group with $|G| \geq 3$. Then for all $k \in \mathbb{N}_0$, we have $\mathcal{U}_{k+1}(\mathcal{B}_{\pm}(G)) = [\lambda_{k+1}(\mathcal{B}_{\pm}(G)), \rho_{k+1}(\mathcal{B}_{\pm}(G))]$ and if $k\mathsf{D}_{\pm}(G) + j \geq 1$, then

$$\lambda_{k\mathsf{D}_{\pm}(G)+j}(\mathcal{B}_{\pm}(G)) = \begin{cases} 2k, & j = 0 \\ 2k+1, & j \in [1, \rho_{2k+1}(\mathcal{B}_{\pm}(G)) - k\mathsf{D}_{\pm}(G)] \\ 2k+2, & j \in [\rho_{2k+1}(\mathcal{B}_{\pm}(G)) - k\mathsf{D}_{\pm}(G) + 1, \mathsf{D}(G)_{\pm} - 1] \end{cases}$$

Lemma

Let G be a finite abelian group. Then $\rho_{2k}(\mathcal{B}(G)) = k\mathsf{D}(G)$ and $\rho_{2k}(\mathcal{B}_{\pm}(G)) = k\mathsf{D}_{\pm}(G)$.

Theorem (Gao+Geroldinger, 2009)

Let G be a cyclic group. Then $\rho_{2k+1}(\mathcal{B}(G)) = k\mathsf{D}(G) + 1$.

Theorem (Fan+Z., 2016)

Let G be a noncyclic group with $\mathsf{D}(G) = \mathsf{D}^*(G)$. Then $\rho_{2k+1}(\mathcal{B}(G)) = \lfloor (2k+1)\mathsf{D}(G) \rfloor$ for all large enough k.

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Proposition (Geroldinger+Halter-Koch+Z., 2022)

Let G be a finite abelian group of odd order such that $\mathsf{D}(G) = \mathsf{D}^*(G) \geq 3$. Then $\rho_k(\mathcal{B}_{\pm}(G)) = \lfloor k\mathsf{D}(G)/2 \rfloor$ for all $k \geq 2$.

Theorem (Geroldinger+Halter-Koch+Z., 2022)

Let G be a finite abelian group of odd order such that $\mathsf{D}(G) = \mathsf{D}^*(G) \geq 3$, and let $k = \ell \mathsf{D}(G) + j \geq 2$, where $\ell \in \mathbb{N}_0$ and $j \in [0, \mathsf{D}(G) - 1]$. Then, we have

$$\mathcal{U}_k(\mathcal{B}_\pm(G)) = \begin{cases} \big[2, \lfloor k\mathsf{D}(G)/2 \rfloor\big], & \text{if } j \in [2, \mathsf{D}(G)-1] \text{ and } \ell = 0; \\ \big[2\ell, \lfloor k\mathsf{D}(G)/2 \rfloor\big], & \text{if } j = 0 \text{ and } \ell \geq 1; \\ \big[2\ell+1, \lfloor k\mathsf{D}(G)/2 \rfloor\big], & \text{if } j \in [1, (\mathsf{D}(G)-1)/2] \text{ and } \ell \geq 1; \\ \big[2\ell+2, \lfloor k\mathsf{D}(G)/2 \rfloor\big], & \text{if } j \in [(\mathsf{D}(G)+1)/2, \mathsf{D}(G)-1] \\ & \qquad \text{and } \ell \geq 1. \end{cases}$$

Set of distances

Definition

- Let $L = \{k_1, k_2, \ldots, k_t\} \subset \mathbb{N}$ with $k_1 < k_2 < \ldots < k_t$. We denote the set of distances of L by $\Delta(L) = \{k_{i+1} - k_i \colon i \in [1, t-1]\}$.
- Let H be a BF-monoid. We denote the set of distances of H

by $$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L).$$

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- Let H be a BF-monoid. We denote the set of distances of H

by $$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L).$$

Theorem (Geroldinger+Schmid, 2017)

Let H be an atomic monoid. Then $\min \Delta(H) = \gcd \Delta(H)$. Furthermore, for every nonempty finite subset $\Delta \subset \mathbb{N}$ with $\min \Delta = \gcd(\Delta)$, there exists a finitely generated Krull monoid H such that $\Delta(H) = \Delta$.

Theorem (Geroldinger + Yuan, 2012)

Let G be a finite abelian group with $|G| \geq 3$. Then

$$\Delta(\mathcal{B}(G)) = [1, \max \Delta(G)] .$$

Lemma

Let G be a cyclic group or an elementary-2 group such that $|G| \geq 3$. Then

$$\Delta(\mathcal{B}(G)) = [1, \mathsf{D}(G) - 2] .$$

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Theorem (Geroldinger+Halter-Koch+Z., 2022)

Let G be prime cyclic of order $|G| = p \geq 3$. Then

$$\Delta\big(\mathcal{B}_{\pm}(G)\big) = [1, p - 2] \,.$$

The Characterization Problem

The Characterization Problem.

Given two finite abelian groups G and G' such that $\mathcal{L}(\mathcal{B}(G)) = \mathcal{L}(\mathcal{B}(G'))$. Does it follow that $G \cong G'$?

Theorem (Geroldinger+Schmid+Z., 2015, 2016, 2017)

Let G be a finite abelian group with $\mathsf{D}(G) \ge 4$, and let G' be an abelian group with $\mathcal{L}(\mathcal{B}(G)) = \mathcal{L}(\mathcal{B}(G'))$. Then G and G' are isomorphic in each of the following cases:

1. $G \cong C_{n_1} \oplus C_{n_2}$ *where* $n_1, n_2 \in \mathbb{N}$ *with* $n_1 \mid n_2$.
2. G *is an elementary 2-group.*
3. $G \cong C_n^r$ *where* $r \le n-3$.
4. $G \cong C_n^r$ *where* $r \ge n-1$ *and* n *is a prime power.*
5. $\mathsf{D}(G) \le 11$.

Problem

Which finite abelian groups G have the following property:

> *If G' is a finite abelian group with $\mathcal{L}\big(\mathcal{B}_{\pm}(G)\big) = \mathcal{L}\big(\mathcal{B}_{\pm}(G')\big)$, then G and G' are isomorphic.*

Problem

Which finite abelian groups G have the following property:

If G' is a finite abelian group with $\mathcal{L}(\mathcal{B}_\pm(G)) = \mathcal{L}(\mathcal{B}_\pm(G'))$, then G and G' are isomorphic.

Theorem (Geroldinger+Halter-Koch+Z., 2022)

Let G be a cyclic group of odd order $|G| \geq 3$. If G' is any finite abelian group of odd order with $\mathcal{L}(\mathcal{B}_\pm(G')) = \mathcal{L}(\mathcal{B}_\pm(G))$, then G and G' are isomorphic.

Thank you for your attention !