

On zero-sum problems over metacyclic groups $C_n \rtimes_s C_2$

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Summary

1. Zero-sum problems
2. The invariants
3. On the group $C_n \rtimes_s C_2$
4. An open problem

Zero-sum problems

Let G be a finite group, written multiplicatively.

The **zero-sum problems** study conditions that guarantee that a given sequence over G admits a non-empty subsequence (possibly subject to prescribed properties such as length, weights, or repetitions) whose terms can be ordered so that their product equals 1, the identity element of G .

The terminology “zero-sum problems” originates from the abelian setting, where an additive notation is used.

Sequences over groups

A **sequence** S over G is a finite and unordered element of the free abelian monoid $\mathcal{F}(G)$, equipped with the concatenation product denoted by \cdot .

$$S = g_1 \cdot \dots \cdot g_k = \prod_{g \in G}^{\bullet} g^{[v_g(S)]} \in \mathcal{F}(G).$$

Remark

$g^2 (\in G)$ denotes the square of g , whereas $g^{[2]} = g \cdot g (\in \mathcal{F}(G))$ denotes a sequence of length two.

$T \in \mathcal{F}(G)$ is a **subsequence** of S if $T \mid S$ as elements of $\mathcal{F}(G)$, i.e., if $v_g(T) \leq v_g(S)$ for every $g \in G$. In this case,

$$S \cdot T^{[-1]} = \prod_{g \in G}^{\bullet} g^{[v_g(S) - v_g(T)]}.$$

If $K \subset G$, we define $S_K = \prod_{g \in K}^{\bullet} g^{[v_g(S)]}$.

$\pi(S) := \{g_{\tau(1)} \cdots g_{\tau(k)}; \tau \text{ is a permutation of } [1, k]\}$ is the **set of products** of S .

$\Pi(S) := \bigcup_{\substack{T|S \\ |T| \geq 1}} \pi(T) \subset G$ is the **set of subproducts** of S .

$\Pi_n(S) := \bigcup_{\substack{T|S \\ |T|=n}} \pi(T) \subset G$ is the **set of n -subproducts** of S .

We say that S is:

- **product-one sequence** if $1 \in \pi(S)$;
- **n -product-one sequence** if $1 \in \pi(S)$ and $|S| = n$;
- **n -product- H sequence** if $\pi(S) \cap H \neq \emptyset$ and $|S| = n$;
- **product-one free** if $1 \notin \Pi(S)$;
- **n -product-one free** if $1 \notin \pi(T)$ for any $T | S$ with $|T| = n$.

The invariants

The **small Davenport constant** of G is defined by

$$d(G) := \sup\{|S|; S \in \mathcal{F}(G) \text{ is product-one free}\}.$$

The **Gao constant** of G is defined by

$$E(G) := \min\{t \in \mathbb{N}; \text{every sequence } S \in \mathcal{F}(G) \text{ with } |S| \geq t \\ \text{has a } |G|\text{-product-one subsequence}\}.$$

- $E(G) \geq d(G) + |G|.$

Gao conjecture: equality holds.

It has been proven for abelian groups.

The direct and inverse problems

The **direct problems** consist of **determining the exact values** of the associated constants, whereas the **inverse problems** consist of **describing the structure** of $(|G|)$ -product-one free sequences of large (or maximal) length.

In this work, we are interested in the direct and inverse problems related to $E(G)$ for certain non-abelian groups G .

On the cyclic groups

Let $C_n = \langle y \mid y^n = 1 \rangle$ be the cyclic group of order n .

Proposition (direct problems for C_n)

$$d(C_n) = n - 1 \quad \text{and} \quad E(C_n) = 2n - 1.$$

Proposition (inverse problem for $E(C_n)$, Gao 1997)

Let $n \geq 2$ and $S \in \mathcal{F}(C_n)$ with $|S| = 2n - k$ and $2 \leq k \leq \lfloor n/2 \rfloor + 2$.
If S is n -product-one free, then $\exists a \cdot b \mid S$ such that $C_n = \langle ab^{-1} \rangle$,
 $\min\{v_a(S), v_b(S)\} \geq n - 2k + 3$ and $v_a(S) + v_b(S) \geq 2n - 2k + 2$.

In particular, $|S| = 2n - 2 \Rightarrow S = (a \cdot b)^{[n-1]}$.

On the dihedral groups

Let $D_{2n} = C_n \rtimes_{-1} C_2 = \langle x, y \mid x^2 = y^n = 1, yx = xy^{-1} \rangle$ be the dihedral group of order $2n$.

Proposition

$$d(D_{2n}) = n \quad \text{and} \quad E(D_{2n}) = 3n.$$

Proposition (inverse problem for $E(D_{2n})$, Oh-Zhong 2020)

Let $n \geq 4$ and $S \in \mathcal{F}(D_{2n})$ of length $|S| = E(D_{2n}) - 1 = 3n - 1$. Then S is $2n$ -product-one free $\iff \exists \alpha, \beta \in D_{2n}, t_1, t_2, t_3 \in \mathbb{Z}$ such that $D_{2n} \cong \langle \alpha, \beta \mid \alpha^2 = \beta^n = 1, \beta\alpha = \alpha\beta^{-1} \rangle$, $\gcd(t_1 - t_2, n) = 1$ and

$$S = (\beta^{t_1})^{[2n-1]} \cdot (\beta^{t_2})^{[n-1]} \cdot \alpha\beta^{t_3}.$$

For $n = 3$, there exists one more sequence: $S = 1^{[5]} \cdot x \cdot xy \cdot xy^2$.

On the group $C_n \rtimes_s C_2$

Let

$$G_{n,s} := C_n \rtimes_s C_2 = \langle x, y \mid x^n = y^2 = 1, yx = xy^s \rangle,$$

where $s^2 \equiv 1 \pmod{n}$ but $s \not\equiv \pm 1 \pmod{n}$.

We have $d(G_{n,s}) = n$. The inverse problem consists of the sequences $\beta^{[n-1]} \cdot \alpha \beta^t$, where $G_{n,s} = \langle \alpha, \beta \mid \alpha^2 = \beta^n = 1, \beta\alpha = \alpha\beta^s \rangle$.

Furthermore, $E(G_{n,s}) \geq d(G_{n,s}) + |G| = 3n$.

It is possible to factorize

$$n = n_1 n_2,$$

where $s \equiv -1 \pmod{n_1}$, $s \equiv 1 \pmod{n_2}$, and $\gcd(n_1, n_2) \in \{1, 2\}$.

If $\gcd(n_1, n_2) = 1$, then $G_{n,s} \cong C_{n_2} \times D_{2n_1} = \langle y^{n_1} \rangle \times \langle x, y^{n_2} \rangle$.

The main result

Proposition

Let n and s be as before. If $S \in \mathcal{F}(G_{n,s})$ has length $|S| = 3n - 1$ and $|S_{\langle y \rangle x}| \geq 2$, then S has a $2n$ -product-one subsequence.

Remark

- (i) Case $|S_{\langle y \rangle x}| = 1$ together with the inverse problem in $C_n \cong \langle y \rangle$ yields the inverse problem.
- (ii) Case $|S_{\langle y \rangle x}| = 0$ implies that $S \in \mathcal{F}(\langle y \rangle)$ and then S has a $2n$ -product-one subsequence.

Theorem

We have $E(G_{n,s}) = 3n$. Moreover, the $2n$ -product-one free sequences of length $3n - 1$ are $S = (\beta^{t_1})^{[2n-1]} \cdot (\beta^{t_2})^{[n-1]} \cdot \alpha\beta^{t_3}$, with $\gcd(t_1 - t_2, n) = 1$.

The cases and the method

The proof of the proposition considers three cases:

- (a) n even (Avelar–Brochero Martínez–R. 2023)
- (b) n odd and $n_1 \geq 5$ (Avelar–Brochero Martínez–R. 2023)
- (c) n odd and $n_1 = 3$ (Oh–R.–Zhao–Zhong 2025+)

We use the inductive method.

The case n even

Let $S \in \mathcal{F}(G_{n,s})$ with $|S| = 3n - 1$ and $|S_{\langle y \rangle x}| \geq 2$. Suppose that S is $2n$ -product-one free.

Let $H = \langle y^{n_1} \rangle \cong C_{n_2}$, $H \triangleleft G_{n,s}$, so that $G_{n,s}/H \cong D_{2n_1}$.

We may decompose $S = T_1 \cdot \dots \cdot T_{3n_2-2} \cdot R$, where the T_i 's are n_1 -product- H subsequences and $|R| = 2n_1 - 2$.

Let $h_i \in \pi(T_i) \subset H$.

If either $h_1 \cdot \dots \cdot h_{3n_2-2} \in \mathcal{F}(H)$ has a $2n_2$ -product-one subsequence or $R \in \mathcal{F}(G_{n,s})$ has a n_1 -product- H subsequence, then S has a $2n$ -product-one subsequence.

If neither occurs, the inverse results over H and $G_{n,s}/H$ force a **very rigid structure** for both $h_1 \cdot \dots \cdot h_{3n_2-2}$ and R .

Analyzing the distribution of terms in $S_{\langle y \rangle x}$ and **performing local replacements of terms**, we may avoid the form of those structures, hence S contains a $2n$ -product-one subsequence. □

The case n odd

Let $S \in \mathcal{F}(G_{n,s})$ with $|S| = 3n - 1$ and $|S_{\langle y \rangle x}| \geq 2$. Suppose that S is $2n$ -product-one free.

Let $H = \langle x, y^{n_2} \rangle \cong D_{2n_1}$, $H \triangleleft G_{n,s}$, so that $G_{n,s}/H \cong C_{n_2}$.

Since $|S| > E(C_{n_2}) = 2n_2 - 1$, we may decompose

$$S = T_1 \cdot \dots \cdot T_{3n_1-1} \cdot R,$$

where the T_i 's are n_2 -product- H subsequences and $|R| = n_2 - 1$.

Since $3n_1 - 1 = E(H) - 1$, the inverse problem ensures that if $h_i \in \pi(T_i)$, then w.l.o.g.

$$(A) \quad h_i = \begin{cases} y^{t_1 n_2} & \text{for } i \in [1, 2n_1 - 1], \\ y^{t_2 n_2} & \text{for } i \in [2n_1, 3n_1 - 2], \\ xy^{t_3 n_2} & \text{for } i = 3n_1 - 1, \end{cases} \quad \text{for } n_1 \geq 3,$$

$$(B) \quad h_1 = \dots = h_5 = 1, h_6 = x, h_7 = xy^{n_2}, h_8 = xy^{2n_2}, \quad \text{for } n_1 = 3.$$

Any decomposition of S as before must satisfy the equalities above.

The case n odd

Rules for the product to commute in $G_{n,s}$:

- $xy^\alpha \cdot xy^\beta = xy^\beta \cdot xy^\alpha \iff \alpha \equiv \beta \pmod{n_1}$,
- $xy^\alpha \cdot y^\gamma = y^\gamma \cdot xy^\alpha \iff \gamma \equiv 0 \pmod{n_1}$.

If **Case (A)** holds, then $\pi(T_i) = \{h_i\}$ and $|(T_i)_{\langle y \rangle x}|$ is even $\forall i \in [1, 3n_1 - 2]$.

In particular, if $|(T_i)_{\langle y \rangle x}| > 0$, then $h_i = 1 \forall i \in [1, 2n_1 - 1]$ or $\forall i \in [2n_1, 3n_1 - 2]$.

If **Case (B)** holds ($n_1 = 3$), then $\pi(T_i) = \{h_i\} \forall i \in [1, 8]$, $|(T_i)_{\langle y \rangle x}|$ is even $\forall i \in [1, 5]$ and odd $\forall i \in [6, 8]$.

The case n odd and $n_1 \geq 5$

We consider three subcases:

- (i) $2 \leq |S_{\langle y \rangle x}| \leq 2n_2$. Using the inverse problem of $E(C_n)$, we show (w.l.o.g.) that $T_1 = (y^a)^{[n_2]}$ and $T_{2n_1} = (y^b)^{[n_2]}$. Choose $g \mid R_{\langle y \rangle x}$. Then $T_1 \cdot T_{2n_1} \cdot g$ has a n_2 -product- H subsequence, whose product belongs to $\langle y \rangle x$. This is a contradiction to the form given in Case (A).
- (ii) $|S_{\langle y \rangle x}| \geq 2n_2 + 1$ and $|S_{\langle y \rangle} \cdot S_{\langle y^{n_1} \rangle}^{[-1]}| \geq n_2 - 1$. Then there are few terms in $S_{\langle y^{n_1} \rangle}$ and many terms in $S_{\langle y \rangle x}$. Thus $|(T_i)_{\langle y \rangle x}| \geq 2$ for some $i \in [1, 3n_1 - 2]$. One can perform replacements giving distinct products h_i , a contradiction.
- (iii) $|S_{\langle y \rangle x}| \geq 2n_2 + 1$ and $|S_{\langle y \rangle} \cdot S_{\langle y^{n_1} \rangle}^{[-1]}| \leq n_2 - 2$. Then S contains many terms in $\langle y \rangle x$, that ensures $h_i = 1$ for many i 's, and many terms in the subgroup $\langle y^{n_1} \rangle$, which ensures other n_1 -product-one subsequences more easily. This implies that S has a $2n$ -product-one subsequence.

The case n odd and $n_1 = 3$: Preliminaries

Recall that $G_{n,s} = C_{3n_2} \rtimes_s C_2 \cong C_{n_2} \times D_6 = \langle y^3 \rangle \times \langle x, y^{n_2} \rangle$.

For $A \subset G$ and $g \in G$, let $gA := \{ga \in G; a \in A\}$.

Let $\text{stab}(A) := \{g \in G; gA = A\}$ be the stabilizer of A .

$\text{stab}(A) \leq G$, and $\text{stab}(A) = G \iff A = G$.

For $H \triangleleft G$, let $\phi_H : G \rightarrow G/H$ denote the natural homomorphism.

Theorem (DeVos-Goddyn-Mohar)

Let G be a finite abelian group, $S \in \mathcal{F}(G)$, $n \in [1, |S|]$, and $H = \text{stab}(\Pi_n(S))$. Then

$$|\Pi_n(S)| \geq \left(\sum_{g \in G/H} \min\{n, v_g(\phi_H(S))\} - n + 1 \right) |H|$$

The case n odd and $n_1 = 3$: A key result

Proposition

Let $G = \langle y \rangle = \langle y^3 \rangle \times \langle y^{n_2} \rangle$, $G_1 = \langle y^3 \rangle$, $G_2 = \langle y^{n_2} \rangle$, $\varphi_i : G \rightarrow G_i$ be projections, and $S \in \mathcal{F}(S)$ be a $7n_2$ -product- G_2 . If for any decomposition $S = T_1 \cdot \dots \cdot T_7$ where each T_i is an n_2 -product- G_2 subsequence, the sequence $h_1 \cdot \dots \cdot h_7$ has no 6-product-one subsequences, where $h_i \in \pi(T_i)$, then $\Pi_{n_2-1}(\varphi_1(S)) = G_1$.

Proof. Let $H = \text{stab}(\Pi_{n_2-1}(\varphi_1(S))) \leq G_1$. Then $H = \langle y^{3r} \rangle$ for some $r \mid n_2$. Set $m := |H| = \frac{n_2}{r}$.

Case $r = 1$: Trivial.

Case $r \geq 5$: If $v_g(\phi_H(\varphi_1(S))) \leq n_2 - 1 \quad \forall g \in G/H$, then DeVos-Goddyn-Mohar Thm implies

$$n_2 = |G_1| \geq |\Pi_{n_2-1}(\varphi_1(S))| \geq (|S| - (n_2 - 1) + 1)|H| \geq 6n_2 + 2,$$

a contradiction.

Similarly, we obtain a contradiction provided $\exists g_1, g_2 \in G_1/H$ such that $v_{g_i}(\phi_H(\varphi_1(S))) \geq n_2$.

Thus \exists exactly one $g \in G_1/H$ such that $v_g(\phi_H(\varphi_1(S))) \geq n_2$.

By DeVos-Goddyn-Mohar Thm again, we obtain

$$v_g(\phi_H(\varphi_1(S))) \geq 7n_2 + 1 - r,$$

and then $\exists t \in [0, r-1]$ such that $\varphi_1(S)$ has at least $7n_2 + 1 - r$ terms in the coset $y^{3t}H$.

Set $S = S_1 \cdot S_2$, where $\varphi_1(S_1) \in \mathcal{F}(y^{3t}H)$ and $\text{supp}(\varphi_1(S_2)) \cap y^{3t}H = \emptyset$.

Since $E(C_m) = 2m - 1$, $\exists V_1 \dots V_{6r+1} \mid S_1$ such that $\pi(\varphi_1(V_i)) = \{y^{3tm}\}$, thus $\prod_{i \in I}^\bullet V_i$ is an n_2 -product- G_2 sequence for every $I \subset [1, 6r+1]$ with $|I| = r$.

Set $T_i = \prod_{j \in [r(i-1)+1, ri]}^\bullet V_j$ for $i \in [1, 6]$ and $T_7 = S \cdot (T_1 \dots T_6)^{[-1]}$.

By performing replacements, we obtain a contradiction. □

Proof of the case n odd and $n_1 = 3$

Recall that $S = T_1 \cdot \dots \cdot T_8 \cdot R$, where the T_i 's are n_2 -product- H subsequences and $|R| = n_2 - 1$.

Let $h_i \in \pi(T_i)$.

Warning! Crude replacements may take us from

$$\text{Case (A): } h_1 \cdot \dots \cdot h_8 = 1^{[5]} \cdot (y^{an_2})^{[2]} \cdot xy^{cn_2}$$

to

$$\text{Case (B): } h_1 \cdot \dots \cdot h_8 = 1^{[5]} \cdot x \cdot xy^{n_2} \cdot xy^{2n_2}$$

and vice versa, without leading to any contradiction.

Assume that $|\{i \in [1, 8]; |(T_i)_{\langle y \rangle_x}| > 0\}|$ is maximal.

We distinguish two cases:

- (i) $|(h_1 \cdot \dots \cdot h_8)_{\langle y^{n_2} \rangle_x}| \geq 2$. Then $h_1 \cdot \dots \cdot h_8$ is in Case (B). By performing replacements, we obtain a contradiction.

(ii) $|(h_1 \cdot \dots \cdot h_8)_{\langle y^{n_2} \rangle_x}| = 1$. Then $h_1 \cdot \dots \cdot h_8$ is in Case (A).

Subcase $|(T_1 \cdot \dots \cdot T_7)_{\langle y \rangle_x}| \geq 1$, say $h \mid (T_1)_{\langle y \rangle_x}$. Then

$$1 \in \pi(T_6 \cdot h \cdot T_7 \cdot (T_1 \cdot h^{[-1]}) \cdot T_2 \cdot T_3 \cdot T_4).$$

Subcase $|(T_1 \cdot \dots \cdot T_7)_{\langle y \rangle_x}| = 0$. Then we use the previous proposition to infer that

$$\Pi_{n_2-1}(\varphi_1(T_1 \cdot \dots \cdot T_7)) = \langle y^3 \rangle.$$

If $h \mid R_{\langle y \rangle_x}$, then $\varphi_1(h) \in \langle y^3 \rangle$ and $\exists T_0 \mid T_1 \cdot \dots \cdot T_7$ of length $|T_0| = n_2 - 1$ such that $\varphi_1(h \cdot T_0)$ is product-one, which contradicts the maximality.

Otherwise, $|R_{\langle y \rangle_x}| = 0$, whence $|S_{\langle y \rangle_x}| = |(T_8)_{\langle y \rangle_x}| \geq 2$. By performing local replacements between $(T_8)_{\langle y \rangle_x}$ and $T_1 \cdot \dots \cdot T_7 \cdot R$, we go back to the previous case.

This finishes the proof. □

An open problem

The direct and inverse problems related to $E(G)$ have already been solved for the non-split metacyclic groups of the form

$$\langle x, y \mid x^2 = y^n, y^{2n} = 1, yx = xy^s \rangle,$$

where $s^2 \equiv 1 \pmod{2n}$.

The only remaining case to complete the solution of the direct and inverse problems for $E(G)$ among all groups having a cyclic subgroup of index 2 occurs when $n = 2^t m$, m is odd, $t \geq 2$, and

$$G = \langle x, y \mid x^2 = y^m, y^n = 1, yx = xy^s \rangle.$$

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That's all, folks. Thank you!