Multiplicative ideal theory in the context of w-module theory: the $w$trace property

Haleh Hamdi

University of Lisbon

Introduction

Fractional ideals

Assume that D is an integral domain with quotient field K.

Fractional ideal

A subset F of K is said to be a fractional ideal of D if

1. F is a D-submodule of K, that is, for any $x, y \in F$ and $d \in D$, $dx - y \in F$, and
2. $dF \subseteq D$ for some nonzero element d of D.

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For $I, J \in F(D)$, the ideal quotient of I by J is the set

$$(I :_K J) := \{x \in K \mid xJ \subseteq I\} \cong \mathrm{Hom}_D(J, I)$$

is a fractional ideal of D.

Star Operations

An essential concept in multiplicative ideal theory is the concept of "star operation" which was introduced by W. Krull in [2] and later first used by R. Gilmer in [1].

Definition

A star operation on D is a map $\star : F(D) \to F(D)$ with the following properties for all $0 \neq x \in K$ and $I, J \in F(D)$:

1. $(xI)_\star = xI_\star$ and $D_\star = D$;
2. $I \subseteq I_\star$;
3. $I \subseteq J \Rightarrow I_\star \subseteq J_\star$;
4. $(I_\star)_\star = I_\star$.

[1] Gilmer, R. "Multiplicative Ideal Theory". Marcel Dekker, 1972.

[2] Krull, W. 1936. "Beiträge zur Arithmetik kommutativer Integritätsbereiche. I. Math. Z. 41, 545-577.

For $I, J \in F(D)$, we have

1. $(IJ)_\star = (I_\star J)_\star$.
2. $(I + J)_\star = (I_\star + J_\star)_\star$.
3. If $I_\star \cap J_\star \neq 0$, then $I_\star \cap J_\star = (I_\star \cap J_\star)_\star$.
4. $(I_\star :_K J_\star) = (I_\star :_K J) = (I_\star :_K J)_\star$.

Some terms related to star operations

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- Let

$$I_{\star_f} := \bigcup J_\star$$

where J ranges over the set of nonzero finitely generated fractional ideals of D contained in I. The star operation $\star$ is called of finite character if for every fractional ideal I of D, $I_\star = I_{\star_f}$

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- A fractional ideal I of D is called $\star$-invertible if there exists $J \in F(D)$ such that $(IJ)_\star = D$.
  Equivalently, $(II^{-1})_\star = D$ where $I^{-1} := (D :_K I)$.

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- A fractional ideal I of D is called $\star$-invertible if there exists $J \in F(D)$ such that $(IJ)_\star = D$. Equivalently, $(II^{-1})_\star = D$ where $I^{-1} := (D :_K I)$.

Theorem

Let I be a nonzero ideal of D and $\star$ be a star operation of finite character on D. Then I is $\star$-invertible iff I is $\star$-finite and ID_P is principal for every maximal $\star$-ideal P of D.

Kang, B.G. 1989. "Prüfer v-multiplication domains and the ring $R[X]_{N_v}$". J. Algebra 123 (1989), 151-170.

Examples of star operations

Some well-known star operations on D are as follows. For $I \in F(D)$,

1. $I_d := I$.
2. $I_v := (I^{-1})^{-1}$ where $I^{-1} := (D :_K I) = \{x \in K \mid xI \subseteq D\}$.
3. $I_t := I_{v_f} = \bigcup\{J_v \mid J \subseteq I \text{ and } J \in f(D)\}$.

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$$I \subseteq I_t \subseteq I_v$$

For any star operation $\star$ on D,

$$I \subseteq I_\star \subseteq I_v,$$

$$I \subseteq I_{\star_f} \subseteq I_t.$$

An example of D with v-Max(D) is empty.

Let D be a valuation domain whose maximal ideal M is not principal. Then $v\text{-Max}(D) = \emptyset$ and $t\text{-Max}(D) = \{M\}$.

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An example of a maximal t-ideal which is not a maximal ideal.

The principal ideal (X) of $\mathbb{Z}[X]$ is a t-invertible prime t-ideal of $\mathbb{Z}[X]$, and hence a maximal t-ideal.
However, (X) is not a maximal ideal in $\mathbb{Z}$ because

$$\mathbb{Z}[X]/(X) \cong \mathbb{Z}.$$

Divisorial ideals

A fractional ideal I of D is called divisorial if

$$I_v := (I^{-1})^{-1} = I.$$

Mori domains

An integral domain D is called Mori if D satisfies the ascending chain condition on divisorial ideals.

Barucci, V. 2000. "Mori domains". In: Non-Noetherian Commutative Ring Theory, in: Mathematics and Its Applications, vol. 520, Kluwer Academic Publishers, 57-73.

Querré, J. 1971. "Sur une propiété des anneaux de Krull". Bulletin des Sciences Mathématiques, 2e Série, 95, 341-354

Characterizations of Mori domains

For an integral domain D, the following conditions are equivalent:

1. D is a Mori domain.
2. Every nonzero ideal I of D is v-finite.
3. Any descending chain of divisorial ideals with nonzero intersection stabilizes.

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An example of a non-Noetherian Mori domain

$$D = \mathbb{Q} + X\mathbb{R}[[X]].$$

Some facts related to Mori domains

1. If every prime ideal of D is v-finite, then D is not necessarily Mori.
   - In $A = \mathbb{Z} + X\mathbb{Q}[[X]]$, every prime ideal is v-finite but A is not a Mori domain because if $p_1, p_2, \ldots$ is an infinite sequence of prime integers, then $p_1 A \supset (p_1 A \cap p_2 A) \supset \ldots$ is an infinite descending chain of divisorial ideals with nonzero intersection.

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2. If D is an integrally closed Mori domain, then $D[X]$ is a Mori domain.
3. Mori domains do not permit primary decomposition of divisorial ideals.
4. Mori domains do not necessarily satisfy the Principal Ideal Theorem.

In 1975, Glaz and Vasconcelos studied the divisibility property of flat ideals. More precisely, they proved that if I is a flat ideal of a commutative ring D, $x \in D$, and J is a finitely generated ideal containing a regular element with $J^{-1} = D$, then $Jx \subseteq I$ implies $x \in I$.

In 1977, they introduced the concept of semi-divisorial ideals to investigate the divisibility properties of flat ideals in integral domains.

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In 1977, they introduced the concept of semi-divisorial ideals to investigate the divisibility properties of flat ideals in integral domains.

Semi-divisorial ideals

An ideal I of D is called semi-divisorial if $(I :_K J) = I$ whenever J is a finitely generated ideal of D with $J^{-1} = D$.

Glaz, S., and Vasconcelos, W.V. (1975). "Flat ideals I." *Comunn. Algebra* 3(6), 531-543.

Glaz, S., and Vasconcelos, W.V. (1977). "Flat ideals II." *Manuscripta Mathematica*, 22(4), 325-341.

In 1980, semi-divisorial ideals were studied by Hedstrom and Houston under the name of F_∞-ideals. They showed that the map

$$I \to I_* = \bigcup\{(I :_K J) \mid J \text{ is a f.g. ideal of } D \text{ with } J^{-1} = D\}$$

is a star operation.

Hedstrom, J.R., and Houston, E.G. (1980). "Some remarks on star-operations." *Journal of Pure and Applied Algebra*, 18, 37-44.

In 1997, Wang and McCasland extensively explored the F_∞-operation in a broader context under the name of the w-operation.
They referred to a finitely generated ideal J of D with $J^{-1} = D$ as a Glaz–Vasconcelos ideal, denoted by $J \in GV(D)$. Therefore,

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They referred to a finitely generated ideal J of D with $J^{-1} = D$ as a Glaz–Vasconcelos ideal, denoted by $J \in GV(D)$. Therefore,

The w-operation

The map $w : F(D) \to F(D)$, defined by

$$I \to I_w := \{x \in K \mid xJ \subseteq I \text{ for some } J \in GV(D)\}$$

is a star operation.

Wang, F., and McCasland, R.L. (1997). "On w-modules over strong Mori domains." *Communications in Algebra*, 25(4), 1285–1306.

Wang, F., and McCasland, R.L. (1999). "On strong Mori domains." *Journal of Pure and Applied Algebra*, 135(2), 155–165.

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4. For any $I \in F(D)$,

$$I_w = \bigcap_{P \in w\text{-Max}(D)} ID_P.$$

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Strong Mori domains

An integral domain D is called strong Mori if the ascending chain condition holds for w-ideals.

An example of a non-Noetherian strong Mori domain

The polynomial ring $R = F[X_1, X_2, \ldots]$ in countably many indeterminates over any field F.

Wang, F., and McCasland, R.L. (1997). "On w-modules over strong Mori domains." *Communications in Algebra*, 25(4), 1285–1306.

Wang, F., and McCasland, R.L. (1999). "On strong Mori domains." *Journal of Pure and Applied Algebra*, 135(2), 155–165.

Characterizations of strong Mori domains

For an integral domain D, the following statements are equivalent:

1. D is a strong Mori domain.
2. Every w-ideal of D is w-finite.
3. Every prime w-ideal of D is w-finite.
4. D_P is Noetherian for every maximal w-ideal P of D and each nonzero element of D lies in only finitely many maximal w-ideals of D.

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1. Every strong Mori domain does satisfy the Principal Ideal Theorem.
2. If D is a strong Mori domain, then $D[X]$ is likewise a strong Mori domain.
3. Every w-ideal of a strong Mori domain has a primary decomposition consisting of w-ideals.

Krull domains

Definition

An integral domain D is called Krull if it satisfies the following three conditions:

(1) $D = \bigcap D_P$, where P ranges over all prime ideals of D of height one.

(2) For every prime ideal P of D of height one, D_P is a discrete valuation ring, that is, a Noetherian local ring whose unique maximal ideal is principal.

(3) Any nonzero element of D lies in only a finite number of prime ideals of height one.

Krull, W. 1931. "Allgemeine Bewertungstheorie". J. Reine Angew. Math., 167. 160-196.

Fossum, R. M. "The Divisor Class Group of a Krull Domain". Vol. 74. Springer Science and Business Media, 2012.

Characterizations of Krull domains

The following statements are equivalent for an integral domain D:

1. D is a Krull domain.
2. D is an integrally closed strong Mori domain.
3. D is a strong Mori domain and D_P is a DVR for each maximal w-ideal P of D.
4. D is a strong Mori Prüfer v-multiplication domain (i.e. every finitely generated ideal of D is t-invertible).
5. Each nonzero ideal of D is w-invertible.

Various types of integrality

Let D be an integral domain with quotient field K.

1. An element $u \in K$ is integral over D iff $uI \subseteq I$ for some nonzero finitely generated ideal I of D.
   The set of all integral elements of D, denoted by $\bar{D}$, is called the integral closure of D.

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   The set of all integral elements of D, denoted by $\bar{D}$, is called the integral closure of D.
2. An element $u \in K$ is almost integral over D iff $uI \subseteq I$ for some fractional ideal I of D. The set of all almost integral elements of D, denoted by D^*, is called the complete integral closure of D.

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The Mori-Nagata Theorem

The integral closure of a Noetherian domain is a Krull domain.

In 2004, another type of integrality has been defined by F. Wang.

3. An element $u \in K$ is w-integral over D iff $uI_w \subseteq I_w$ for some nonzero finitely generated ideal I of D. The set of all w-integral elements of D, denoted by D^w, is called w-integral closure of D.

It is clear that

$$D \subseteq \bar{D} \subseteq D^w \subseteq D^*.$$

Theorem

Let D be a strong Mori domain. Then $D^w = D^*$ is a Krull domain.

[1] Wang, F. 2004. "On induced operations and UMT-domains". Sichuan Shifan Daxue Xuebao Kexue Ban 27, 1-9.

wTrace Property

Trace Properties

Let $\star$ be a star operation on D. We say that D is a $\star$TP domain (resp., $\star$RTP domain) if, for each nonzero ideal I of D, either $(II^{-1})_\star = D$ or $(II^{-1})_\star$ is a prime ideal (resp., radical ideal) of D.

$$\star\text{TP domains} \Longrightarrow \star\text{RTP domains.}$$

Mimouni, A. "Star-operations and trace properties. In: Badawi, A., (ed). Trends in Commutative Rings Research. New York, 2002: Nova Science Publishers, Inc., 77-92.

This section is based on a joint work with Gyu Whan Chang.

wTP domains

An integral domain D is called a wTP domain if for each nonzero ideal I of D, either $(II^{-1})_w = D$ or $(II^{-1})_w$ is a prime ideal.

wRTP domains

An integral domain D is called a wRTP domain if for each nonzero ideal I of D, either $(II^{-1})_w = D$ or $(II^{-1})_w$ is a radical ideal.

$$\text{Krull domains} \Longrightarrow w\text{TP domains} \Longrightarrow w\text{RTP domains.}$$

Some properties of wTP and wRTP domains

Theorem

An integral domain D is a TP domain (resp., an RTP domain) iff D is a wTP domain (resp., wRTP domain) all of whose nonzero maximal ideals are t-ideals.

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1. D_S is a wTP domain (resp., wRTP domain) for any multiplicative set S of D.
2. D_P is a TP domain (resp., an RTP domain) for each maximal t-ideal P of D.

wTrace Properties of Strong Mori Domains

Theorem

Let D be a strong Mori domain that is not a field. Then D is a wTP domain iff either D is a Krull domain, or

(i) $t\text{-dim}(D) = 1$,

(ii) D has a unique non-t-invertible maximal t-ideal M, and

(iii) $M^{-1} = D^*$.

- The t-dimension of D, denoted by $t\text{-dim}(D)$, is defined by

$$\sup\{n \mid P_1 \subsetneq \cdots \subsetneq P_n \text{ for some prime } t\text{-ideals } P_i \text{ of } D\}.$$

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Let D be a strong Mori domain that is not a field. Then D is a wRTP domain iff D_P is a TP domain for all $P \in w$-Max(D).

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Theorem

Let D be a strong Mori domain that is not a field. Then D is a wRTP domain iff $t\text{-dim}(D) = 1$ and for each prime t-ideal P of D, either PD_P is principal or $(D_P : PD_P) = \overline{D_P}$.

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Let D be a strong Mori domain that is not a field. Then D is a wRTP domain iff $t\text{-dim}(D) = 1$ and for each prime t-ideal P of D, either PD_P is principal or $(D_P : PD_P) = \overline{D_P}$.

Theorem

Let D be a weakly Matlis domain. Then D is a wRTP domain iff D_P is an RTP domain for each maximal t-ideal P of D.

- An integral domain D is called weakly Matlis if every nonzero nonunit element of D belongs to at most a finite number of maximal t-ideals of D, and each prime t-ideal of D is contained in a unique maximal t-ideal.

*w*Trace Properties of P*v*MDs

Theorem

Let D be a PvMD of t-dimension one. Then the following statements are equivalent:

1. D is a wTP domain.
2. D is a wRTP domain with at most one non-t-invertible maximal t-ideal.
3. D has at most one non-t-invertible maximal t-ideal and every nonzero nonunit element of D belongs to at most a finite number of maximal t-ideals of D.

Trace properties in polynomial rings

- Any polynomial ring over a field is a TP-domain.

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- Any polynomial ring over a field is a TP-domain.
- Consider the following three conditions for an integral domain D.
  (1) $D[X]$ is a wTP domain.
  (2) $D[X]_{N_v}$ is a TP domain where $N_v = \{f \in D[X] \mid f \neq 0 \text{ and } c(f)_v = D\}$.
  (3) D is a wTP domain.

  Then the implications of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold.

Trace properties in polynomial rings

- Any polynomial ring over a field is a TP-domain.
- Consider the following three conditions for an integral domain D.
  (1) $D[X]$ is a wTP domain.
  (2) $D[X]_{N_v}$ is a TP domain where $N_v = \{f \in D[X] \mid f \neq 0 \text{ and } c(f)_v = D\}$.
  (3) D is a wTP domain.

  Then the implications of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold.

- If D is a strong Mori domain or a PvMD, then the aforementioned statements are equivalent:

Future research topic!

Let I be a nonzero fractional ideal of D. Then

- I is called a strong ideal if $II^{-1} = I$.
- I is called strongly divisorial if I is strong and divisorial

Now, we say that I is a strong w-ideal if I is strong and a w-ideal, i.e., $I_w = I = II^{-1}$.

Key Properties

Every strong w-ideal of D is a prime ideal (resp., radical ideal) iff D is a wTP domain (resp., wRTP domain).

Let D be a wTP domain and S the set of all strongly divisorial ideals of D. Then

$$D^* = \bigcup \{P^{-1} \mid P \in S \cap \mathrm{Spec}(D) \text{ or } P = D\}.$$

• What are the trace properties of commutative cancellative monoids?

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Thank you very much for your attention.