On the Additive Monoid of Simple Extension Semirings

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Part I: The Valuation Property

—Alan Yao

Overview

Start with a real overview of the two topics we'll discuss…

- *The simple semiring extension* $\mathbb{N}_0[\alpha]$ *has been actively studied in recent literature (by Chapman, Gotti, Polo, et al.)*
- *Correa-Morris and Gotti (2022) provide the first systematic study*
- *We completely characterize several properties of this semiring, most notably the valuation and atomicity properties*
- *Alan will discuss valuation, and Timothy will discuss atomicity*

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Goal. Understand the valuation property for semirings $\mathbb{N}_0[\rho]$

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  - $\mathbb{N}_0[\rho]$ is the smallest semiring containing $\mathbb{N}_0$ and ρ

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- **Remark.** $\mathbb{N}_0[\rho] \cong \mathbb{N}_0[x]$ if and only if ρ is transcedental

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Exclusively focus on the additive monoid of $\mathbb{N}_\circ[\alpha]$, denoted as $M_\alpha = (\mathbb{N}_\circ[\alpha], +)$

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M_α has only nonnegative elements, so M_α is reduced

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- $M_{\sqrt{2}}$ is not a valuation monoid. Since $\sqrt{2} > 1$, we have $\sqrt{2} \nmid 1$. Since 1 is the smallest positive element of $M_{\sqrt{2}}$ and $\sqrt{2} - 1 < 1$, we know $1 \nmid \sqrt{2}$

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We therefore focus on **positive algebraic** $\alpha < 1$.

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- Notably, for $d = 2$, $\alpha = \varphi^{-1} = \dfrac{\sqrt{5}-1}{2}$, the reciprocal of the golden ratio.

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- Repeating this allows us to rewrite every element of M_α into a fixed window of d consecutive exponents
- Fixing $a, b \in M_\alpha$, for all sufficiently large r, we can write

$$a = \sum_{i=0}^{d-1} c_i \alpha^{r+i}, \qquad b = \sum_{i=0}^{d-1} c_i' \alpha^{r+i},$$

with $c_i, c_i' \in \mathbb{N}_0$

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$$a - b = \sum_{i=1}^{d-1} (G_0 + G_i)\alpha^{r+i} + G_0 \alpha^{r+d}$$

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- If the nonzero G_i all have the same sign, then immediately either $a - b \in M_\alpha$ or $b - a \in M_\alpha$
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$$a - b = \sum_{i=1}^{d-1} (G_0 + G_i)\alpha^{r+i} + G_0 \alpha^{r+d}$$

- We will show that by repeatedly using the Fibonacci recurrence, all the nonzero coefficients will be the same sign at some point

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- If F_n is the Fibonacci sequence of order d, with the first nonzero term as $F_{d-1} = 1$, the first and last coefficents nicely turn out to be

$$\sum_{j=0}^{d-1} F_{s+d-j-1} G_j > 0, \qquad \sum_{j=0}^{d-1} F_{s+d-j-2} G_j < 0,$$

and if we choose a $0 \leqslant \ell \leqslant d-1$ with G_ℓ that is positive, this becomes

$$-\sum_{\substack{0 \leqslant j \leqslant d-1 \\ j \neq \ell}} \frac{F_{s+d-j-1}}{F_{s+d-\ell-1}} G_j < G_\ell < -\sum_{\substack{0 \leqslant j \leqslant d-1 \\ j \neq \ell}} \frac{F_{s+d-j-2}}{F_{s+d-\ell-2}} G_j$$

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Descartes' Rule of Signs

Theorem (Descartes' Rule of Signs)

The number of sign changes in $f(x) \in \mathbb{R}[x]$ has the same parity and is at least the number of positive roots (counting multiplicity).

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Valuation $\implies$ α has one positive conjugate (itself)

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A Characterization of Valuation

Theorem (Chen, Gotti, Lu, and Yao 2025)

If α is algebraic and there exists a simplified polynomial $p(x) \in x\mathbb{N}_0[x] - 1$ such that $p(\alpha) = 0$, then M_α is a valuation monoid.

A Characterization of Valuation (cont.)

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- *Proof Sketch.* When d is large, the d^{th} roots of consecutive integers are close together, so their reciprocals are too: $\{\sqrt[d]{\frac{1}{n}} : n, d \in \mathbb{N}\}$ is dense in $(0,1)$
- However, the minimal polynomial of $\sqrt[d]{\frac{1}{n}}$ is $nx^d - 1$ and does not produce valuation monoids
  - Consider the shift $\Longrightarrow (n-1)x^d + x^{d-1} - 1$
  - It satisfies all the conditions to produce a valuation monoid

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