Semi-ideas About Semi-domains

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Outline

1. Preliminaries on Semidomains
2. Polynomial Extensions of Semidomains
3. Goldbach Theorems for Semidomains
4. Integer-Valued Polynomials on Semidomains

Semidomains

Definition. A semiring S is a (nonempty) set endowed with two binary operations denoted by '$+$' and '$\cdot$' and called addition and multiplication, respectively, such that the following conditions hold:

1. $(S, +)$ is a commutative monoid with its identity element denoted by 0;
2. $(S, \cdot)$ is a commutative semigroup with an identity element denoted by 1;
3. $b \cdot (c + d) = b \cdot c + b \cdot d$ for all $b, c, d \in S$.

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Definition. A semidomain is a subsemiring of an integral domain.

Remark: For a semidomain S, the additive group $\text{gp}(S, +)$ is the smallest integral domain containing S; we denote it by $\mathcal{G}(S)$.

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Examples of Semidomains

- integral domains,
- Puiseux monoids that are closed under multiplication and contain 1,
- $\mathbb{N}_0$, $\mathbb{N}_0[x]$, $\mathbb{N}_0[x, x^{-1}]$, $\mathbb{N}_0[[x]]$

Factorization Properties of Semidomains

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Definitions

(a) The set of additive (resp., multiplicative) atoms of S is denoted by $\mathcal{A}_+(S)$ (resp., $\mathcal{A}(S)$).

(b) A semidomain S is atomic if the monoid S^* is atomic.

(c) A semidomain S is a bounded factorization semidomain (BFS) if the monoid S^* is a BFM.

(d) A semidomain S is a finite factorization semidomain (FFS) if the monoid S^* is an FFM.

(e) A semidomain S is a unique factorization semidomain (UFS) if the monoid S^* is a UFM.

Polynomial Semidomains

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- For a polynomial $f = c_n x^{k_n} + \cdots + c_0 x^{k_0}$ written in canonical form, we set $\mathrm{Supp}(f) := \{k_n, \ldots, k_0\}$.

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Unique Factorization Property

Theorem (folklore)

Let R be an integral domain. Then R is a UFD if and only if $R[x]$ is a UFD.

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Example. In the semidomain $\mathbb{N}_0[x]$, we have two factorizations of $x^5 + x^4 + x^3 + x^2 + x + 1$, namely,

$$(x + 1)(x^4 + x^2 + 1) \quad \text{and} \quad (x^2 + x + 1)(x^3 + 1).$$

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Theorem (Gotti-P., 2022)

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Open Question (Baeth-Chapman-Gotti)

Is $\mathbb{N}_0$ the only "honest" semidomain S satisfying that $(S, +)$ and S^* are both UFMs?

Bounded and Finite Factorization Properties

Example. Let M be a BFM torsion-free monoid that is not an FFM. Then $\mathbb{N}_0[x; M]$ is a BFS that is not an FFS.

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What about the power series extensions of semidomains?

Example. Consider the power series semidomain $\mathbb{N}_0[[x]]$. It does not satisfy the ACCP. Indeed,

$$\sum_{n=0}^{\infty} x^{n\cdot 2^k} = \left(1 + x^{2^k}\right) \sum_{n=0}^{\infty} x^{n\cdot 2^{k+1}}.$$

Elasticity

Let M be an atomic monoid. The elasticity of an element $b \in M \setminus \mathcal{U}(M)$, denoted by $\rho(b)$, is defined as

$$\rho(b) = \frac{\sup \mathsf{L}(b)}{\inf \mathsf{L}(b)}.$$

By convention, we set $\rho(u) = 1$ for every $u \in M^\times$. In addition, the elasticity of the monoid M is defined to be

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On the other hand, the set of elasticities of M is $R(M) := \{\rho(b) \mid b \in M\}$, and M is said to have full elasticity provided that $R(M) = (\mathbb{Q} \cup \{\infty\}) \cap [1, \rho(M)]$.

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Proposition (Gotti-P., 2022)

Let S be a semidomain such that $S[x]$ is atomic. Then $S[x]$ has full and infinite elasticity provided that $(S, +)$ is reduced.

Goldbach Conjecture for Polynomials

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Every odd monic polynomial f of degree $n \geq 2$ over every finite field $\mathbb{F}_q$ (except the case $f = x^2 + \alpha$ with q even) can be expressed as the sum of three irreducible polynomials.

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Theorem (Pollack, 2011)

Let D be a Noetherian domain with infinitely many maximal ideals. Every polynomial $f \in D[x]$ can be expressed as the sum of two irreducible polynomials.

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- Being able to subtract makes the problem easier.

$$\begin{aligned} 10 &= 5 + 5 \\ &= 3 + 7 \end{aligned}$$

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- We do not recover the Goldbach conjecture by considering polynomials of degree 0 in the previous statement.

Goldbach Conjecture for $\mathbb{N}_0[x, x^{-1}]$

Theorem (Liao-P., 2023)

Every polynomial $f \in \mathbb{N}_0[x, x^{-1}]$ can be written as the sum of two irreducibles provided that $f(1) > 3$ and $|\mathrm{Supp}(f)| > 1$.

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- Since $\mathbb{N}_0[x, x^{-1}]^\times = \{x^k \mid k \in \mathbb{Z}\}$, we can think of a polynomial in $\mathbb{N}_0[x, x^{-1}]$ as a formal sum of units. Then, as in the Goldbach conjecture, proving this involves partitioning a fixed set of units into two subsets, each one of which represents an irreducible.

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- Note that $f \in \mathbb{N}_0[x, x^{-1}]$ is irreducible when $f(1)$ is a prime number. Therefore, if the Goldbach conjecture were true, then our statement would hold for Laurent polynomials $f \in \mathbb{N}_0[x, x^{-1}]$ satisfying that $f(1)$ is an even number strictly greater than 2.

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- A similar statement does not hold for polynomials with positive integer coefficients as $x^5 + x^4 + x^3 + x^2$ cannot be written as the sum of two irreducibles.

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  Ex: The polynomial $x^2 + x + 1$ cannot be expressed as the sum of two irreducibles.

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Open Question

Can we write every element of $\mathbb{N}_0[[x, x^{-1}]]$ as the sum of at most two irreducibles?

Goldbach Conjecture for Laurent Polynomials

Theorem (Kaplan-P., 202?)

Let S be an additively reduced and additively atomic semidomain. The following statements are equivalent:

1. $\mathcal{A}_+(S) = S^\times$;
2. every $f \in S[x, x^{-1}]$ with $|\text{Supp}(f)| > 1$ can be expressed as the sum of at most 2 irreducibles;
3. there exists $k \in \mathbb{N}$ such that every $f \in S[x, x^{-1}]$ with $|\text{Supp}(f)| > 1$ can be expressed as the sum of at most k irreducibles.

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Moreover, if any of the previous statements hold and $f \in S[x, x^{-1}]$ does not have one of the following forms:

(a) $f = ax^{k_0} + bx^{k_1}$, where either $a \in S^\times$ or $b \in S^\times$;

(b) $f = ax^{k_0} + bx^{k_1} + cx^{k_2}$, where $a, b, c \in S^\times$,

then f is the sum of exactly two irreducible polynomials in $S[x, x^{-1}]$.

Goldbach Conjecture for Polynomial Semidomains

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Open Question

Can we write every element of $S[[x, x^{-1}]]$ as the sum of at most two irreducibles (assuming S is additively reduced and additively atomic)?

Number of Goldbach Decompositions

Example. Consider the additive monoid $M = \langle (\frac{2}{3})^k \mid k \in \mathbb{Z} \rangle$, which is clearly reduced. It is known that M is atomic and $\mathcal{A}(M) = \{r^k \mid k \in \mathbb{Z}\}$. Observe that M is a semidomain.

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$$f = \left[\left(\frac{2}{3} \right)^n x + 1 \right] + [s_n x + 1],$$

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Proposition (Kaplan-P., 202?)

Let S be an additively reduced and additively atomic semidomain for which $\mathcal{A}_+(S) = S^\times$. Suppose that $f \in S[[x, x^{-1}]]$ is not a polynomial. Then we can write f as the sum of at most three irreducibles in, at least, $2^{\aleph_0}$ ways.

Integer-Valued Polynomials on Semidomains

Definition. Let S be a semidomain with quotient field $\mathcal{F}(S)$, and let $\text{Int}(S)$ be the set of integer-valued polynomials on S, that is,

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Open Question

Let S be a semidomain that is not an integral domain. Is $\operatorname{Int}(S) \neq S[x]$?

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Example. Consider the semidomain $\text{Int}(\mathbb{R}_{\geq 0})$ whose elements we refer to as positive-real-valued polynomials.

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References

1. F. Campanini and A. Facchini: *Factorizations of polynomials with integral non-negative coefficients*, Semigroup Forum **99** (2019) 317–332.
2. F. Gotti and H. Polo: *On the arithmetic of polynomial semidomains*, Forum Math. **35** (2023) 1179–1197.
3. D. R. Hayes: *A Goldbach theorem for polynomials with integral coefficients*, Amer. Math. Monthly **72** (1965) 45–46.
4. G. Effinger and D. R. Hayes: *A complete solution to the polynomial* 3*-primes problem*, Bull. Amer. Math. Soc. **24** (1991) 363–369.
5. S. Liao and H. Polo: *A Goldbach theorem for Laurent polynomials with positive integer coefficients*, Amer. Math. Monthly (to appear).

The End

Thank you!