Premons and Factorization under Local Finiteness Conditions

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Outline

Factorization problems

There is a great variety of problems (from all areas of mathematics) involving, in one way or another, the decomposition of *certain* elements of a monoid in terms of *certain* other elements (**building blocks**) that, in a sense, cannot be broken down into smaller pieces:

- Additive decompositions of the elements of a ring into units [Barroero et al., 2011].
- Cyclic decompositions of permutations in the symmetric group of degree n.
- Idempotent factorizations of the singular elements of a monoid, with emphasis on monoids of zero divisors arising from rings [Cossu et al., 2018; Cossu & T., 2021];
- Factorizations of the elements of a finitely generated group in terms of the elements of a fixed (finite, symmetric) generating set (as in geometric group theory).
- Various decompositions of ideals into prime, primary, radical, or other special types of ideals [Fontana, Houston & Lucas, 2013; Olberding & Reinhart, 2019 & 2020].

Most of these problems (and many others) fall far beyond the scope of the **classical theory of factorization** (Slide 14), which has been a motivation for a handful of past & present members of the AlgNT group at University of Graz to enlarge, as much as possible, the (somewhat narrow) boundaries of this research area with the long-term goal of building up a **unifying theory**.

The classical theory of factorization

Throughout, H is a multiplicatively written monoid and we denote by $H^\times$ its group of units; H need not be commutative, cancellative, or whatever.

In the *classical theory*, the building blocks used all through the "factorization process" have been *mainly* atoms[(1)] in the sense of [Cohn, 1968 & 1973]; and two are the basic questions one is constantly juggling with:

1) Check whether every non-unit of H factors as a product of atoms[(2)] (i.e., H is atomic); and if not, characterize which elements do.

2) Assuming the atomicity of H, qualify & quantify — by a number of "invariants" — the departure of H from "factoriality" (however defined).

Answering these questions when H is *commutative* & *cancellative*, has led to a rich theory [Geroldinger & Halter-Koch, 2006; Geroldinger & Zhong, 2020], *partly* extended in recent years to unit-cancellative[(3)] monoids [Geroldinger, 2013; Smertnig 2013 & 2019; Baeth & Smertnig, 2015, 2018 & 2021; Fan et al., 2017; Fan & T., 2018; Garcia-Elsener et al., 2019].

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(1) An atom of H is a non-unit $a \in H$ s.t. $a \neq xy$ for all non-units $x, y \in H$.

(2) A non-empty product of atoms cannot be a unit, see Lemma 2.2(i) in [Fan & T., 2018].

(3) We say H is unit-cancellative if $xy \neq x \neq yx$ for all $x, y \in H$ with $y \notin H^\times$.

Moving away from the comfort zone

Even for cancellative but non-commutative monoids, these extensions (of the classical theory) are however *not* very satisfactory from many points of view:

- They work nicely if there exists a suitable "transfer morphism" [Smertnig, 2013; Bachman et al., 2014; Baeth & Smertnig, 2015; Fan & T., 2018; T., 2019] to a commutative and [unit-]cancellative monoid.
- Otherwise, it is typical to observe a "breakdown" of the theory, to the extent that either the monoids under consideration are not atomic (also in situations where they should *morally* be), or most of the classical invariants associated with atomic factorizations (or with analogous decompositions defined in terms of a different type of building blocks) blow up in a predictable fashion and lose most of their significance.

Needless to say that things become worse and worse as one moves further and further from the "comfort zone" of cancellative monoids (as happens in the presence of non-trivial idempotents or in rings with (non-trivial) zero divisors).

To better illustrate this point, we'll briefly discuss four examples that are emblematic of the many things that can go wrong when leaving the comfort zone. But first a quick review of free monoids and presentations is in order.

Free monoids

We write $\mathscr{F}(X)$ for the free monoid on a set X; use the symbols $*_X$ and ε_X, resp., for the operation and the identity of $\mathscr{F}(X)$; and refer to an element of $\mathscr{F}(X)$ as an X-word[(4)]. The length $\|\mathfrak{u}\|_X$ of an X-word $\mathfrak{u}$ is the unique $k \in \mathbb{N} := \{0, 1, 2, \dots\}$ s.t. $\mathfrak{u} \in X^{\times k}$; and we call ε_X the empty X-word.[(5)] If $\mathfrak{u}$ is an X-word of *positive* length k, then $\mathfrak{u} = u_1 *_X \cdots *_X u_k$ for some uniquely determined $u_1, \dots, u_k \in X$.

We rely on free monoids to encode the notion of "factorization". In particular, an atomic factorization of an element $x \in H$ is an $\mathscr{A}(H)$-word $\mathfrak{a}$ such that $\pi_H(\mathfrak{a}) = x$, where $\mathscr{A}(H)$ is the set of atoms of H and π_H is the factorization hom of H, i.e., the unique extension of id_H to a monoid hom $\mathscr{F}(H) \to H$.

Free monoids are also a forgery of (counter)examples. For, let R be a binary rel on $\mathscr{F}(X)$. We denote by $\mathrm{Mon}\langle X \mid R\rangle$ the quotient of $\mathscr{F}(X)$ by the smallest (monoid) congruence $\equiv_R$ on $\mathscr{F}(X)$ containing R. We write $\mathrm{Mon}\langle X \mid R\rangle$ multiplicatively and call it a presentation; the elements of X are (called) the generators, and each pair $(\mathfrak{q}, \mathfrak{q}') \in R$ is a defining relation. In particular, $\mathrm{Mon}\langle X \mid R\rangle$ is a finite presentation if X and R are both finite sets.

(4) The underlying set of $\mathscr{F}(X)$ consists of all finite tuples of elements of X; and $\mathfrak{u} *_X \mathfrak{v}$ is the concatenation of two such tuples $\mathfrak{u}$ and $\mathfrak{v}$. In particular, the identity of $\mathscr{F}(X)$ is the empty tuple.

(5) If there is no risk of confusion, we simply say "word" instead of X-word and drop the "X".

The Dirty Four

We are now ready for the examples promised in the last lines of Slide 5.

- EXAMPLE 1. The monoid given by the presentation Mon$\langle A \mid R \rangle$, where $A = \{a, b\}$ with $a \neq b$ and $R := \{(a, b * a * b)\} \subseteq \mathscr{F}(A) \times \mathscr{F}(A)$, is reduced[(6)], 2-generated, and cancellative[(7)], but is *not* atomic: What goes wrong here is that the congruence class of a mod $\equiv_R$ is neither a unit nor an atom.
- EXAMPLE 2. The monoid given by the presentation Mon$\langle A \mid R' \rangle$, where R' is the set $\{(a^{*2}, b * a^{*2} * b)\} \subseteq \mathscr{F}(A) \times \mathscr{F}(A)$, is reduced, 2-generated, atomic, and cancellative, but is not BF[(8)]: The problem is that $a^{*2} \equiv_{R'} b^{*k} * a^{*2} * b^{*k}$ for all $k \in \mathbb{N}$; and the congruence classes of a and b mod $\equiv_{R'}$ are atoms.
- EXAMPLE 3. The multiplicative monoid of the integers modulo p^n, where p is a prime and $n \geq 2$, is atomic (cf. Slide 24); but the atomic factorizations of zero are not essentially unique in the classical sense (i.e., up to order and associates), although each atom is prime[(9)]: This is basically an instance of a bigger problem with rings with zero divisors.
- EXAMPLE 4. The reduced power monoid $\mathcal{P}_{\mathrm{fin},1}(H)$ of H [Fan & T., 2018], that is, the (reduced) monoid obtained by endowing the finite subsets of H containing 1_H with the operation of setwise multiplication induced by H on its own power set, is atomic iff $1_H \neq x^2 \neq x$ for every non-identity $x \in H$ (Theorem 3.9 in [Antoniou & T., 2021]).

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(6) A monoid is reduced if its group of units is trivial (i.e., the unique unit is the identity).

(7) The Adian graphs are linear graphs of order 2, so we can apply Adian's embedding theorem.

(8) That is, atomic and with the atomic factorizations of each non-unit bounded in length.

(9) A prime of H is a non-unit $c \in H$ s.t., if $c \mid_H ab$ for some $a, b \in H$, then $c \mid_H a$ or $c \mid_H b$, where $\mid_H$ is the divisibility preorder on H (i.e., $x \mid_H y$ iff $y = uxv$ for some $u, v \in H$).

Pondering the events

A close inspection of the examples on the previous slide reveals that there are at least two things that make atoms and atomic factorizations (as per their classical definition) not really fit for the study of the arithmetic of non-commutative or "highly non-cancellative" monoids.

1) THE ATOMIC MIRAGE: If the goal is to generalize the (existence part of the) fundamental theorem of arithmetic (FTA) to the extent that *every non-unit of H factors as a product of elements from a certain set $A \subseteq H$*, then it is a mistake (at least from a conceptual point of view) to take atoms as the "default choice" for A (Examples 1 and 4).
2) THE BLOW-UP PHENOMENON: Assuming that H is atomic, the (classical) atomic factorizations of a non-unit $x \in H$ are unbounded in length if, e.g., $x = uxv$ for some $u, v \in H$ s.t. u or v is a non-unit[10] (Examples 2 and 3); and analogous considerations apply to the case where building blocks other than atoms are used all along the factorization process.

Both of these issues have been recognized for a long time; and especially in the commutative setting, various solutions by various authors have been proposed to overcome (or at least mitigate) their undesirable effects.

[10] This is especially true when H is *not* unit-cancellative.

Outline

Anderson & Valdes-Leon's work and ACCs

It was apparently first realized in [Anderson & Valdes-Leon, 1996 & 1997] that an effective alternative to atoms is provided by the following notion (see also [Ağargün et al., 2001; Frei & Frisch, 2009; Anderson & Chun, 2011 & 2013]):

Definition 1 (T., 2022, Sect. 1).

An irreducible (or irred) of H is a non-unit $a \in H$ s.t. $a \neq xy$ for all non-units $x, y \in H$ with $HxH \neq HaH \neq HyH$.

Anderson & Valdes-Leon focus entirely on the commutative case, where it is evident that every atom is irred but not the other way around; nevertheless, there are a number of precious insights one can gain from their work.

To start with, we recall that H satisfies the ascending chain condition (ACC) on principal right ideals (shortly, ACCPR) if there is no H-valued sequence $a_0, a_1, \ldots$ s.t. $a_i H \subsetneq a_{i+1} H$ for all $i \in \mathbb{N}$.

The ACCPL and the ACCP are defined similarly, with principal *right* ideals (i.e., sets of the form aH) replaced, resp., by principal *left* ideals (i.e., sets of the form Ha) and principal *two-sided* ideals (i.e., sets of the form HaH).

Standing on the shoulders of the classics

The ACCPR, the ACCPL, and the ACCP are known to play a key role in the classical theory; and in this regard, a simple but quite interesting result from [Anderson & Valdes-Leon, 1996] is the following:

Anderson & Valdes-Leon's (factorization) theorem.

If a *commutative* monoid (e.g., the *multiplicative monoid* of a commutative ring) satisfies the ACCP, then every non-unit is a product of irreds.

This result is reminiscent of a related theorem that, to my knowledge, was first published (if not first proved) by P.M. Cohn (see Proposition 1.2.5 in [Cohn, 1985], Proposition 0.9.3 in [Cohn, 2006], and the historical notes therein):

Cohn's (factorization) theorem.

If a *cancellative* monoid (e.g., the multiplicative monoid of the non-zero elements of a domain) satisfies the ACCPR and the ACCPL, then every non-unit factors as a product of atoms.

Cohn's theorem was extended to *unit-cancellative* monoids in [Y. Fan & T., 2018]; for *commutative* domains, it famously appeared in a 1968 paper of Cohn where it is however stated *incorrectly* as an iff [Grams, 1974; T., 2022e].

Echoes from the outer world

Cohn's and Anderson & Valdes-Leon's theorems are "foundational results" for the classical theory — before getting to prove anything else, you probably want to know *what factors into what*.

There are, however, a number of "factorization theorems" in maths that are not only beyond the scope of Cohn's and Anderson & Valdes-Leon's theorems, but also beyond the paradigm itself of the classical theory (where, among other things, units are *never* allowed as building blocks):

- Every permutation of a finite k-element set is a (functional) composition of k or fewer transpositions (this is folklore in group theory, combinatorics, etc.).
- Every non-invertible matrix in the multiplicative monoid of the (full) ring of n-by-n matrices with entries in a field factors as a product of idempotent matrices [J. A. Erdos, 1968], and later work has revealed that the same holds for a wider class of commutative or non-commutative rings [Ballantine, 1978; R. J. H. Dawlings, 1981; Laffey, 1983 & 1989; Alahmadi et al., 2014; Cossu et al., 2018; Cossu & T., 2021e].
- In a variety of rings, every element is the *sum* of finitely many units (see, e.g., F. Barroero et al.'s 2011 survey on additive unit representations).

The question arises: Is there any "sensible and natural" abstraction of the (classical) ACCs as well as of the notions of atom, irred, prime, etc., that makes it possible to bring the results from the above list (and the factorization problems they originate from) under the umbrella of a unifying theory?

Preorders enter the scene

The key idea, first envisioned in [T., 2022] and further developed in [Cossu & T., 2021e & 2022e], is to graft the language of preorders[11] onto that of mons.

Definition 2 (T., 2022, Definitions 3.4 and 3.6).

A premon (or premonoid) is a pair $\mathcal{H} = (H, \preceq)$ consisting of a monoid H and a preorder $\preceq$ on H. A $\preceq$-unit (of H) is then an element $u \in H$ s.t. $1_H \preceq u \preceq 1_H$ (we also say that u is $\preceq$-equivalent to 1_H). An element that is not a $\preceq$-unit is a $\preceq$-non-unit. Accordingly, we refer to a $\preceq$-non-unit $a \in H$ as

- a $\preceq$-irreducible (or $\preceq$-irred) if $a \neq xy$ for all $\preceq$-non-units $x \prec a$ & $y \prec a$;
- a $\preceq$-atom if $a \neq xy$ for all $\preceq$-non-units $x, y \in H$;
- a $\preceq$-quark if there is no $\preceq$-non-unit $b \in H$ with $b \prec a$;
- a $\preceq$-prime if $a \preceq xy$, for some $x, y \in H$, implies $a \preceq x$ or $a \preceq y$.

In particular, note that the divisibility preorder $|_H$ is a preorder; and if H is a Dedekind-finite monoid (i.e., $xy = 1_H$ iff $yx = 1_H$), then $|_H$-unit $=$ unit, $|_H$-atom $=$ atom, $|_H$-irred $=$ irred (Definition 1) , and $|_H$-prime $=$ prime.

(11) A preorder on a set X is a reflexive, transitive rel $\preceq$ (read 'p') on X. We write $x \prec y$ when $x \preceq y$ and $y \not\preceq x$, and we call $(X, \preceq)$ a preset. If $x \preceq y \preceq x$ implies $x = y$, then $\preceq$ is an order.

Inclusive and comprehensive

We denote the set of $\preceq$-units of a premon $\mathcal{H} = (H, \preceq)$ by $\mathcal{H}^\times$; that of $\preceq$-irreds by $\mathscr{I}(\mathcal{H})$; the set of $\preceq$-atoms by $\mathscr{A}(\mathcal{H})$; and that of $\preceq$-quarks by $\mathscr{Q}(\mathcal{H})$.

When, in particular, $\mathcal{H}$ is the divisibility premon $(H, |_H)$ of the monoid H, let us simply write $\mathscr{I}_\mathrm{d}(H)$ for $\mathscr{I}(\mathcal{H})$, $\mathscr{A}_\mathrm{d}(H)$ for $\mathscr{A}(\mathcal{H})$, and $\mathscr{Q}_\mathrm{d}(H)$ for $\mathscr{Q}(\mathcal{H})$.

It is easily seen that $\mathscr{Q}(\mathcal{H}) \cup \mathscr{A}(\mathcal{H}) \subseteq \mathscr{I}(\mathcal{H})$; and we have from Slide 13 that, in the Dedekind-finite case, $\mathscr{I}_\mathrm{d}(H)$ is the set of irreds of H (Definition 1) and $\mathscr{A}_\mathrm{d}(H)$ is the set $\mathscr{A}(H)$ of (ordinary) atoms of H. Moreover:

Proposition 3 (T., 2022, Corollary 4.4).

Let H be an acyclic monoid, meaning that $x \neq uxv$ for all $u, v, x \in H$ with $u \notin H^\times$ or $v \notin H^\times$. Then H is Dedekind-finite (and hence every $|_H$-unit is a unit); and we have $\mathscr{I}_\mathrm{d}(H) = \mathscr{Q}_\mathrm{d}(H) = \mathscr{A}_\mathrm{d}(H) = \mathscr{A}(H)$.

Based on the above, we will formally understand the **classical theory of factorization** (CTF) as the study of the *arithmetic of the divisibility premon of a Dedekind-finite monoid*. Note that every [unit-]cancellative or commutative monoid is Dedekind-finite, so that the theory of factorization as presented either in [Geroldinger & Halter-Koch, 2006; Geroldinger & Zhong, 2020] or in the work of D. D. Anderson et al. is entirely within the scope of the CTF.

Existential questions

Preorders also provide a natural way to extend the classical ACCs:

Definition 4.

We say that a preorder $\preceq_X$ on a set X is artinian or satisfies the descending chain condition (DCC) if every $\preceq_X$-non-increasing (infinite) sequence in X is $\preceq_X$-non-decreasing from a certain point on (and hence "stabilizes").

So we have all the ingredients to cook the following generalization of the FTA, which is extremely abstract and, for this very reason, quite easy to prove:

Theorem 5 (T., 2022, Theorem 3.10; cf. Cossu & T., 2021e).

If $\preceq$ is an artinian preorder on (the carrier set of) a monoid H, then every $\preceq$-non-unit of H factors as a product of $\preceq$-irreds.

In a way, Theorem 5 lays the atomic mirage (Slide 8) to rest, insofar as it shows that it is the notion of "irreducible" to be central to the study of factorization: The notions of "atom", "quark", or "prime" are somewhat secondary, though still of great interest due, e.g., to the fact that understanding the interrelation between $\preceq$-irreds, $\preceq$-atoms, $\preceq$-quarks, and $\preceq$-primes for a specific preorder $\preceq$ is often pivotal to a deeper comprehension of various phenomena.

Back to the classical theory

Theorem 5, along with the remarks on the bottom of Slide 13, leads *at once* to a non-commutative generalization of Anderson & Valdes-Leon's theorem:[(12)]

Corollary 6 (T., 2022, Corollary 4.1).

If H is a Dedekind-finite monoid satisfying the ACCP, then every non-unit of H is a product of irreds (as defined on Slide 10).

It also results in a refinement of Cohn's theorem:

Theorem 7 (T., 2022, Corollary 4.6).

The following conditions are equivalent, and each implies the atomicity of H:

(a) H is unit-cancellative and satisfies the ACCPR and the ACCPL.

(b) H is acyclic and satisfies the ACCP.

Question [T., 2022, §5]: Is a cancellative f.g. monoid with the ACCP acyclic?

(12) H satisfies the ACCP iff the divisibility preorder $|_H$ is artinian, while it satisfies the ACCPR iff the "divides from the *left*" preorder $\vdash_H$ is artinian (and similarly for the ACCPL vs the "divides from the *right*" preorder $\dashv_H$), where $a \vdash_H b$ iff $b \in aH$ (and $a \dashv_H b$ iff $b \in Ha$).

A sharp razor

The next result can be viewed as a sort of converse to Theorem 5 and shows that the latter is, in a certain sense, "best possible": In a monoid, proving that every element of a given set S factors through the elements of a prescribed set A of building blocks is equivalent to the artinianity of a suitable preorder.

Proposition 8 (Cossu & T., 2022e, Proposition 2.8).

Let A and S be subsets of a monoid H with $1_H \notin A \cup S$. TFAE:

- **(a)** Every element of S factors as a non-empty product of elements of A.
- **(b)** There exists a "strongly artinian" preorder $\preceq$ on H such that every $x \in S$ is a $\preceq$-non-unit and an element $a \in H$ is $\preceq$-irreducible if and only if it is a $\preceq$-quark, if and only if $a \in A$.
- **(c)** There exists an artinian preorder $\preceq$ on H such that every $x \in S$ is a $\preceq$-non-unit and every $\preceq$-irred is an element of A.

With that said, there are in fact many possible ways to improve on Theorem 5: E.g., one could check if the "factorizations" (however defined) of an element are all bounded in length, or if there are only finitely many of them that are "essentially different", or if the same conditions hold true for a "restricted class" of factorizations. We'll formalize these ideas in the next section.

Outline

A new paradigm

Solutions to the blow-up phenomenon in a *commutative* setting have been proposed by many authors over the years [Fletcher, 1969; Bouvier, 1974a & 1974b; Galovich, 1978; Geroldinger & Lettl, 1990; Chun et al., 2011]. To my knowledge, the only approaches to the problem in a *non-commutative* setting are due to [Cohn, 1963] and [Brungs, 1969] (cf. [Facchini & Fassina, 2017]): Fine for rings, they do not carry over to monoids in any obvious way.

A totally different idea was set forth in [Antoniou & T., 2021](13): Instead of letting an atomic factorization of $x \in H$ be *any* $\mathscr{A}(H)$-word $\mathfrak{a} \in \pi_H^{-1}(x)$, consider a preorder $\sqsubseteq$ on $\mathscr{F}(\mathscr{A}(H))$ and discard the atomic factorizations of x that are *not* $\sqsubseteq$-minimal(14). (Recall that $\mathscr{A}(H)$ is the set of atoms of H.)

More in detail, let $\sqsubseteq_H$ be the binary rel on $\mathscr{F}(H)$ defined by $\mathfrak{a} \sqsubseteq_H \mathfrak{b}$ iff there is an injection $\sigma\colon [\![1, \|\mathfrak{a}\|_H]\!] \to [\![1, \|\mathfrak{b}\|_H]\!]$ s.t. $\mathfrak{a}[i] \in H^\times \mathfrak{b}[\sigma(i)] H^\times$ for every $i \in [\![1, \|\mathfrak{a}\|_H]\!]$. It turns out that $\sqsubseteq_H$ is a preorder; and an $\mathscr{A}(H)$-word $\mathfrak{a}$ is a minimal atomic factorization of x if $\mathfrak{a}$ is $\sqsubseteq_H$-minimal in $\pi_H^{-1}(x) \cap \mathscr{F}(\mathscr{A}(H))$.

(13) The paper was written in spring 2018, when Austin was visiting the Algebra & NT group at University of Graz as a PhD student of Alan Loper and I was there with a post-doc position funded by the Austrian FWF; but its publication has had a long and "eventful" history.

(14) Given a preset $(X, \preceq)$, we say that $x \in X$ is a $\preceq$-minimal element of a set $Y \subseteq X$ if $x \in Y$ and there is no element $y \in Y$ with $y \prec x$.

Factorizations and minimal factorizations

The same approach has now a natural generalization to premons:

Definition 9 (Cossu & T., 2022e, Definition 3.1).

Given a premon $\mathcal{H} = (H, \preceq)$, we denote by $\sqsubseteq_\mathcal{H}$ the binary rel on $\mathscr{F}(H)$ defined by $\mathfrak{a} \sqsubseteq_\mathcal{H} \mathfrak{b}$, for some H-words $\mathfrak{a}$ and $\mathfrak{b}$, iff there is an injection $\sigma\colon [\![1, \|\mathfrak{a}\|_H]\!] \to [\![1, \|\mathfrak{b}\|_H]\!]$ such that $\mathfrak{a}[i] \preceq \mathfrak{b}[\sigma(i)] \preceq \mathfrak{a}[i]$ for every $i \in [\![1, \|\mathfrak{a}\|_H]\!]$.

Since the composition of two injections is injective, $\sqsubseteq_\mathcal{H}$ is a preorder (called $\preceq$-shuffling) on $\mathscr{F}(H)$ and we can talk of $\sqsubseteq_\mathcal{H}$-minimality & $\sqsubseteq_\mathcal{H}$-equivalence:

Definition 10 (Cossu & T., 2022e, Definition 3.2, items (1) and (2)).

A $\preceq$-factorization of an element $x \in H$ is an $\mathscr{I}(\mathcal{H})$-word $\mathfrak{a} \in \pi_H^{-1}(x)$ and we set $\mathcal{Z}_\mathcal{H}(x) := \pi_H^{-1}(x) \cap \mathscr{F}(\mathscr{I}(\mathcal{H}))$. A minimal $\preceq$-factorization of x is then a $\sqsubseteq_\mathcal{H}$-minimal word in $\mathcal{Z}_\mathcal{H}(x)$, namely, an $\mathscr{I}(\mathcal{H})$-word $\mathfrak{a} \in \pi_H^{-1}(x)$ with the additional property that there is no $\mathscr{I}(\mathcal{H})$-word $\mathfrak{b} \in \pi_H^{-1}(x)$ such that $\mathfrak{b} \sqsubset_\mathcal{H} \mathfrak{a}$. We denote the set of minimal $\preceq$-factorizations of x by $\mathcal{Z}_\mathcal{H}^{\mathsf{m}}(x)$, and we put

$$\mathsf{L}_\mathcal{H}(x) := \{\|\mathfrak{a}\|_H : \mathfrak{a} \in \mathcal{Z}_\mathcal{H}(x)\} \quad \text{and} \quad \mathsf{L}_\mathcal{H}^{\mathsf{m}}(x) := \{\|\mathfrak{a}\|_H : \mathfrak{a} \in \mathcal{Z}_\mathcal{H}^{\mathsf{m}}(x)\}.$$

B[m]F-ness, F[m]F-ness, etc.

In the notation of Definition 10, we refer to $\mathsf{L}_{\mathcal{H}}(x)$ and $\mathsf{L}^{\mathsf{m}}_{\mathcal{H}}(x)$, resp., as the set of lengths and the set of minimal lengths of x (relative to the premon $\mathcal{H}$).

Definition 11 (Cossu & T., 2022e, Definition 3.2(4)).

We say that the premon $\mathcal{H} = (H, \preceq)$ is

- factorable if every $\preceq$-non-unit has at least one $\preceq$-factorization;
- BF-factorable (resp., BmF-factorable) if the set of lengths (resp., of minimal lengths) of each $\preceq$-non-unit is finite and non-empty;
- HF-factorable (resp., HmF-factorable) if the same sets of lengths (resp., of minimal lengths) are all singletons;
- FF-factorable (resp., FmF-factorable) if the quotient of $\mathcal{Z}_H(x)$ (resp., of $\mathcal{Z}^{\mathsf{m}}_H(x)$) by the rel of $\sqsubseteq_{\mathcal{H}}$-equivalence (properly restricted) is finite and non-empty for every $\preceq$-non-unit x;
- UF-factorable (resp., UmF-factorable) if the same quotients are singletons.

In particular, note that, since $\sqsubseteq_{\mathcal{H}}$ is an *artinian* preorder, the premon $\mathcal{H}$ is factorable iff every $\preceq$-non-unit has a minimal $\preceq$-factorization.

Atomic analogues

Atomic $\preceq$-factorizations and minimal atomic $\preceq$-factorizations are now defined in a similar way, with $\mathscr{I}(\mathcal{H})$ replaced by $\mathscr{A}(\mathcal{H})$ in Definition 11.

Given $x \in H$, we use $\mathcal{Z}_\mathcal{H}(x\,;\mathscr{A}(\mathcal{H}))$ for the set of atomic $\preceq$-factorizations and $\mathcal{Z}^{\mathsf{m}}_\mathcal{H}(x\,;\mathscr{A}(\mathcal{H}))$ for the set of minimal atomic $\preceq$-factorizations of x. Then

$$\mathsf{L}_\mathcal{H}(x\,;\mathscr{A}(\mathcal{H})) := \big\{\|\mathfrak{a}\|_H : \mathfrak{a} \in \mathcal{Z}_\mathcal{H}(x\,;\mathscr{A}(\mathcal{H}))\big\}$$

and

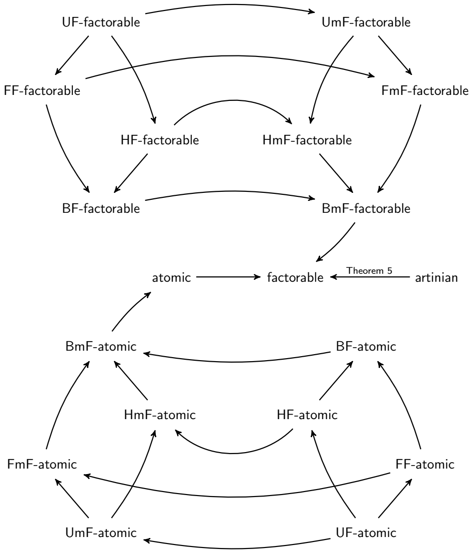
$$\mathsf{L}^{\mathsf{m}}_\mathcal{H}(x\,;\mathscr{A}(\mathcal{H})) := \big\{\|\mathfrak{a}\|_H : \mathfrak{a} \in \mathcal{Z}^{\mathsf{m}}_\mathcal{H}(x\,;\mathscr{A}(\mathcal{H}))\big\},$$

are, resp., the set of atomic lengths and the set of minimal atomic lengths of x (relative to the premon $\mathcal{H}$). Accordingly, we will talk of atomic premons, BF-atomic (resp., BmF-atomic) premonos, and so on and forth.

Since every $\preceq$-atom is a $\preceq$-irred, it is clear that an atomic (resp., minimal atomic) $\preceq$-factorization is also a $\preceq$-factorization (resp., minimal $\preceq$-factorization). Moreover, UF-ness implies both FF-ness and HF-ness, and each of these implies BF-ness (and it goes the same with the minimal versions).

Overall, this leads to the diagram on the next slide, where none of the implications represented by the arrows can, in general, be reversed. (Recall from Theorem 5 that every artinian premon is factorable.)

They say a picture speaks a thousand words



An educational example

Let H be the multiplicative monoid of the integers modulo p^n and set $\mathcal{H} = (H, |_H)$, where $p \in \mathbb{N}^+$ is a prime and n be an integer ≥ 2 (cf. Example 3 on Slide 7). Given $x \in \mathbb{Z}$, we write $\bar{x}$ for the residue class of x modulo p^n.

The units of H are the residue classes modulo p^n of the integers between 1 and p^n that are not divisible by p; and the (ordinary) atoms of H are the elements of the form $\bar{p}\,u$ with $u \in H^\times$ (note that H is Dedekind-finite and hence every $|_H$-unit is a unit and every $|_H$-atom is an atom, as observed on Slide 13).

Each non-zero non-unit $x \in H$ can be uniquely written as $\bar{p}^{\,k} u$ for some $k \in [\![1, n-1]\!]$ and $u \in H^\times$, so every irred is an atom. The $|_H$-factorizations of x are therefore the non-empty H-words $\bar{p}\,u_1 * \cdots * \bar{p}\,u_k$ of length k with $u_1, \ldots, u_k \in H$ and $u_1 \cdots u_k = u$ (which implies that the u_i's are all units). On the other hand, the $|_H$-factorizations of $\bar{0}$ are all and only the length-l H-words of the form $\bar{p}\,v_1 * \cdots * \bar{p}\,v_l$ with $l \geq n$ and $v_1, \ldots, v_l \in H^\times$.

So, H is UmF-atomic: All the minimal $|_H$-factorizations of a non-unit $x \in H$ are $\sqsubseteq_{\mathcal{H}}$-equivalent (in particular, the minimal $|_H$-factorizations of $\bar{0}$ are the H-words $\bar{p}\,v_1 * \cdots * \bar{p}\,v_n$ with $v_1, \ldots, v_n \in H^\times$). On the other hand, H is not BF-atomic: The set of $|_H$-factorizations of $\bar{0}$ contains the H-word $\bar{p}^{\,*(n+k)}$ for every $k \in \mathbb{N}$ (and infinitely many of them are pairwise $\sqsubseteq_{\mathcal{H}}$-inequivalent).

Another educational example

Let $\mathscr{P}(S)$ be the premon obtained by endowing the power set of a set S with the binary operation $\cup_S$ sending a pair of subsets of S to their union and the inclusion order $\subseteq_S$ defined by $X \subseteq_S Y$ iff $X \subseteq Y \subseteq S$.

It is immediate that the only $\subseteq_S$-unit is the empty set and hence the $\subseteq_S$-irreducibles are the one-element subsets of S, which, in addition, are all $\subseteq_S$-quarks. On the other hand, the set of $\subseteq_S$-atoms is empty, because $X = X \cup X$ for every set X.

It follows that the premon $\mathscr{P}(S)$ is atomic iff $S = \varnothing$; and it is factorable iff it is BmF-factorable, iff it is FmF-factorable, iff it is UmF-factorable, iff $|S| < \infty$.

In particular, recall that the $\subseteq_S$-irreds are the one-element subsets of S and hence every finite $X \subseteq S$ has a unique minimal $\subseteq_S$-factorization; and note that, if a is an arbitrary element of S (and hence S is non-empty), then the length-n word $\{a\} * \cdots * \{a\}$ is a $\subseteq_S$-factorization of $\{a\}$ for all $n \in \mathbb{N}^+$.

By the way, note that $\mathscr{P}(S)$ is, in fact, the divisibility premon of the power set of S endowed with the operation $\cup_S$, since for all sets X and Y we have $X \subseteq Y$ iff $Y = X \cup (Y \setminus X)$.

A pointwise version of a classical idea

To study the arithmetic of a monoid H, it is often convenient to do so through the lens of a certain class $\mathcal{S}$ of "arithmetically smaller" submonoids of H.

In the standard approach, $\mathcal{S}$ is the class of divisor-closed submons $[\![x]\!]_H$ of H generated by the units $x \in H$. However, it is often the case that a monoid has no *non-trivial* divisor-closed submonoids or the monoids $[\![x]\!]_H$ are still "too large". An effective alternative is then offered by the following *observations*:

Proposition 12 (Cossu & T., 2022e, Proposition 3.5).

Let $\mathcal{H} = (H, \preceq)$ be a premon and $\mathcal{K} = (K, \preceq_K)$ be a subpremon of $\mathcal{H}$ containing the divisors (in H) of a fixed $x \in H$ (and hence x itself). Then:

(i) $\mathcal{Z}_{\mathcal{H}}(x) = \mathcal{Z}_{\mathcal{K}}(x)$ and $\mathcal{Z}_{\mathcal{H}}(x; \mathscr{A}(\mathcal{H})) = \mathcal{Z}_{\mathcal{K}}(x; \mathscr{A}(\mathcal{K}))$.

(ii) $\mathcal{Z}^{\mathrm{m}}_{\mathcal{H}}(x) = \mathcal{Z}^{\mathrm{m}}_{\mathcal{K}}(x)$ and $\mathcal{Z}^{\mathrm{m}}_{\mathcal{H}}(x; \mathscr{A}(\mathcal{H})) = \mathcal{Z}^{\mathrm{m}}_{\mathcal{K}}(x; \mathscr{A}(\mathcal{K}))$.

(iii) $\mathsf{L}_{\mathcal{H}}(x) = \mathsf{L}_{\mathcal{K}}(x)$ and $\mathsf{L}_{\mathcal{H}}(x; \mathscr{A}(\mathcal{H})) = \mathsf{L}_{\mathcal{K}}(x; \mathscr{A}(\mathcal{K}))$.

(iv) $\mathsf{L}^{\mathrm{m}}_{\mathcal{H}}(x) = \mathsf{L}^{\mathrm{m}}_{\mathcal{K}}(x)$ and $\mathsf{L}^{\mathrm{m}}_{\mathcal{H}}(x; \mathscr{A}(\mathcal{H})) = \mathsf{L}^{\mathrm{m}}_{\mathcal{K}}(x; \mathscr{A}(\mathcal{K}))$.

Here, saying that $(K, \preceq_K)$ is a subpremon of $\mathcal{H}$ means that K is a submonoid of H and $\preceq_K$ is the restriction of the preorder $\preceq$ to $K \times K$.

The divisor-closed case

Specializing Proposition 12 to a divisor-closed submonoid leads to:

Corollary 13 (Cossu & T., 2022e, Corollary 3.6).

Let $\mathcal{H} = (H, \preceq)$ be a premon and $\mathcal{K} = (K, \preceq_K)$ be a subpremon of $\mathcal{H}$ s.t. K is a divisor-closed submon. Then $\mathscr{I}(\mathcal{K}) = K \cap \mathscr{I}(\mathcal{H})$ and $\mathscr{A}(\mathcal{K}) = K \cap \mathscr{A}(\mathcal{H})$.

Given a premon $\mathcal{H} = (H, \preceq)$ and an element $x \in H$, we will denote by $[\![x]\!]_{\mathcal{H}}$ the subpremon of $\mathcal{H}$ whose "ground monoid" is $[\![x]\!]_H$, and call $[\![x]\!]_{\mathcal{H}}$ the divisor-closed subpremon of $\mathcal{H}$ generated by x.

Corollary 14 (Cossu & T., 2022e, Corollary 3.7).

A premon $\mathcal{H} = (H, \preceq)$ satisfies any of the properties specified in the nodes of the diagram on Slide 23 if and only if so does $[\![x]\!]_{\mathcal{H}}$ for every $\preceq$-non-unit x.

When $\preceq$ is the divisibility preorder on H, these are in turn generalizations of basic results in the classical theory (for the part concerning factorizations and sets of lengths relative to the (ordinary) atoms of H), upon recalling from Lemma 2.2(i) in [Fan & T., 2018] that the existence itself of an atom in a monoid H is a sufficient condition for H to be Dedekind-finite.

Outline

Finitely generated monoids and beyond

Throughout, $\mathcal{H} = (H, \preceq)$ is a (multiplicatively written) premon.

Our next goal: Sufficient conditions for $\mathcal{H}$ to be B[m]F- or F[m]F-factorable.

We start with the following def., where "f.g." stands for "finitely generated" and $\langle A \rangle_H$ for the submonoid of H generated by a set $A \subseteq H$:

Definition 15 (Cossu & T., 2022e, Definition 4.1).

We denote by $\langle\!\langle x \rangle\!\rangle_H$ the submon of H generated by the divisors of an element $x \in H$ and by $\langle\!\langle x \rangle\!\rangle_{\mathcal{H}}$ the subpremon of $\mathcal{H}$ obtained by endowing $\langle\!\langle x \rangle\!\rangle_H$ with the restriction of the preorder $\preceq$. We call $\langle\!\langle x \rangle\!\rangle_{\mathcal{H}}$ the germ of $\mathcal{H}$ at x.

The premon $\mathcal{H}$ is f.g. if H is an f.g. mon; f.g.u. if there is a finite $A \subseteq H$ s.t. $H = \langle \mathcal{H}^\times A \, \mathcal{H}^\times \rangle_H$; locally f.g.u. or l.f.g.u. (resp., locally f.g. or l.f.g.) if, for each $\preceq$-non-unit x, the premon $[\![x]\!]_{\mathcal{H}}$ is f.g.u. (resp., f.g.); and weakly l.f.g.u. (resp., weakly l.f.g.) if the germ $\langle\!\langle x \rangle\!\rangle_{\mathcal{H}}$ is f.g.u. (resp., f.g.).

Lastly, $\mathcal{H}$ is locally of finite type or loft if, for each $\preceq$-non-unit x, there is a finite $A_x \subseteq \mathscr{I}(\mathcal{H})$ s.t. every $\preceq$-factorization of x is $\sqsubseteq_{\mathcal{H}}$-equiv. to an A_x-word.

Specializing to the divisibility premon

In particular, we say the *monoid* H is l.f.g., f.g.u., [weakly] l.f.g.u., or loft if so is the divisibility premon $(H, |_H)$ of H.

This provides a natural generalization (from cancellative & commutative to arbitrary monoids) of the classical notion of locally finitely generated monoid (viz., Definition 2.7.6.5 in [Geroldinger & Halter-Koch, 2006]).

Cancellative & commutative l.f.g.u. monoids are central to the classical theory: Among other things, they are FF-atomic, which is a fundamental step for *further and deeper* developments; and the conclusion carries over to *unit-cancellative* commutative l.f.g.u. monoids [Fan et al., 2018].

How to extend this result to *non*-unit-cancellative or *non*-commutative monoids, and even farther to the setting of premons? The starting point is:

Proposition 16 (Cossu & T., 2022e, Proposition 4.3).

F.g.u. $\Longrightarrow$ l.f.g.u. $\Longrightarrow$ weakly l.f.g.u (and f.g. $\Longrightarrow$ l.f.g. $\Longrightarrow$ weakly l.f.g.).

Why significant? One half of the answer is, "Because Proposition 12 applies to the germs of $\mathcal{H}$." Another part of the answer is related to the **next question**: "Are there any (interesting) weakly l.f.g.u. [pre]mons that are not l.f.g.u.?"

A short list is not a shortlist

In fact, there are many examples of weakly l.f.g.u. monoids that are not l.f.g.u.:

- An affine monoid (i.e., a submonoid of $(\mathbb{N}^{\times k}, +)$ with $k \in \mathbb{N}^+$) is weakly l.f.g.u., but need not be l.f.g.u. More generally, the same holds for certain finitely primary mons (Definition 2.9.1 in [Geroldinger & Halter-Koch, 2006]).
- By Theorem 3.3.1 in [Geroldinger, Gotti, & T., 2021], a Puiseux monoid (i.e., a submonoid of the non-negative rational numbers under addition) is f.g. iff it is (isomorphic to) a numerical monoid. On the other hand, Puiseux monoids are primary and hence have no non-trivial divisor-closed submonoids. So, a Puiseux monoid is l.f.g.[u.] iff it is a numerical monoid. Yet, many non-f.g. Puiseux monoids are weakly l.f.g.[u.] (e.g., this happens if 0 is not a limit point of the set of *positive* elements).
- The monoid of product-one sequences over a multiplicatively-written group G with support in a subset $G_0 \subseteq G$ is weakly l.f.g.u., but need not be l.f.g.u. when G is non-abelian.
- Following [Fan & T., 2018], the reduced power monoid $\mathcal{P}_{\text{fin},1}(H)$ of H (i.e., the monoid obtained by endowing the family of all *finite* subsets of H containing the identity 1_H with the operation of setwise multiplication) is weakly l.f.g.u., but need not be l.f.g.u.

Natural examples of weakly l.f.g.u. premons that are not divisibility premons come from "monoids of sets" (including monoids of ideals). E.g.:

- The (reduced) monoid of non-zero ideals of a finite molecularization domain (FMD) is a weakly l.f.g. premon when coordered by inclusion. By results of [Hetzel, Lawson, & Reinhart, 2021], examples of FMDs include (i) the polynomial ring over a principal order in an algebraic number field and (ii) every 1-dimensional Noetherian commutative FF-domain with trivial Picard group. (I don't know if, in general, these monoids are l.f.g.[u.])

A positive attitude to existence

It remains to see if there is anything interesting one can prove. For, we will need the monoid operation to have a certain compatibility with the preorder:

Definition 17 (Cossu & T., 2022e, Definition 2.3).

The premon $\mathcal{H}$ is a preordered (resp., strongly preordered) mon if $x \preceq y$ (resp., $x \prec y$) implies $uxv \preceq uyv$ (resp., $uxv \prec uyv$) for all $u, v \in H$; a positive (resp., strongly positive) mon if $\mathcal{H}$ is a preordered (resp., strongly preordered) mon with $1_H \preceq H$; a weakly positive mon if $\mathcal{H}^\times x \mathcal{H}^\times \preceq x \preceq HxH$ for every $x \in H$.

In particular, the divisibility premon of H is a weakly positive monoid when H is *Dedekind-finite* and hence in the setting of the classical theory (Slide 14).

Theorem 18 (Cossu & T., 2022e, Theorem 4.7).

Let $\mathcal{H} = (H, \preceq)$ be an f.g.u. weakly positive monoid. There then exists a finite set A of $\preceq$-irreds s.t. every $\preceq$-non-unit can be written as a product of elements of $\mathscr{I}(\mathcal{H}) \cap \mathcal{H}^\times A \mathcal{H}^\times$. In particular, $\mathcal{H}$ is factorable.

This is *not* a consequence of Theorem 5 in any obvious way, not even when $\mathcal{H}$ is a divisibility premon: Theorem 18 applies to the monoid H from Example 1 on Slide 7, but Theorem 5 does not (since H does not satisfy the ACCP).

Higman's lemma

An immediate consequence of Theorem 18 is the following:

Corollary 19 (Cossu & T., 2022e, Theorem 4.8).

Every weakly l.f.g.u., weakly positive monoid is factorable.

Another question arises: Is there anything *beyond existence*?

Given a set X, we say an X-word $\mathfrak{u}$ is a scattered subword of an X-word $\mathfrak{v}$ if there is a (strictly) increasing function $\sigma \colon [\![1, \|\mathfrak{u}\|_X]\!] \to [\![1, \|\mathfrak{v}\|_X]\!]$ s.t.

$$\mathfrak{u}[i] = \mathfrak{v}[\sigma(i)], \qquad \text{for each } i \in [\![1, \|\mathfrak{u}\|_X]\!].$$

Our interest in scattered subwords is mainly due to the following result:

Higman's lemma (Higman, 1952).

If X is a finite set, then every infinite sequence of X-words contains an infinite subsequence each of whose terms is a scattered subword of the next.

This is a non-commutative generalization of Dickson's lemma (an old acquaintance) and, in our approach, factorizations are encoded by words, so...

BmF-ness and FmF-ness

...it shouldn't come as a surprise that Higman's lemma is going to play a role in what's coming next. In particular, we have the following:

Theorem 20 (Cossu & T., 2022e, Theorem 4.12).

Every weakly l.f.g.u., weakly positive monoid $\mathcal{H} = (H, \preceq)$ such that the $\preceq$-irred are $\preceq$-atoms, is loft and hence FmF-atomic.

Corollary 21 (Cossu & T., 2022e, Corollary 4.13).

Every weakly l.f.g.u., strongly positive monoid is FF-atomic.

Sketch of proof.

If $\mathcal{H}$ is a strongly positive monoid, then it is positive and every $\preceq$-irred is a $\preceq$-atom. So we can use Theorem 20 and conclude in the first place that $\mathcal{H}$ is FmF-atomic. It remains to see that FmF can be lifted to FF: For, assume the contrary and use Higman's lemma to reach a contradiction, using that a loft premon is FmF-factorable iff it is BmF-factorable [Cossu & T., 2022e., Thm 4.11]. ■

It turns out that Dickson's lemma is still enough for Theorem 20 (at the price of making its proof slightly, and unnecessarily, longer), while the proof of Corollary 20 is using Higman's lemma in a rather essential way.

A classical result (revised)

Lastly, some applications to the divisibility premon of a Dedekind-finite mon H (and hence to the classical theory). For, we'll denote by $\mathscr{I}(H)$ the set of irreds (i.e., $|_H$-irreds) and by $\mathscr{A}(H)$ the set of $|_H$-atoms: We've already noted (Slide 14) that $\mathscr{A}(H)$ is the set of (ordinary) atoms of H (by Dedekind-finiteness).

Theorem 22 (Cossu & T., 2022e, Theorem 5.1).

Every Dedekind-finite, weakly l.f.g.u. monoid H is factorable. If, in addition, every irred of H is an (ordinary) atom, then H is FmF-atomic.

Proof.

By def., H is a weakly l.f.g.u. mon iff $(H, |_H)$ is a weakly l.f.g.u. premon. On the other hand, $(H, |_H)$ is a weakly positive mon (by Dedekind-finiteness of H). So, H is a factorable mon by Corollary 19; and if $\mathscr{I}(H) = \mathscr{A}(H)$, then Thm 20 applies and hence H is FmF-atomic. ■

Corollary 23 (Cossu & T., 2022e, Corollaries 5.2 & 5.3).

Every acyclic, weakly l.f.g.u. monoid is FmF-atomic; and hence every unit-cancellative, weakly l.f.g.u., commutative monoid is FF-atomic.

Outline

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